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Global Solutions of the Boltzmann Equation with the External Force

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1 Introduction and Main Theorem

The Boltzmann equation in the field of a potential force is written as

$$\frac{\partial f}{\partial t}+\xi\cdot\nabla_x f-\nabla_x\Phi(x)\cdot\nabla_\xi f=Q(f,f). \tag{1.1}$$

Here $f=f(t,x,\xi)$ is the unknown scalar function which stands for the distributional density of gas particles at time t in the phase space of position $x\in\mathbb{R}^3$ and velocity $\xi\in\mathbb{R}^3$, while Φ is a given potential function and Q is the *collision* operator. We restrict ourself to the hard sphere model for Q.

We also assume that Φ is t-independent, i.e. $\Phi=\Phi(x)$. Then, the *local Maxwellian* given by

$$\bar{M}(x,\xi)=\frac{\bar{\rho}}{(2\pi R\bar{\theta})^{3/2}}\exp\Big\{-\frac{1}{R\bar{\theta}}\Big(\Phi(x)+\frac{|\xi|^2}{2}\Big)\Big\}\equiv\mathrm{M}_{[\hat{\rho}(x),0,\bar{\theta}]}(\xi), \tag{1.2}$$

where $R>0$ is the gas constant and $\bar{\rho},\bar{\theta}>0$ are some constants, is a stationary solution to (1.1), which describes, physically, the distributional density of a gas in an equilibrium state with the mass density

$$\hat{\rho}(x)=\bar{\rho}\exp\Big\{-\frac{\Phi(x)}{R\bar{\theta}}\Big\}, \tag{1.3}$$

zero bulk velocity, and absolute temperature $\bar{\theta}$.

Our aim is to show that this local Maxwellian is asymptotically stable. More precisely, we consider the Cauchy problem of (1.1) for $t \geq 0$ and $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ with the initial condition

$$f(0, x, \xi) = f_0(x, \xi), \qquad (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{1.4}$$

For a constant $\theta_- > 0$, let

$$M_-(\xi) = \frac{1}{(2\pi R\theta_-)^{3/2}} \exp\Big(-\frac{|\xi|^2}{2R\theta_-}\Big),$$

be the Maxwellian specified by θ_-. We will prove the

Main Theorem. *For any positive constants $\bar{\rho}, \bar{\theta}$ and for any integer $N \geq 4$, there exist a small positive constant ϵ and a positive constant $\theta_- \in (\bar{\theta}/2, \bar{\theta})$ such that if the potential Φ and the initial data of the form $f_0 = \bar{M} + M_-^{1/2} g_0$ satisfy*

$$||\Phi||_{L^2_x} + \sum_{1 \leq |\alpha| \leq N+1} ||\partial_x^\alpha \Phi||_{L^2_x} + \sum_{|\alpha|+|\beta| \leq N} ||\partial_x^\alpha \partial_\xi^\beta g_0||_{L^2_{x,\xi}} \leq \epsilon, \tag{1.5}$$

then, the Cauchy problem (1.1), (1.4) has a unique classical solution in the large in time of the form

$$f = \bar{M} + M_-^{1/2} g,$$

with g satisfying

$$\partial_t^\gamma \partial_x^\alpha \partial_\xi^\beta g \in C(\overline{\mathbb{R}}_+ \; ; \; L^2_{x,\xi}) \cap L^\infty(\mathbb{R}_+ \; ; \; L^2_{x,\xi}),$$

for any $\gamma + |\alpha| + |\beta| \leq N$, and

$$(1 + |\xi|)^{1/2} \partial_t^\gamma \partial_x^\alpha \partial_\xi^\beta g \in L^2_{t,x,\xi}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3).$$

for any $\gamma + |\alpha| > 0, \gamma + |\alpha| + |\beta| \leq N$. Moreover, the asymptotic property

$$\sup_{x \in \mathbb{R}^3} ||\partial_t^\gamma \partial_x^\alpha \partial_\xi^\beta g(t, x, \cdot)||_{L^2_\xi} \to 0 \qquad (t \to \infty),$$

holds for any $\gamma + |\alpha| + |\beta| \leq N - 4$.

Remark 1. A similar result was announced in [1] on the L^∞_ξ solution under the additional assumption that the support of Φ is compact. On the other hand, the compressible Navier-Stokes equations with the potential force was solved in [7] on the same global existence and asymptotic property under the same assumption on Φ as in our theorem. Recently, the initial value problem of the Vlasov-Poisson-Boltzmann system has been solved globally in time in the torus in [3], and in the whole space in [9]. See also [4] for the Vlasov-Maxwell-Boltzmann system.

Remark 2. The assumption (1.5) requires, among others, that $\Phi \in L^2_x$, but this does not contradict to the fact that the potential Φ is unique only up to an additive constant, because this constant can be absorbed into the constant $\bar{\rho}$.

Our proof relies on the energy method based on the macro-micro (fluid dynamic-kinetic) decomposition of the Boltzmann equation which was developed recently in [6].

The energy estimates for the macroscopic (fluid) component of f are obtained with the H-theorem for the lower order derivatives and by the usual integration by parts on the differential equation for higher order derivatives. Both estimates contain Sobolev norms of the microscopic (kinetic) component of f. It should be noted that if these norms are dropped, our estimates coincide with those derived in [7] for the compressible Navier-Stokes equation.

The norms of the microscopic component can be estimated by virtue of the microscopic H-theorem of a new type, i.e., the negative definiteness of the linearized collision operator with different weight functions on the space of microscopic components, and again by the integration by parts on the differential equations for derivatives of the microscopic component.

This technique has been developed in [6] for the force-free case, where the energy estimates can be closed only with (t, x) derivatives of f. In our case, however, ξ derivatives should be also included. Recently, in [5], another L^2 energy method has been proposed for the Boltzmann equation. Although the technique is quite different from [6], it applies also to our case, to deduce the same result.

The global existence is concluded by combining the local existence and the energy estimates. Our local solutions should be, therefore, in consistence with our energy estimates, that is, they should be L^2 solutions with respect to ξ as well as (t, x). Such solutions can be constructed by using the L^2 estimate of Q derived in [2] and time local L^2 energy estimates.

The present paper is organised as follows. In the next section, it is shown that the macro-micro decomposition of [6] works also in the presence of the external force. Actually, we will derive a system of fluid-type equations governing the macroscopic component whose main parts consist of the compressible Navier-Stokes equations, and the equation governing the microscopic component. These equations will be used in §3 to establish an a priori estimate for L^2 Sobolev norm of the solution. The norms of lower order derivatives of the macroscopic component are estimated by the celebrated H-theorem of the Boltzmann equation and those of higher order derivatives by applying the L^2-energy technique developed for the compressible Navier-Stokes equations in [7]. The L^2-energy method is also shown to work for norms of the microscopic component thanks to a microscopic version of the H-theorem. Since the computation is delicate and lengthy, only an outline is presented. The detaill will appear in [8]. The a priori estimate in §3 is combined with the local existence theorem established in §4, to conclude the global existence theorem under the smallness conditions both on the initial data and potential function.theorem. L^2-local solutions are constructed by the contraction mapping principle based on local energy estimates. It is noted that the smallness conditions on the initial data and potential function are necessary even for the local result.

2 Macro-Micro Decomposition

The decomposition of [6] has two significances: It made possible to develop a new theory of the Boltzmann equation, that is, the L^2-theory based on the energy method familiar in the theory of partial differential equations, and it provided a new method to reveal the mathematical and physical relation between the Boltzmann equation and the compressible Navier-Stokes equations.

Originally, it was developed for the force-free case, but it works also for our case. Thus, we decompose the solution $f(t,x,\xi)$ of (1.1) into the macroscopic (fluid) component specified by the local Maxwellian $\mathbf{M} = \mathbf{M}(t,x,\xi) = \mathbf{M}_{[\rho,u,\theta]}(\xi)$ and the microscopic (kinetic) component $\mathbf{G} = \mathbf{G}(t,x,\xi)$:

$$f(t,x,\xi) = \mathbf{M}(t,x,\xi) + \mathbf{G}(t,x,\xi). \tag{2.1}$$

Here,

$$\mathbf{M} \equiv \mathbf{M}_{[\rho,u,\theta]}(t,x,\xi) \equiv \frac{\rho(t,x)}{\sqrt{(2\pi R\theta(t,x))^3}} \exp\left(-\frac{|\xi - u(t,x)|^2}{2R\theta(t,x)}\right), \tag{2.2}$$

is the local Maxwellian with the macroscopic density $\rho(t,x)$, bulk velocity $u(t,x) = (u_1,u_2,u_3)$, and temperature θ associated with this particular f by

$$\begin{cases} \rho(t,x) \equiv \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \\ \rho(t,x)u_i(t,x) \equiv \int_{\mathbf{R}^3} \psi_i(\xi)f(t,x,\xi)d\xi \text{ for } i = 1,2,3, \\ \left[\rho\left(\mathbf{E} + \frac{1}{2}|u|^2\right)\right](t,x) \equiv \int_{\mathbf{R}^3} \psi_4(\xi)f(t,x,\xi)d\xi, \\ \mathbf{E} = \frac{3}{2}R\theta, \end{cases} \tag{2.3}$$

$\mathbf{E}$ being the inner energy density. And $\psi_\alpha(\xi)$, $\alpha = 0,1,\cdots,4$, are the *collision invariants*,

$$\psi_0(\xi) \equiv 1, \quad \psi_i(\xi) \equiv \xi_i \text{ for } i = 1,2,3, \quad \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \tag{2.4}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_\alpha(\xi)Q(h,g)d\xi = 0, \text{ for } \alpha = 0,1,2,3,4.$$

With this local Maxwellian, we now define the inner product

$$\langle h, g\rangle_{\mathbf{M}} \equiv \int_{\mathbf{R}^3} \frac{1}{\mathbf{M}} h(\xi) g(\xi) d\xi.$$

Then, the functions

$$(2.5)\qquad \begin{cases} \chi_0(\xi;\rho,u,\theta) & \equiv \frac{1}{\sqrt{\rho}}\mathbf{M}, \\ \chi_i(\xi;\rho,u,\theta) & \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}}\mathbf{M} \ \text{ for } \ i = 1,2,3, \\ \chi_4(\xi;\rho,u,\theta) & \equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi-u|^2}{R\theta} - 3\right)\mathbf{M}, \\ \langle \chi_i, \chi_j\rangle_{\mathbf{M}} & = \delta_{ij}, \ \text{ for } i,j = 0,1,2,3,4, \end{cases}$$

form an orthonormal basis of the space of macroscopic (fluid) components of the solution, so that

$$(2.6)\qquad \mathbf{P}_0 h \equiv \sum_{j=0}^{4} \langle h, \chi_j\rangle_{\mathbf{M}} \chi_j, \qquad \mathbf{P}_1 h \equiv h - \mathbf{P}_0 h,$$

define orthogonal self-adjoint projections w.r.t. the inner product $\langle\cdot,\cdot\rangle_{\mathbf{M}}$. $\mathbf{P}_0$ is called the macroscopic projection and $\mathbf{P}_1$ the microscopic projection, respectively. A function $h(\xi)$ is called microscopic or kinetic if it has no fluid components, that is,

$$(2.7)\qquad \int_{\mathbf{R}^3} h(\xi)\psi_\alpha(\xi) d\xi = 0, \ \text{ for } \alpha = 0,1,2,3,4.$$

It is clear that such a function is in the range of the microscopic projection $\mathbf{P}_1$. Notice that the decomposition (2.1) satisfies

$$(2.8)\qquad \mathbf{P}_0 f = \mathbf{M}, \qquad \mathbf{P}_1 f = \mathbf{G},$$

Plug (2.1) into (1.1) to deduce

$$(2.9)\qquad (\mathbf{M}+\mathbf{G})_t + \xi\cdot\nabla_x(\mathbf{M}+\mathbf{G}) - \nabla_x\Phi\cdot\nabla_\xi(\mathbf{M}+\mathbf{G}) = 2Q(\mathbf{G},\mathbf{M}) + Q(\mathbf{G},\mathbf{G}).$$

By applying $\mathbf{P}_0$ to (2.9), we have

$$(2.10)\qquad \mathbf{M}_t + \mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{M}\Big) + \mathbf{P}_0\Big(\xi\cdot\nabla_x\mathbf{G}\Big) - \nabla_x\Phi\cdot\nabla_\xi\mathbf{M} = 0.$$

where we have used the fact

$$(2.11)\qquad \mathbf{P}_0 f_t = (\mathbf{P}_0 f)_t = \mathbf{M}_t.$$

Notice that (2.11) holds only for the particular f specifying $\mathbf{M}$ and $\mathbf{P}_0$, but does not if $\mathbf{P}_0$ is defined with (ρ, u, θ) independent of f.

As usual, the system of five conservation laws follow by taking the inner product of (2.10) and the collision invariants $\psi_\alpha(\xi)$:

$$
\tag{2.12}
\begin{cases}
\rho_t + \operatorname{div}_x \rho u = 0, \\
(\rho u_i)_t + \sum\limits_{j=1}^{3} (\rho u_i u_j)_{x_j} + p_{x_i} - \hat{p}_{x_i} + (\rho - \hat{\rho})\,\Phi_{x_i} \\
\qquad\qquad = -\displaystyle\int_{\mathbf{R}^3} \psi_i\,(\xi \cdot \nabla_x \mathbf{G})\, d\xi, \quad i = 1, 2, 3, \\
\left[\rho(\tfrac{1}{2}|u|^2 + \mathbf{E})\right]_t + \sum\limits_{j=1}^{3} \left(u_j\Big(\rho\left(\tfrac{1}{2}|u|^2 + \mathbf{E}\right) + p\Big)\right)_{x_j} + \rho u \cdot \nabla_x \Phi \\
\qquad\qquad = -\displaystyle\int_{\mathbf{R}^3} \psi_4\,(\xi \cdot \nabla_x \mathbf{G})\, d\xi.
\end{cases}
$$

Here, p is the pressure for the monatomic gases: $p = \frac{2}{3}\rho\mathbf{E} = R\rho\theta$, and we have used the fact that for $\hat{\rho}(x) = \bar{\rho}e^{-\Phi(x)/(R\bar{\theta})}$, $\bar{\theta}$ in (1.2),

$$
\hat{p}(x) = R\hat{\rho}(x)\bar{\theta}, \qquad \hat{p}_{x_i} + \hat{\rho}\Phi_{x_i} = 0.
$$

On the other hand, the microscopic equation for $\mathbf{G}$ is obtained by applying $\mathbf{P}_1$ to (2.9) and using again (2.11):

$$
\tag{2.13}
\mathbf{G}_t + \mathbf{P}_1\Big(\xi \cdot \nabla_x \mathbf{G} + \xi \cdot \nabla_x \mathbf{M}\Big) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} = L_{\mathbf{M}}\mathbf{G} + Q(\mathbf{G}, \mathbf{G}),
$$

where

$$
L_{\mathbf{M}} g = L_{[\rho,u,\theta]} g \equiv Q\Big(\mathbf{M} + g, \mathbf{M} + g\Big) - Q(g, g).
$$

It is classical that $L_{\mathbf{M}}$ is self-adjoint and negative definite on the space of microscopic components w.r.t. the inner product $< \cdot, \cdot >_{\mathbf{M}}$. Therefore, it has a bounded inverse and (2.13) can be rewritten as

$$
\tag{2.14}
\begin{aligned}
\mathbf{G} &= L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\Big) \\
&\qquad + L_{\mathbf{M}}^{-1}\Big(\mathbf{G}_t + \mathbf{P}_1(\xi \cdot \nabla_x \mathbf{G}) - \nabla_x \Phi \cdot \nabla_\xi \mathbf{G} - Q(\mathbf{G}, \mathbf{G})\Big) \\
&:= L_{\mathbf{M}}^{-1}\Big(\mathbf{P}_1(\xi \cdot \nabla_x \mathbf{M})\Big) + \Theta,
\end{aligned}
$$

Set

$$\begin{cases} A_j(\xi) = & \frac{1}{2}(|\xi|^2-5)\xi_j, \quad j=1,2,3, \\ B_{ij}(\xi) = & \xi_i\xi_j - \frac{1}{3}\delta_{ij}|\xi|^2, \quad i,j=1,2,3, \\ \mu(\theta) = & -R\theta \int_{\mathbf{R}^3} B_{ij}\left(\frac{\xi}{\sqrt{R\theta}}\right) L^{-1}_{\mathbf{M}_{[1,u,\theta]}} \left(B_{ij}\left(\frac{\xi}{\sqrt{R\theta}}\right)\mathbf{M}_{[1,u,\theta]}\right) d\xi > 0, \quad i \neq j, \\ \kappa(\theta) = & -R^2\theta \int_{\mathbf{R}^3} A_l\left(\frac{\xi}{\sqrt{R\theta}}\right) L^{-1}_{\mathbf{M}_{[1,u,\theta]}} \left(A_l\left(\frac{\xi}{\sqrt{R\theta}}\right)\mathbf{M}_{[1,u,\theta]}\right) d\xi > 0. \end{cases}$$

After straightforward but lengthy computations, we have

$$\begin{cases} -\int_{\mathbf{R}^3} \psi_i \xi \cdot \nabla_x L^{-1}_{\mathbf{M}}\Big(\mathbf{P}_1(\xi\cdot\nabla_x \mathbf{M})\Big) d\xi = & \sum_{j=1}^{3} \left[\mu(\theta)\left(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right]_{x_j}, \quad i=1,2,3, \\ -\int_{\mathbf{R}^3} \psi_4 \xi \cdot \nabla_x L^{-1}_{\mathbf{M}}\Big(\mathbf{P}_1(\xi\cdot\nabla_x \mathbf{M})\Big) d\xi = & \sum_{j=1}^{3} \left(\kappa(\theta)\theta_{x_j}\right)_{x_j} \\ & + \sum_{i,j=1}^{3} \left\{\mu(\theta)u_i\left(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right\}_{x_j}. \end{cases}$$

Plugging these relations and (2.14) into (2.12) yields now another representation of the equation (1.1) which contains a system of fluid-type equations:

$$\begin{aligned} & \rho_t + \mathrm{div}_x(\rho u) = 0, \\ & (\rho u_i)_t + \sum_{j=1}^{3} (\rho u_i u_j)_{x_j} + (p-\hat{p})_{x_i} + (\rho - \hat{\rho})\,\Phi_{x_i} \\ & \qquad = \sum_{j=1}^{3} [\,\mu(\theta)\,(\,u_{ix_j} + u_{jx_i} - \frac{2}{3}\delta_{ij}\mathrm{div}_x u\,)\,]_{x_j} \\ & \qquad - \int_{\mathbf{R}^3} \psi_i\,(\xi\cdot\nabla_x\Theta)\,d\xi, \quad i=1,2,3, \\ & \left[\rho(\frac{1}{2}|u|^2+\mathbf{E})\right]_t + \sum_{j=1}^{3}\left(u_j\Big(\rho\left(\frac{1}{2}|u|^2+\mathbf{E}\right)+p\Big)\right)_{x_j} \\ & \qquad + \rho u\cdot\nabla_x\Phi = \sum_{i,j=1}^{3}\left\{\mu(\theta)u_i\left(u_{ix_j}+u_{jx_i}-\frac{2}{3}\delta_{ij}\mathrm{div}_x u\right)\right\}_{x_j} \\ & \qquad + \sum_{j=1}^{3}\left(\kappa(\theta)\theta_{x_j}\right)_{x_j} - \int_{\mathbf{R}^3}\psi_4\,(\xi\cdot\nabla_x\Theta)\,d\xi. \end{aligned} \tag{2.15}$$

Notice that if one drops all the terms containing Θ, then (2.15) reduces to the system of the commpressible Navier-Stokes equations with the external force, which has been solved in

[7] on the existece of global solutions in an L^2 Sobolev space by using the energy method. This suggests the possibility of developing the L^2-theory of the Boltzmann equation. This is true, indeed, as seen below.

3 Energy Estimates

The construction of global L^2 solutions relies on an a priori estimate for the norm

(3.1)
$$N(t)^2 = \sup_{0\le\tau\le t}\Big(\sum_{|\gamma|\le 4} ||\partial^\gamma_{t,x}(\rho-\hat{\rho},u,\theta-\overline{\theta})(\tau)||^2_{L^2_x} + \sum_{|\gamma|+|\beta|\le 4} ||\partial^\gamma_{t,x}\partial^\beta_\xi \mathbf{G}(\tau)||_{L^2_{\tilde{\mathbf{M}}_-,x,\xi}}\Big)$$
$$+ \sum_{1\le|\gamma|\le 4}\int_0^t ||\partial^\gamma_{t,x}(\rho-\hat{\rho},u,\theta-\overline{\theta})(\tau)||^2_{L^2_x} d\tau + \sum_{|\gamma|+|\beta|\le 4}\int_0^t ||\nu_{\mathbf{M}}^{1/2}\partial^\gamma_{t,x}\partial^\beta_\xi \mathbf{G}(\tau)||^2_{L^2_{\tilde{\mathbf{M}}_-,x,\xi}} d\tau.$$

The lower order norms of (ρ, u, θ) involved here are controlled by the celebrated H-Theorem

$$\int_{\mathbb{R}^3} Q(f,f)\ln f d\xi \le 0, \tag{3.2}$$

and the higher order norms by the L^2-method developed in [7], while the estimate for $\mathbf{G}$ are derived by using the microscopic H-Theorem of a new type, namely, the negative definiteness of $L_{\mathbf{M}}$ with different weight functions,

$$-\int_{\mathbf{R}^3} \frac{\mathbf{G}L_{\mathbf{M}}\mathbf{G}}{\tilde{\mathbf{M}}} d\xi \ge \overline{\sigma}\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)\mathbf{G}^2}{\tilde{\mathbf{M}}} d\xi, \tag{3.3}$$

which holds if two Maxwellians $\mathbf{M}, \tilde{\mathbf{M}}$ are rather close to each other, with a constant $\bar{\sigma} > 0$ depending only on $(u, \tilde{u}, \theta, \tilde{\theta})$, see [6] for a detail.

We also need the L^2 estimate of the nonlinear operator Q established in [2]: $\exists C > 0$ such that

$$\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)^{-1}Q(f,g)^2}{\tilde{\mathbf{M}}} d\xi \le C\left\{\int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int_{\mathbf{R}^3} \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int_{\mathbf{R}^3} \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int_{\mathbf{R}^3} \frac{\nu_{\mathbf{M}}(\xi)g^2}{\tilde{\mathbf{M}}} d\xi\right\}, \tag{3.4}$$

where $\tilde{\mathbf{M}}$ is any Maxwellian such that the above integrals are well defined.

In the above, the function $\nu_{\mathbf{M}}(\xi)$ is the collision frequency associated with the classical decomposition of the operator $L_{\mathbf{M}}$,

$$L_{\mathbf{M}} = -\nu_{\mathbf{M}}(\xi) \times + K_{\mathbf{M}}, \tag{3.5}$$

where $K_{\mathbf{M}}(\cdot) = -K_{1\mathbf{M}}(\cdot) + K_{2\mathbf{M}}(\cdot)$ is a symmetric L^2-compact integral operator. The explicit expressions of $\nu_{\mathbf{M}}$ and the integral kernels of $K_{i\mathbf{M}}(\cdot)$ are

$$\begin{cases} \nu_{\mathbf{M}}(\xi) &= \frac{2\rho}{\sqrt{2\pi R\theta}}\left\{\left(\frac{R\theta}{|\xi-u|} + |\xi-u|\right)\int_0^{|\xi-u|}\exp\left(-\frac{y^2}{2R\theta}\right)dy + R\theta\exp\left(-\frac{|\xi-u|^2}{2R\theta}\right)\right\}, \\ k_{1\mathbf{M}}(\xi,\xi_*) &= \frac{\pi\rho}{\sqrt{(2\pi R\theta)^3}}|\xi-\xi_*|\exp\left(-\frac{|\xi-u|^2}{4R\theta} - \frac{|\xi_*-u|^2}{4R\theta}\right), \\ k_{2\mathbf{M}}(\xi,\xi_*) &= \frac{2\rho}{\sqrt{2\pi R\theta}}|\xi-\xi_*|^{-1}\exp\left(-\frac{|\xi-\xi_*|^2}{8R\theta} - \frac{(|\xi|^2-|\xi_*|^2)^2}{8R\theta|\xi-\xi_*|^2}\right). \end{cases}$$

In the below, it is crucial that

$$\nu_{\mathbf{M}}(\xi) \sim (1+|\xi|) \qquad \text{as} \quad |\xi| \to \infty.$$

3.1 Lower Order Estimate

Introduce the entropy

$$(3.6) \qquad -\frac{3}{2}\rho S \equiv \int_{\mathbf{R}^3} \mathbf{M}\ln\mathbf{M}d\xi.$$

By a direct computation, we see that

$$S = -\frac{2}{3}\ln\rho + \ln(2\pi R\theta) + 1, \qquad p = R\rho\theta = k\rho^{\frac{5}{3}}e^S, \qquad k = 1/(2\pi e).$$

A convex entropy-entropy flux pair (η, q) around the sationary solution $\overline{\mathbf{M}} = \mathbf{M}_{[\hat\rho(x),0,\bar\theta]}$ of (1.2) is then given by

$$(3.7) \qquad \begin{cases} \eta = \frac{3}{2}\left\{\rho\theta - \bar\theta\rho S + \rho\left[\left(\bar S - \frac{5}{3}\right)\bar\theta + \frac{|u|^2}{2}\right] + \frac{2}{3}\hat\rho\bar\theta\right\}, \\ q_j = u_j\eta + u_j\left(\rho\theta - \hat\rho\bar\theta\right), j = 1,2,3, \end{cases}$$

which satisfies, for some constant $C > 1$ and with $\hat E = \frac{3}{2}R\bar\theta$,

$$(3.8) \qquad C^{-1}\Big(|\rho-\hat\rho|^2 + |\rho u|^2 + |\rho E - \hat\rho\hat E|^2\Big) \le \eta \le C\Big(|\rho-\hat\rho|^2 + |\rho u|^2 + |\rho E - \hat\rho\hat E|^2\Big),$$

On the other hand, multiplying (2.10) by $\log\mathbf{M}$ and integrating w.r.t. ξ by parts, we get

$$(3.9) -\frac{3}{2}(\rho S)_t - \frac{3}{2}\mathrm{div}_x(\rho u S) + \nabla_x\left(\int_{\mathbf{R}^3}(\xi\ln\mathbf{M})\mathbf{G}d\xi\right) = \int_{\mathbf{R}^3}\frac{\mathbf{G}\mathbf{P}_1\big(\xi\cdot\nabla_x\mathbf{M}\big)}{\mathbf{M}}d\xi,$$

which can be rewritten in terms of the pair (η, q) as

$$\begin{aligned}\eta_t + \mathrm{div}_x q = \; & -\nabla_x \left(\int_{\mathbf{R}^3} \left(\xi \mathbf{G} \ln \mathbf{M} + \tfrac{3}{2}\psi_4 \xi \mathbf{G} \right) d\xi \right) - \tfrac{3}{2}\rho u \cdot \nabla_x \Phi \\ & + \int_{\mathbf{R}^3} \tfrac{\mathbf{P}_1(\xi\cdot\nabla_x \mathbf{M})\mathbf{G}}{\mathbf{M}} d\xi. \end{aligned} \tag{3.10}$$

In the below, we assume that (1.5) holds for $\epsilon \in (0, \epsilon_1]$ with some fixed $\epsilon_1 > 0$ (say $\epsilon_1 = 1$) and similarly that $N(t) \leq \delta_1$ with some fixed $\delta_1 > 0$ (say $\delta_1 = 1$). Plug (2.13) into (3.10) and integrate w.r.t. t, x. Owing to the mass conservation law $(2.12)_1$, it holds that

$$\int_{\mathbf{R}^3} \rho u \cdot \nabla_x \Phi dx = -\int_{\mathbf{R}^3} \nabla_x \cdot (\rho u) \Phi dx = \frac{d}{dt} \int_{\mathbf{R}^3} (\rho - \hat{\rho}) \Phi dx,$$

and as a consequence, we get

$$\begin{aligned} & \int_{\mathbf{R}^3} \left(\eta + \tfrac{3}{2}(\rho - \hat{\rho})\,\Phi\right)(t) dx + \int_0^t \int_{\mathbf{R}^3} |\nabla_x(u,\theta)|^2 dx d\tau \leq O(1)\epsilon^2 \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{\nu_{\mathbf{M}}(\xi)}{\mathbf{M}} \Big(|\mathbf{G}_t|^2 + |\nabla_x \mathbf{G}|^2 + \epsilon |\nabla_\xi \mathbf{G}|^2 + N(t) |\mathbf{G}|^2 \Big)(t, x, \xi) d\xi dx d\tau. \end{aligned} \tag{3.11}$$

Here, $N(t)$ comes by estimating $Q(\mathbf{G}, \mathbf{G})$ by (3.4). Notice also that

$$\int_{\mathbf{R}^3} |\rho - \hat{\rho}| \Phi dx \leq \epsilon \| \rho - \hat{\rho} \|,$$

so that if ϵ is small, this is absorbed into the entropy term $\int \eta dx$.

The microscopic component $\mathbf{G}$ can be estimated by taking the inner product of (2.13) and $\mathbf{G}$ and using (3.3), to result in

$$\begin{aligned} & \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{|\mathbf{G}|^2}{\mathbf{M}} d\xi dx + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}} d\xi dx d\tau \\ & \leq O(1)\epsilon^2 + O(1)(\epsilon + N(t)) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \left(|\nabla_x(u,\theta)|^2 + \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\nabla_x \mathbf{G}|^2}{\mathbf{M}} \right) dx d\tau, \end{aligned} \tag{3.12}$$

where the uncommon weight $\mathbf{M}_-$ is needed to contorl the terms like $\partial^\alpha_{t,x,\xi}(\log \mathbf{M})$, $|\alpha| = 1$. A similar calculation with the weight $\mathbf{M}$ replaced by $\mathbf{M}_-$ gives,

$$\begin{aligned} & \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{|\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx + \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ & \leq O(1)\epsilon^2 + (\epsilon + N(t)) \int_0^t \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\mathbf{G}|^2}{\mathbf{M}_-} d\xi dx d\tau \\ & + O(1) \int_0^t \int_{\mathbf{R}^3} \left(|\nabla_x(\rho - \hat{\rho}, u, \theta)|^2 + \int_{\mathbf{R}^3} \tfrac{\nu(\xi)|\nabla_x \mathbf{G}|^2}{\mathbf{M}_-} \right) dx d\tau. \end{aligned} \tag{3.13}$$

Note that an extra term for $\partial\rho$ appears here. (3.11)-(3.13) give the complete lower order energy estimates.

3.2 Higer Order Estimates

The estimate for $\partial_{t,x}^{\gamma}(\rho-\hat{\rho},u,\theta-\bar{\theta})$ can be deduced from (2.15) by proceeding just in the same way as in [7], while the estimates for $\partial_{t,x}^{\gamma}\partial_{\xi}^{\beta}\mathbf{G}$ can be derived from (2.13) applied by $\partial_{t,x}^{\gamma}\partial_{\xi}^{\beta}$ in a similar way as in (3.12) and (3.13), by knowing that the commutator

$$[\partial_{\xi}^{\beta},L_{\mathbf{M}}]=\partial_{\xi}^{\beta}L_{\mathbf{M}}-L_{\mathbf{M}}\partial_{\xi}^{\beta}$$

is bounded in $L^2_{\mathbf{M},\xi}$ and that the quantity $\partial_{\xi}^{\beta}Q(f,g)$ enjoyes a similar estimate as (3.4).

The computation is very much delicate and lengthy, however, and so, only the final result is presented here. See [8] for the detail.

$$\begin{aligned}
&\sum_{|\gamma|\le 4}\int_{\mathbf{R}^3}\left|\partial^{\gamma}(\rho-\overline{\rho},u,\theta-\overline{\theta})\right|^2dx+\sum_{|\gamma|+|\beta|\le 4}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{|\partial^{\gamma}\partial^{\beta}\mathbf{G}|^2}{\mathbf{M}}d\xi dx\\
&+\int_0^t\int_{\mathbf{R}^3}\left(\sum_{1\le|\gamma|\le 4}|\partial^{\gamma}(\rho-\overline{\rho},u,\theta)|^2+\sum_{|\gamma|+|\beta|\le 4}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma}\partial^{\beta}\mathbf{G}|^2}{\mathbf{M}}d\xi\right)dxd\tau\\
\le\;& O(1)\epsilon^2+O(1)(\epsilon+N(t))\sum_{|\gamma|+|\beta|\le 4}\int_0^t\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\nu(\xi)|\partial^{\gamma}\partial^{\beta}\mathbf{G}|^2}{\mathbf{M}_-}d\xi dxd\tau.
\end{aligned}\tag{3.14}$$

As in (3.13), the weight $\mathbf{M}$ can be replaced by $\mathbf{M}_-$, but again, at the cost of the extra term of the form

$$\sum_{1\le|\gamma|\le 4}\int_0^t\int_{\mathbf{R}^3}|\partial^{\gamma}(\rho-\hat{\rho},u,\theta)|^2\,dxd\tau.$$

on the right hand side.

3.3 A Priori Estimate

An appropriate linear combination of all the estimates obtained so far yields an inequality of the form

$$N(t)^2\le C_1\epsilon^2+C_2(\epsilon+N(t))N(t)^2,$$

for some constants $C_1,C_2>0$ independent of ϵ and $N(t)$. It is easy to see that there exists a positive constant ϵ_2 such that for each $\epsilon\in(0,\epsilon_2)$, the cubic equation

$$C_2x^3-(1-C_2\,\epsilon)x^2+C_1\,\epsilon^2=0$$

has positive solutions. Denote the smallest one by δ_2, and set $\epsilon_0=\min(\epsilon_1,\epsilon_2)$ and $\delta_0=\min(\delta_1.\delta_2)$, to conclude that $N(t)\le\delta_0$ holds for each $\epsilon\in(0,\epsilon_0)$.

4 Local Solution

Put $f = \overline{\mathbf{M}} + g$ and consider the Cauchy problem

$$
(4.1) \qquad \begin{cases} g_t + \xi \cdot \nabla_x g - \nabla_x \Phi \cdot \nabla_\xi g = L_{\overline{\mathbf{M}}} g + Q(g,g), \\ g|_{t=0} = g_0(x,\xi). \end{cases}
$$

In order to construct local solutions, we first analyse, for each given point $(t_0, x_0, \xi_0) \in \mathbf{R}^+ \times \mathbf{R}^3 \times \mathbf{R}^3$, the backward bi-characteristic curve $(X(t), \Xi(t)) \equiv (X, \Xi)(t; t_0, x_0, \xi_0)$ of (4.1) passing through the point (t_0, x_0, ξ_0) which is given by

$$
(4.2) \qquad \begin{cases} \frac{dX(t)}{dt} = & \Xi(t), \\ \frac{d\Xi(t)}{dt} = & -\nabla_x \Phi(X(t)), \\ (X(t), \Xi(t))|_{t=t_0} = & (x_0, \xi_0). \end{cases}
$$

Since Φ is assumed sufficiently smooth with bounded derivatives, a smooth bi-characterisitic curve exists uniquely for all time t.

Recall the decomposition

$$
\begin{aligned}
&L_{\overline{\mathbf{M}}} = -\nu_{\overline{\mathbf{M}}}(x,\xi) + K_{\overline{\mathbf{M}}}, \\
&\nu_0(1+|\xi|) \le \nu_{\overline{\mathbf{M}}}(x,\xi) \le \nu_1(1+|\xi|), \qquad K_{\overline{\mathbf{M}}} : \text{bounded on } L^2_{\overline{\mathbf{M}},\xi},
\end{aligned}
$$

and define the operator $U(t)$ by

$$
U(t)g_0 = \exp\Big\{-\int_0^t \nu_{\overline{M}}(X(s,t,x,\xi), \Xi(s,t,x,\xi))ds\Big\} g_0(X(0,t,x,\xi), \Xi(0.t,x,\xi)).
$$

Then, (4.1) can be rewritten formally in the form of the integral equation

$$
(4.3) \qquad \begin{aligned} g(t) &= U(t)g_0 + \int_0^t U(t-s)\Big\{K_{\overline{\mathbf{M}}} g(s) + Q(g(s), g(s)\Big\}ds. \\ &\equiv V(g)(t). \end{aligned}
$$

We shall show that the nonlinear map V is a contraction map in the energy space

$$
(4.4) \qquad X_{T,a} = \left\{ g(t,x,\xi) \;\middle|\; \begin{array}{c} \frac{\partial^\alpha \partial^\beta g(t,x,\xi)}{\sqrt{\mathbf{M}_-(\xi)}} \in \mathbf{BC}_t\left([0,T), L^2_{x,\xi}\left(\mathbf{R}^3 \times \mathbf{R}^3\right)\right) \\ |||g||| \le a, \quad |\alpha| + |\beta| \le 4 \end{array} \right\},
$$

$$
(4.5) \qquad \begin{aligned} |||g||| = &\sup_{0 \le t \le T} \left\{ \sum_{|\alpha|+|\beta| \le 4} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|\partial^\alpha \partial^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx \right\} \\ &+ \sum_{|\alpha|+|\beta| \le 4} \int_0^T \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{(1+|\xi|)|\partial^\alpha \partial^\beta g(t,x,\xi)|^2}{\mathbf{M}_-(\xi)} d\xi dx d\tau, \end{aligned}
$$

provided that $T, a > 0$ are chosen sufficiently small.

To this end, put $h = V(g)$. First, assume that $g \in X_{T,a}$. Then, h is well defined (the integral on the right hand side of (4.3) converges). If, in addition, g and g_0 are assumed sufficently smooth, then evidently, h is also sufficiently smooth and solves the partial differential equation

$$\left\{ \begin{aligned} & h_t + \xi \cdot \nabla_x h - \nabla_x \Phi \cdot \nabla_\xi h + \nu_{\overline{\mathbf{M}}}(\xi) h = K_{\overline{\mathbf{M}}} g + Q(g,g), \\ & h|_{t=0} = g_0(x,\xi). \end{aligned} \right. \tag{4.6}$$

The extra smoothness assumptions on g, g_0 will be removed later, by appealing to the limiting argument. See below the estimate (4.12).

Now, multiply (4.6) by $h/\mathbf{M}_-$ and integrate w.r.t. x, ξ to deduce

$$\frac{d}{dt}||h(t)||^2 + 2(1 - C_1\epsilon)||\nu_{\overline{\mathbf{M}}}^{1/2} h(t)||^2 \le C_2||g||||h|| + ||\nu_{\overline{\mathbf{M}}}^{-1/2} Q(g(t), g(t))||||\nu_{\overline{\mathbf{M}}}^{1/2} h(t)||,$$

where ϵ is like in (1.5) for Φ, or more precisely, it is assumed that

$$\sum_{1\le|\alpha|\le 5} ||\partial_x^\alpha \Phi||_{L_x^3} \le \epsilon, \tag{4.7}$$

while $C_1, C_2 > 0$ are constants independent of g, h, and the norm is

$$||h||^2 = \int_{\mathbf{R}}^3 \int_{\mathbf{R}}^3 \frac{h^2}{\mathbf{M}_-} dx d\xi.$$

Finally, we apply $\partial_{t,x}^\gamma \partial_\xi^\beta$ to (4.6) and proceed as above. Fix $\epsilon > 0$ so that $1 - C_1\epsilon > 0$, or,

$$\epsilon < 1/(2C_1). \tag{4.8}$$

Then, by virtue of (3.4), we get for $|\gamma| + |\beta| \le 4$,

$$\begin{aligned} & \frac{d}{dt}||\partial_{t,x}^\gamma \partial_\xi^\beta h(t)||^2 + ||\nu_{\overline{\mathbf{M}}}^{1/2} \partial_{t,x}^\gamma \partial_\xi^\beta h(t)||^2 \\ & \qquad \le C_3 \sum_{|\gamma|+|\beta|\le 4} ||\partial_{t,x}^\gamma \partial_\xi^\beta g||^2 \Big(1 + ||\nu_{\overline{\mathbf{M}}}^{1/2} \partial_{t,x}^\gamma \partial_\xi^\beta g(t)||^2\Big). \end{aligned}$$

Integrating this w.r.t. t yields

$$|||h||| = |||V(g)||| \le |||g_0|||_0 + C_3|||g|||(T + |||g|||), \tag{4.9}$$

where the norm $|||g|||$ is as in (4.5) and $|||g_0|||_0$ is defined also by (4.5) but with $T = 0$. Consider the quadratic equation $C_3 a^2 - (1 - C_3 T)a + |||g_0|||_0 = 0$. This has two positive solutions if $1 - C_3 T > 0$ and $D = (1 - C_3T)^2 - 4C_3|||g_0|||_0 > 0$, that is, if

$$T < \frac{1}{C_3}, \qquad |||g_0|||_0 \le \frac{(1 - C_3T)^2}{4C_3}, \tag{4.10}$$

and the smaller one is given by

$$a = \frac{1}{2C_3}(1 - C_3 T - \sqrt{D}).$$

Under this choice of T, g_0, a, (4.9) implis that

$$|||g||| \leq a \Longrightarrow |||V(g)||| \leq a.$$

Further, let $g' \in X_{T,a}$ and put $h = V(g) - V(g')$. It is easy to see that if g' is also assumed sufficiently smooth, h solves

$$\begin{cases} h_t + \xi \cdot \nabla_x h - \nabla_x \Phi \cdot \nabla_\xi h + \nu_{\overline{\mathbf{M}}}(\xi) h = K_{\overline{\mathbf{M}}}(g - g') + Q(g + g', g - g'), \\ h|_{t=0} = 0. \end{cases} \tag{4.11}$$

Quite similarly as in (4.9), we can deduce

$$|||h||| = |||V(g) - V(g')||| \leq C_3 |||g + g'||| |||g - g'|||, \tag{4.12}$$

with the same constant C_3 as in (4.9). Since we are assuming $g, g' \in X_{T,a}$, we get

$$|||V(g) - V(g')||| \leq \mu |||g - g'|||, \qquad \mu = 2C_3 a = 1 - \sqrt{D} < 1. \tag{4.13}$$

We can now remove the extra smoothness assumption on g, g', g_0 required in the arguments given so far. Indeed, any $g \in X_{T,a}$ can be approximated by a sequence $\{g_n\} \subset C_0^\infty$ in the norm of $X_{T,a}$ and (4.13) gives

$$|||V(g_n) - V(g_m)||| \leq \mu |||g_n - g_m|||,$$

which shows that $V(g_n)$ converges. The limit is of course $V(g)$, and by the limiting argument, we see that the estimates (4.9), (4.12), and (4.13) hold without extra smoothness assumption.

Since $X_{T,a}$ is a complete metric space with the metric induced by the norm $||| \cdot |||$ and since $\mu \in (0, 1)$, the above argument implies that the contraction mapping theorem applies and V has a unique fixed point in $X_{T,a}$. This fixed point is the desired local solution.

Note that this local existence result requires the smallness conditions both on the potential Φ and initial data g_0, as seen by (4.8) and (4.10).

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