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# The equivalence among three Kantorovich type inequalities

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**Abstract**

First we discuss Hölder-McCarthy type inequality and Kantorovich type inequality including the case that the exponents of $(T^{p}x,x)$ and $(Tx,x)^{q}$ are different. Among three regions of $p$-$q$ plane, we derive these inequalities in a region from those in the other regions.

## 1 Introduction

An operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ and also $T$ is said to be strictly positive (denoted by $T>0$) if $T$ is positive and invertible.

**Theorem A.** (Hölder-McCarthy inequality)
*Let $A$ be a strictly positive operator on a Hilbert space $H$. For any unit vector $x$,*

$$\begin{array}{lrcl} \text{(a)} & (A^{\lambda}x,x) & \geq & (Ax,x)^{\lambda} \text{ for any } \lambda>1, \\ \text{(b)} & (A^{\lambda}x,x) & \geq & (Ax,x)^{\lambda} \text{ for any } \lambda<0, \\ \text{(c)} & (A^{\lambda}x,x) & \leq & (Ax,x)^{\lambda} \text{ for any } \lambda\in[0,1]. \end{array}$$

**Theorem B.** (original Kantorovich inequality)
*Let $A$ be a positive operator on a Hilbert space $H$ such that $MI\geq A\geq mI>0$. For any unit vector $x$ ,*

$$\begin{aligned} (A^{-1}x,x) &\leq \frac{(m+M)^{2}}{4mM}(Ax,x)^{-1}, \\ (A^{2}x,x) &\leq \frac{(m+M)^{2}}{4mM}(Ax,x)^{2}. \end{aligned}$$

The original Kantorovich inequality is extended as follows [3][6]:

**Theorem C.** *Let $T$ be a strictly positive operator on a Hilbert space $H$ such that $MI\geq T\geq mI>0$. Then for any unit vector $x$,*

(a) *If* $p > 1$, $q > 1$,

$$K(m, M, p, q)(Tx, x)^q \geq (T^p x, x),$$

(b) *If* $p < 0$, $q < 0$,

$$K(m, M, p, q)(Tx, x)^q \geq (T^p x, x),$$

(c) *If* $0 < p < 1$, $0 < q < 1$,

$$K(m, M, p, q)(Tx, x)^q \leq (T^p x, x),$$

*where Kantorovich constant*

$$K(m, M, p, q) = \begin{cases} K^{(1)}(m, M, p, q) & \text{if case 1 holds} \\ m^{p-q} & \text{if case 2 holds} \\ M^{p-q} & \text{if case 3 holds,} \end{cases}$$

$$K^{(1)}(m, M, p, q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left\{ \frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right\}^q.$$

$$\begin{cases} \text{case 1.} & \begin{cases} m^{p-1}q \leq \dfrac{M^p - m^p}{M - m} \leq M^{p-1}q & \text{(a),(b)} \\ m^{p-1}q \geq \dfrac{M^p - m^p}{M - m} \geq M^{p-1}q & \text{(c)} \end{cases} \\ \text{case 2.} & \begin{cases} m^{p-1}q > \dfrac{M^p - m^p}{M - m} & \text{(a), (b)} \\ m^{p-1}q < \dfrac{M^p - m^p}{M - m} & \text{(c)} \end{cases} \\ \text{case 3.} & \begin{cases} M^{p-1}q < \dfrac{M^p - m^p}{M - m} & \text{(a), (b)} \\ M^{p-1}q > \dfrac{M^p - m^p}{M - m} & \text{(c).} \end{cases} \end{cases}$$

As an application of Theorem C, the following chaotic order version is obtained.

**Theorem D.** *Let $T$ be strictly positive operator on a Hilbert space $H$ such that $MI \geq T \geq mI > 0$ and let $h = \dfrac{M}{m} > 1$. Then for any unit vector $x$,*

$$SR_h(p, q, m, M)\Delta_x(T^q) \geq (T^p x, x) \qquad \text{for } p > 0 \text{ and } q > 0,$$

*where Specht ratio*

$$SR_h(p, q, m, M) = \begin{cases} m^{p-q} \dfrac{h^{\frac{q}{h^p - 1}}}{e \log h^{\frac{q}{h^p - 1}}} & \text{if } q \leq \dfrac{h^p - 1}{\log h} \leq qh^p \\ m^{p-q} & \text{if } \dfrac{h^p - 1}{\log h} \leq q \\ M^{p-q} & \text{if } qh^p \leq \dfrac{h^p - 1}{\log h}, \end{cases}$$

*determinant* $\Delta_x(T) = \exp(((\log T)x, x))$.

## 2 Results

**Theorem 2.1** *Let $T$ be a strictly positive operator on a Hilbert space $H$ such that $MI \geq T \geq mI > 0$. Then for any unit vector $x$,*
(a) *If $p > 1$, $q > 1$,*

$$K(m,M,p,q)(Tx,x)^q \geq (T^p x,x) \geq C(m,M,p,q)(Tx,x)^q,$$

(b) *If $p < 0$, $q < 0$,*

$$K(m,M,p,q)(Tx,x)^q \geq (T^p x,x) \geq C(m,M,p,q)(Tx,x)^q,$$

(c) *If $0 < p < 1$, $0 < q < 1$,*

$$K(m,M,p,q)(Tx,x)^q \leq (T^p x,x) \leq C(m,M,p,q)(Tx,x)^q,$$

*where*

$$C = C(m,M,p,q) = \begin{cases} \min\{m^{p-q}, M^{p-q}\} & (a),(b) \\ \max\{m^{p-q}, M^{p-q}\} & (c), \end{cases}$$

*Kantorovich constant*

$$K = K(m,M,p,q) = \begin{cases} K^{(1)}(m,M,p,q) & \text{if case 1 holds} \\ m^{p-q} & \text{if case 2 holds} \\ M^{p-q} & \text{if case 3 holds,} \end{cases}$$

$$K^{(1)}(m,M,p,q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left\{ \frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right\}^q .$$

$$\begin{cases} \text{case 1.} & \begin{cases} m^{p-1}q \leq \dfrac{M^p - m^p}{M-m} \leq M^{p-1}q & \text{(a),(b)} \\ m^{p-1}q \geq \dfrac{M^p - m^p}{M-m} \geq M^{p-1}q & \text{(c)} \end{cases} \\ \text{case 2.} & \begin{cases} m^{p-1}q > \dfrac{M^p - m^p}{M-m} & (a),(b) \\ m^{p-1}q < \dfrac{M^p - m^p}{M-m} & \text{(c)} \end{cases} \\ \text{case 3.} & \begin{cases} M^{p-1}q < \dfrac{M^p - m^p}{M-m} & (a),(b) \\ M^{p-1}q > \dfrac{M^p - m^p}{M-m} & \text{(c).} \end{cases} \end{cases}$$

**Theorem 2.2** *In Theorem 2.1,(a),(b) and (c) are derived from each other.*

**Proposition 2.3** *For any $p, q$,*

$$(1)\quad K(m, M, p+1, q+1) = K\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right),$$

$$(2)\quad K(m, M, p, q)^{-\frac{1}{q}} = K\left(m^p, M^p, \frac{1}{p}, \frac{1}{q}\right).$$

*$K$ can be replaced by $C$.*

**Theorem 2.4** *Under the assumption of Theorem 2.1, the following inequalities are derived from each other.*

• *If $p > \frac{1}{2}$ and $q > \frac{1}{2}$,*
$(d) K(m, M, \frac{1}{2}+p, \frac{1}{2}+q)\,(Tx, x)^{\frac{1}{2}+q} \geq (T^{\frac{1}{2}+p}x, x), \geq C(m, M, \frac{1}{2}+p, \frac{1}{2}+q)\,(Tx, x)^{\frac{1}{2}+q}$
$\Longleftrightarrow$
$(e) K(m, M, \frac{1}{2}-p, \frac{1}{2}-q)\,(Tx, x)^{\frac{1}{2}-q} \geq (T^{\frac{1}{2}-p}x, x) \geq C(m, M, \frac{1}{2}-p, \frac{1}{2}-q)\,(Tx, x)^{\frac{1}{2}-q}$.

•*If $0 < |p| < \frac{1}{2}$ and $0 < |q| < \frac{1}{2}$,*
$(d') K(m, M, \frac{1}{2}+p, \frac{1}{2}+q)\,(Tx, x)^{\frac{1}{2}+q} \leq (T^{\frac{1}{2}+p}x, x), \leq C(m, M, \frac{1}{2}+p, \frac{1}{2}+q)\,(Tx, x)^{\frac{1}{2}+q}$
$\Longleftrightarrow$
$(e') K(m, M, \frac{1}{2}-p, \frac{1}{2}-q)\,(Tx, x)^{\frac{1}{2}-q} \leq (T^{\frac{1}{2}-p}x, x) \leq C(m, M, \frac{1}{2}-p, \frac{1}{2}-q)\,(Tx, x)^{\frac{1}{2}-q}$.

The relation (a)$\Leftrightarrow$(b) in Theorem 2.1 is included in Theorem 2.4.

**Proposition 2.5** *For every $p, q$,*

$$K\left(m, M, \frac{1}{2}+p, \frac{1}{2}+q\right) = K\left(\frac{1}{M}, \frac{1}{m}, \frac{1}{2}-p, \frac{1}{2}-q\right).$$

*$K$ can be replaced by $C$.*

As an application of Theorem 2.1, the following chaotic order version is obtained.

**Theorem 2.6** *Let $T$ be a strictly positive operator on a Hilbert space $H$ such that $MI \geq T \geq mI > 0$ and $h = \dfrac{M}{m} > 1$. Then for any unit vector $x$,*
*(a) If $p > 0$, $q > 0$,*

$$SR_h(m, M, p, q)\Delta_x(T^q) \geq (T^p x, x) \geq C(m, M, p, q)\Delta_x(T^q),$$

*(b) If $p < 0$, $q < 0$ ,*

$$SR_h(m, M, p, q)\Delta_x(T^q) \geq (T^p x, x) \geq C(m, M, p, q)\Delta_x(T^q),$$

*where*

$$\textit{determinant} \quad \Delta_x(T) = \exp(((\log T)x, x)),$$
$$C(m, M, p, q) = \min\{m^{p-q}, M^{p-q}\},$$

*Specht ratio type constant*

$$SR_h(m, M, p, q) = \begin{cases} m^{p-q} \dfrac{h^{\frac{q}{h^p-1}}}{e \log h^{\frac{q}{h^p-1}}} & \textit{if } q \le \dfrac{h^p-1}{\log h} \le qh^p \\ m^{p-q} & \textit{if } \dfrac{h^p-1}{\log h} \le q \\ M^{p-q} & \textit{if } qh^p \le \dfrac{h^p-1}{\log h}, \end{cases}$$

*these are case* 1, *case* 2 *and case* 3.

By Theorem 2.1 $(a)$, we get

$$K(m_1, M_1, np, nq)\left(1 + \frac{\log T}{n}x, x\right)^{nq} \ge \left(\left(1 + \frac{\log T}{n}\right)^{np} x, x\right)$$
$$\ge C(m_1, M_1, np, nq)\left(1 + \frac{\log T}{n}x, x\right)^{nq}, \qquad (2.1)$$

$$\text{where } m_1 = 1 + \frac{\log m}{n}, \ M_1 = 1 + \frac{\log M}{n}$$

$n$ tending to $\infty$, we obtain

$$SR_h(m, M, p, q)\Delta_x(T^q) \ge (T^p x, x) \ge C(m, M, p, q)\Delta_x(T^q)$$

On (2.1) operators appear in the form $\log T$. So if we discuss an inequality concerning two operators, it is natural that the assumption about order between these operators are given by chaotic order. Therefore we call Theorem 2.6 chaotic order version. The inequality (2.1) is a very good bridge between usual order and chaotic order. (cf. Proposition 2.9)

**Theorem 2.7** *In Theorem* 2.6, (a),(b) *and* (c) *are derived from each other.*

**Proposition 2.8** *For any* $p, q$,

$$SR_h(m, M, p, q) = SR_h\left(\frac{1}{M}, \frac{1}{m}, -p, -q\right).$$

$SR_h$ *can be replaced by* $C$.

We can see that Kantorovich constant is point symmetric with respect to $(\frac{1}{2}, \frac{1}{2})$, and Specht ratio is point symmetric with respect to $(0, 0)$ on $p$-$q$ plain. These fact are reflected in the proofs that an inequality in a region is derived from one in the other regions of $p$-$q$ plane.

**Proposition 2.9** *For any $p, q$,*

$$(1)\ \lim_{n\to\infty} K\left(1+\frac{\log m}{n}, 1+\frac{\log M}{n}, np, nq\right) = SR_h(m, M, p, q),$$
$$(2)\ \lim_{r\downarrow 0} K\left(m^r, M^r, 1+\frac{p}{r}, 1+\frac{q}{r}\right) = SR_h(m, M, p, q),$$
$$(2')\ \lim_{r\uparrow 0} K\left(M^r, m^r, \frac{p}{r}, \frac{q}{r}\right) = SR_h(m, M, p, q).$$

*$K$ and $SR_h$ can be replaced by $C$.*

# 3 Proof of results

**Proof of Theorem 2.1.** The left hand side inequality (Kantorovich type inequality ($p$-$q$ version)) is proved by Furuta and Giga [3][4]. We prove the right hand side inequality (Hölder-McCarthy type inequality ($p$-$q$ version)).

It is clear that $C = C(m, M, p, q)$ satisfies this inequality by using original Hölder-McCarthy inequality. The best possibility of $C = C(m, M, p, q)$ is shown as follows.

In the proof of Theorem 2.1[3], for a convex function $f$,

$$(f(T)x, x) \le f(m) + \frac{f(M) - f(m)}{M - m}((Tx, x) - m).$$

Then

$$(f(T)x, x) \le \left\{(Tx, x)^{-q}\left(f(m) + \frac{f(M) - f(m)}{M - m}((Tx, x) - m)\right)\right\}(Tx, x)^q. \quad (3.1)$$

for $q > 1$. Put

$$h(t) = t^{-q}\left(f(m) + \frac{f(M) - f(m)}{M - m}(t - m)\right).$$

The function $h(t)$ is a concave function. We consider $f(T) = T^p$ $(p > 1)$. The statement until here is in the proof of Theorem 2.1 in [3].

By (3.1), $C(m, M, p, q)$ must be $\le h((Tx, x))$. And $\min_{m \le (Tx,x) \le M} h((Tx, x)) = \min(m^p, M^p)$ because $f(t)$ is concave. Meanwhile $\min(m^p, M^p)$ satisfy the Hölder-McCarthy type inequality ($p$-$q$ version). Then $C(m, M, p, q) = \min(m^p, M^p)$ is the best possible constant in (a). (b) and (c) are proved similarly. ///

**Proof of Theorem 2.2.** The relation of Kantorovich type inequality among three regions of $p$-$q$ plane is proved by Furuta[4]. As for the Hölder-McCarthy type inequality ($p$-$q$ version), we will state later.

Proposition 2.3 is proved by calculation.

For the proof of Theorem 2.4, we prepare the following two lemmas.

**Lemma 3.1** *If* $p > \frac{1}{2}$, $q > \frac{1}{2}$,

$$(d)\begin{cases} m^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) \le \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m} \le M^{(\frac{1}{2}+p)-1}(\dfrac{1}{2}+q) \\ m^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) > \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m} \\ M^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) < \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m}, \end{cases}$$

$$\iff$$

$$(e)\begin{cases} \left(\dfrac{1}{M}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) \le \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}} \le \left(\dfrac{1}{m}\right)^{(\frac{1}{2}-p)-1}(\dfrac{1}{2}-q) \\ \left(\dfrac{1}{M}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) > \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}} \\ \left(\dfrac{1}{m}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) < \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}}, \end{cases}$$

*these are case* 1, *case* 2 *and case* 3 *in Theorem* 2.1.

**Lemma 3.2** *If* $0 < |p| < \frac{1}{2}$, $0 < |q| < \frac{1}{2}$,

$$(d')\begin{cases} m^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) \ge \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m} \ge M^{(\frac{1}{2}+p)-1}(\dfrac{1}{2}+q) \\ m^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) < \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m} \\ M^{(\frac{1}{2}+p)-1}(\frac{1}{2}+q) > \dfrac{M^{\frac{1}{2}+p}-m^{\frac{1}{2}+p}}{M-m}, \end{cases}$$

$$\iff$$

$$(e')\begin{cases} \left(\dfrac{1}{M}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) \ge \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}} \ge \left(\dfrac{1}{m}\right)^{(\frac{1}{2}-p)-1}(\dfrac{1}{2}-q) \\ \left(\dfrac{1}{M}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) < \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}} \\ \left(\dfrac{1}{m}\right)^{(\frac{1}{2}-p)-1}(\frac{1}{2}-q) > \dfrac{\left(\frac{1}{m}\right)^{\frac{1}{2}-p}-\left(\frac{1}{M}\right)^{\frac{1}{2}-p}}{\frac{1}{m}-\frac{1}{M}}, \end{cases}$$

*these are case* 1, *case* 2 *and case* 3 *in Theorem* 2.1.

**Proof of Theorem 2.4.**

**(d)⇒(e)**

Suppose $p > \frac{1}{2}, q > \frac{1}{2}$, put $y = \dfrac{T^{-\frac{1}{2}}x}{\|T^{-\frac{1}{2}}x\|}$.

Let $k = K\left(m, M, \frac{1}{2}+p, \frac{1}{2}+q\right)$.

$$k\left(T\frac{T^{-\frac{1}{2}}x}{\|T^{-\frac{1}{2}}x\|}, \frac{T^{-\frac{1}{2}}x}{\|T^{-\frac{1}{2}}x\|}\right)^{\frac{1}{2}+q} \geq \left(T^{\frac{1}{2}+p}\frac{T^{-\frac{1}{2}}x}{\|T^{-\frac{1}{2}}x\|}, \frac{T^{-\frac{1}{2}}x}{\|T^{-\frac{1}{2}}x\|}\right)$$

$$k\left(\frac{x}{\|T^{-\frac{1}{2}}x\|}, \frac{x}{\|T^{-\frac{1}{2}}x\|}\right)^{\frac{1}{2}+q} \geq \left(T^{-\frac{1}{2}+p}\frac{x}{\|T^{-\frac{1}{2}}x\|}, \frac{x}{\|T^{-\frac{1}{2}}x\|}\right)$$

$$k\frac{\|T^{-\frac{1}{2}}x\|^2}{\|T^{-\frac{1}{2}}x\|^{2(\frac{1}{2}+q)}} \geq (T^{-\frac{1}{2}+p}x, x)$$

$$k\|T^{-\frac{1}{2}}x\|^{2(1-\frac{1}{2}-q)} \geq (T^{-\frac{1}{2}+p}x, x)$$

$$k(T^{-1}x, x)^{\frac{1}{2}-q} \geq (T^{-\frac{1}{2}+p}x, x).$$

By putting $S = T^{-1}$, we obtain

$$k(Sx, x)^{\frac{1}{2}-q} \geq (S^{\frac{1}{2}-p}x, x).$$

By using Proposition 2.5,

$$K\left(\frac{1}{M}, \frac{1}{m}, \frac{1}{2} - p, \frac{1}{2} - q\right)(Sx, x)^{\frac{1}{2}-q} \geq (S^{\frac{1}{2}-p}x, x).$$

Since $\frac{1}{m}I \geq S \geq \frac{1}{M}I > 0$, this means the inequarity in Theorem 2.4 (e). The classification case 1,2,3 in (e) is guaranteed by Lemma 3.1.

**(e)⇒(d)** is obtained similarly.

As for the right hand side, replace $K$ by $C$ and reverse the sense of inequarity.

**(d′)⇔(e′)** is obtained similarly. ///

The proof of the part of Hölder-McCarthy type inequality in Theorem 2.2 is proved similarly to Theorem 2.4.

Proposition 2.5 is proved by calculation.

**Proof of Theorem 2.6.** This theorem is proved by Theorem 2.1 and Proposition 2.9. The left hand side inequality (Specht ratio type inequality ($p$-$q$ version)) is proved by Furuta and Giga [3][4].

For the proof of Theorem 2.7, we prepare the following lemma.

**Lemma 3.3** *Put $h = \frac{M}{m}$. For any $p, q$,*

$$(a)\begin{cases} q \leq \dfrac{h^p - 1}{\log h} \leq qh^p \\ q > \dfrac{h^p - 1}{\log h} \\ qh^p < \dfrac{h^p - 1}{\log h} \end{cases} \iff (b)\begin{cases} -q \leq \dfrac{h^{-p} - 1}{\log h} \leq -qh^{-p} \\ -q > \dfrac{h^{-p} - 1}{\log h} \\ -qh^{-p} < \dfrac{h^{-p} - 1}{\log h}, \end{cases}$$

*these are case 1, case 2 and case 3 in Theorem 2.6.*

**Proof of Theorem 2.7.**

**(a)⇒(b) in Theorem 2.6**

Suppose $p > 0,\ q > 0$,

$$SR_h(m, M, p, q)\Delta_x(T^q) \geq (T^p x, x).$$

Put $S = T^{-1}$.

$$SR_h(m, M, p, q)\Delta_x(S^{-q}) \geq (S^{-p}x, x).$$

By using Proposition 2.8 ,

$$SR_h(\frac{1}{M}, \frac{1}{m}, -p, -q)\Delta_x(S^{-q}) \geq (S^{-p}x, x).$$

Since $\frac{1}{m}I \geq S \geq \frac{1}{M}I > 0$, this means the inequality in Theorem 2.6 (b). The classification case 1,2,3 in (b) is guaranteed by Lemma 3.3. /

**(b)⇒(a)** is obtained similarly.

As for the right hand side, replace $K$ by $C$ and reverse the sense of inequarity. ///

Proposition 2.8 is proved by calculation.

**Proof of Proposition 2.9.**

(1) is proved by Furuta-Giga[3].

(2) $p$-$p$ version is proved by Yamazaki-Yanagida [7].

We state the proof of (2) $p$–$q$ version.

$$\begin{aligned}
&K\left(m^r, M^r, 1+\frac{p}{r}, 1+\frac{q}{r}\right)\\
&= \frac{m^r M^{r(1+\frac{p}{r})} - M^r m^{r(1+\frac{p}{r})}}{\frac{q}{r}(M^r - m^r)} \left\{ \frac{\frac{q}{r}(M^{r(1+\frac{p}{r})} - m^{r(1+\frac{p}{r})})}{(1+\frac{p}{r})m^r M^{r(1+\frac{p}{r})} - M^r m^{r(1+\frac{p}{r})}} \right\}^{1+\frac{p}{r}}\\
&= m^{r+p} \frac{\left(\frac{M}{m}\right)^{r+p} - \left(\frac{M}{m}\right)^r}{\left\{\left(\frac{M}{m}\right)^r - 1\right\}} \left\{ \frac{q\left\{\left(\frac{M}{m}\right)^{r+p} - 1\right\}}{(r+p)\left\{\left(\frac{M}{m}\right)^{r+p} - \left(\frac{M}{m}\right)^r\right\} m^r} \right\}^{1+\frac{q}{r}}\\
&= m^{p-q} \frac{h^{r+p}-1}{r+p} \frac{r}{h^r - 1} \left(\frac{h^{r+p}-1}{h^p - 1}\right)^{\frac{q}{r}} \left(1 + \frac{r}{q}\right)^{-\frac{r}{q}} h^{-q}. \qquad (3.2)
\end{aligned}$$

$$\frac{h^{r+p}-1}{r+p} \to \frac{h^p - 1}{q} \text{ as } r \to 0, \qquad \frac{r}{h^r - 1} \to \frac{1}{\log h} \text{ as } r \to 0,$$

$$\text{and } \left(1 + \frac{r}{q}\right)^{-\frac{r}{q}} \to \frac{1}{e} \text{ as } r \to 0.$$

The following calculation is in the proof of Lemma 11 in [7].

$$\left(\frac{h^{r+p}-1}{h^p - 1}\right)^{\frac{q}{r}} \to h^{\frac{qh^p}{h^p - 1}} \text{ as } r \to 0.$$

Therefore (3.2) tends to $m^{p-q}\dfrac{h^{\frac{q}{h^p-1}}}{e\log h^{\frac{q}{h^p-1}}} = SR_h(m, M, p, q)$ as $r \to 0$.

(2)($p$-$q$ version) is thus proved. (2′) is led immediately from (2) by the fact (a) and (b) are derived from each other in Theorem 2.1. As for $C(m, M, p, q)$, we can prove it by calculation. ///

We show Figure 1 and Figure 2 below. In Figure 1, Kantorovich type constant is bigger than 1 on part and less than 1 on part in $p$-$q$ plane. In Figure 2, Hölder-McCarthytype constant is expressed similarly to Figure 1.

We remember the following original fact. For $(T^p x, x)$ and $(Tx, x)^p$, $\geq$ holds and $<$ does not hold when $p \geq 1$. We feel it is a big difference. On the other hand, in the case $1 \leq m < M$ or $m < M \leq 1$, comparing Figure 1 and Figure 2, the differnce between Kantorovich type constant and Hölder-McCarthy type constant is that the boundarys between bigger than 1 and less than 1 on $p$-$q$ plane are curved or straight only. I am so impressed by this fact.

$<1, \quad =1, \quad >1$

$K(m, M, p, q)$ $\qquad$ $C(m, M, p, q)$

$1 \leq m < M$

$m < M \leq 1$

$m < 1 < M$

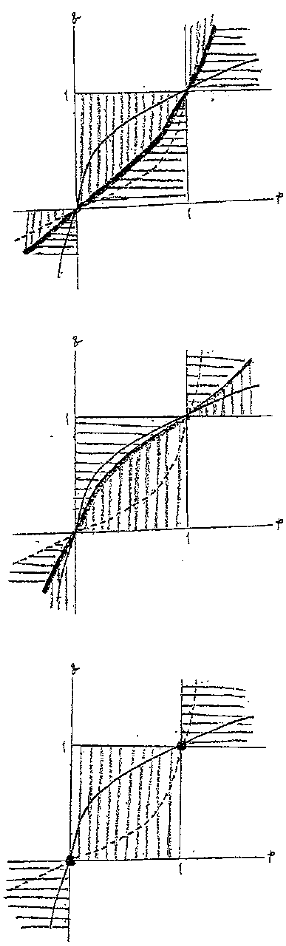

Figure 1

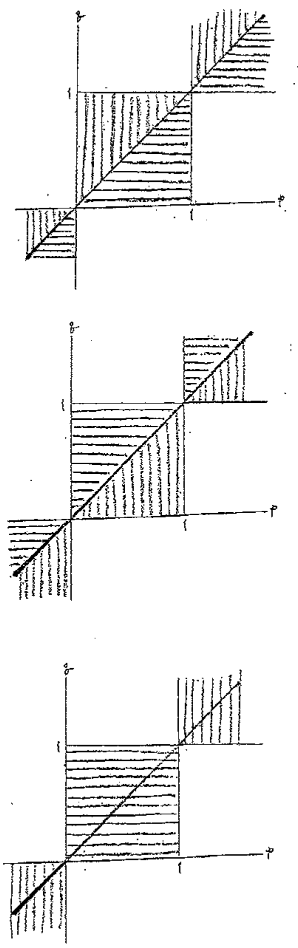

Figure 2

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