数理解析研究所講究録 1427 巻 2005 年 54-64

ON POWERS OF MATRICES PRESERVING A SELF-DUAL CONE

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ヒルベルト空間に正錐から誘導される順序を定義し、この順序を保存する意味でヒルベルト空間上の線形写像に順序を定義する。本稿では冪に関する不等式について考察を行う。

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space with an inner product (,). A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be self-dual if $\mathcal{H}^+ = \{v \in \mathcal{H} | (v, w) \geq 0 \ \forall w \in \mathcal{H}^+\}$. Let A, B be bounded linear operators on $\mathcal{H}$. For a fixed self-dual cone $\mathcal{H}^+$, we denote

$$A \trianglerighteq B$$

if $(A - B)(\mathcal{H}^+) \subset \mathcal{H}^+$. The relation '$\trianglerighteq$' defines an ordered vector space on the set of all bounded linear operators on $\mathcal{H}$, since $\mathcal{H}^+$ is a total set in $\mathcal{H}$. In fact, an arbitrary element v of $\mathcal{H}$ can be written uniquely in the form

$$v = v_1 - v_2 + i(v_3 - v_4)$$

for $v_1, \cdots, v_4 \in \mathcal{H}^+$ and $(v_1, v_2) = (v_3, v_4) = 0$. This order is compatible with operator multiplication. In fact, since

$$AA' - BB' = A(A' - B') + (A - B)B',$$

we obtain that if $A \trianglerighteq B \trianglerighteq O$ and $A' \trianglerighteq B' \trianglerighteq O$, then $AA' \trianglerighteq BB' \trianglerighteq O$. This yields immediately $A^m \trianglerighteq B^m \trianglerighteq O$ for $m = 1, 2, \cdots$. On the contrary, this property does not hold in the case of usual order '$\geq$' for operators. Namely, for two bounded hermitian operators A, B on $\mathcal{H}$, when $A - B$ is a positive semi-definite operator, we write $A \geq B$. It is well-known as the Löwner-Heinz inequality [**1**, bf 3] that if $A \geq B \geq O$, then $A^x \geq B^x \geq O$ for all $x \in [0, 1]$.

2. The case of general self-dual cones in a Hilbert space

The set of all bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $L(\mathcal{H})$.

Theorem 2.1. *Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and A, B in $L(\mathcal{H})$ with $A \geq O, B \geq O$ and $A \trianglerighteq B \trianglerighteq O$ satisfying the following conditions:*
(i) *A and B are compact.*
(ii) *$\overline{A(\mathcal{S})} \subset (\mathcal{H}^+)^\circ$ and $\overline{(A-B)(\mathcal{S})} \subset (\mathcal{H}^+)^\circ$, where $\mathcal{S}$ denotes the set of all unit vectors in $\mathcal{H}^+$.*
Then there exists a number $s > 0$ such that $A^x \trianglerighteq B^x \trianglerighteq O$ for all $x \in [s, \infty)$.

Proof. We shall prove the second inequality. Since B is a compact operator, $\overline{B(\mathcal{S})}$ is compact. By the condition (ii) there exists a number $\varepsilon \in (0,1)$ such that for every $v \in \mathcal{S}$ an ε-neighborhood $U(Bv; \varepsilon)$ of Bv is contained in $\mathcal{H}^+$. Indeed, if such a neighborhood is not contained in $\mathcal{H}^+$, then for every natural number n there exists $v_n \in \mathcal{S}$ such that $U(Bv_n; \frac{1}{n}) \cap (\mathcal{H}^+)^c \neq \emptyset$. Since B is compact, there exists a subsequence $\{Bv_{n_k}\}$ converging to some $w_0 \in \overline{B(\mathcal{S})}$. This implies $v_0 \in \overline{(\mathcal{H}^+)^c}$, a contradiction.

Now, consider a map: $x \mapsto A^x, x \in \mathbb{R}$. By the norm continuity of the map, there exists a number $\mu \in (0,1)$ such that

$$\| A - A^x \| < \varepsilon$$

holds for all $x \in (1-\mu, 1+\mu)$, hence

$$\| Bv - B^x v \| \leq \| B - B^x \| \| v \| = \| B - B^x \| < \varepsilon.$$

This means $B^x v \in \mathcal{H}^+$, i.e., $B^x \trianglerighteq O$. Hence $B^{mx} \trianglerighteq O$ for $m = 1, 2, \cdots$. Setting $m_0 = \left[\frac{1}{\varepsilon}\right] + 1$, we have $B^x \trianglerighteq O$ for all $x \geq m_0(1-\varepsilon)$.

The first inequality in the statement can be proved similarly by considering a map: $x \mapsto A^x - B^x$.

Theorem 2.2. *Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and A in $L(\mathcal{H})$ with $A \geq O$ and $A \trianglerighteq O$ satisfying the following conditions:*
(i) *$A$is invertible.*
(ii) *$A(\mathcal{H}^+) \subsetneq \mathcal{H}^+$.*
Then $A^{-\lambda} \not\trianglerighteq O$ for all $\lambda > O$.

Proof. Suppose that $A^{-\lambda_0} \trianglerighteq O$ for some $\lambda_0 > 0$. In the case where λ_0 is a rational number, we choose $m, n \in \mathbb{N}$ with $m - n\lambda_0 = -1$. It follows by assumption that

$$A^{-1} = A^m A^{-n\lambda_0} \trianglerighteq O.$$

This means that A is an order isomorphism, i.e., $A(\mathcal{H}^+) = \mathcal{H}^+$, a contradiction. It is known that if λ_0 is an irrational number then the set $\{m - n\lambda_0 | m, n \in \mathbb{N}\}$

is dense in $\mathbb{R}$. We choose a sequence $\{r_n\}$ from the dense set converging to -1. Then

$$A^{-1} = \lim_{n\to\infty} A^{r_n} \trianglerighteq O.$$

Similarly, we get the contradiction.

Remark. (cf. [4]) For a facial homogeneous cone $\mathcal{H}^+$, $A(\mathcal{H}^+) = \mathcal{H}^+$ implies $A^x \trianglerighteq O$ for all $x \in \mathbb{R}$.

3. The case of finetely generated self-dual cones

From the rest of this manuscript we deal with a finite dimensional Hilbert space. In this section we consider the case of finitely generated self-dual cones. We first prove the following lemma:

Lemma 3.1. *Let a_i, be real numbers and λ_i be positive numbers with $1 \leq i \leq n$. Put*

$$f(x) = a_1\lambda_1^x + \cdots + a_n\lambda_n^x$$

for $x > 0$. Suppose that there exists an unbounded increasing sequence $\{x_m\}$ such that

$$f(x_m) \geq 0, \ m = 1, 2, \cdots .$$

Then $f(x)$ is identically 0, or there exists $s > 0$ such that

$$f(x) > 0, \ x \in [s, \infty).$$

If, in particular,

$$f(x_m) = 0, \ m = 1, 2, \cdots ,$$

then $f(x)$ is identically 0.

Proof. Let $f(x_m) \geq 0$ for all $m \in \mathbb{N}$. Suppose that $f(x)$ is not identically 0. We may assume $\lambda_1 > \cdots > \lambda_n > 0$ and $a_1 \neq 0$. Since

$$\frac{f(x)}{\lambda_1^x} = a_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^x + \cdots + a_n\left(\frac{\lambda_n}{\lambda_1}\right)^x,$$

it follows that

$$a_1 = \lim_{x\to\infty} \frac{f(x)}{\lambda_1^x}.$$

Hence we have $a_1 \geq 0$, and so $a_1 > 0$ by the assumption. By the continuity of the function, we obtain the desired results. It is now immediate that the latter statement holds.

Theorem 3.2. *Let $\mathcal{H}^+$ be a self-dual cone generated by a finite set in an n-dimensional Euclidean space $\mathcal{H}$. If $A, B \in L(\mathcal{H})$ satisfy $A \geq O, B \geq O$ and $A \trianglerighteq B \trianglerighteq O$, then there exists a number $s > 0$ such that $A^x \trianglerighteq B^x \trianglerighteq O$ for all $x \in [s, \infty)$.*

Proof. Suppose that $\mathcal{H}^+$ is a self-dual cone and

$$\mathcal{H}^+ = \{c_1 v_1 + \cdots + c_m v_m | c_1, \cdots, c_m \geq 0, v_1, \cdots, v_m \in \mathcal{H}^+\}$$

where $\{v_1, \cdots, v_m\}$ is linearly independent. By the assumption we have $A^k - B^k \trianglerighteq O$ for $k = 1, 2, \cdots$. Then $((A^k - B^k)v_i, v_j) \geq 0$ for all i, j. Put

$$f_{ij}(x) = ((A^x - B^x)v_i, v_j).$$

We write

$$A^x = U \begin{pmatrix} \alpha_1^x & & 0 \\ & \ddots & \\ 0 & & \alpha_n^x \end{pmatrix} U^{-1}, B^x = V \begin{pmatrix} \alpha_{n+1}^x & & 0 \\ & \ddots & \\ 0 & & \alpha_{2n}^x \end{pmatrix} V^{-1},$$

for $\alpha_i \geq 0$ and unitaries U, V. Let $\{\beta_1, \cdots, \beta_\ell\}$ be the set of all distinct elements of $\{\alpha_1, \cdots, \alpha_{2n}\}$. Then we can write as $f_{ij}(x) = a_1\beta_1^x + \cdots + a_\ell \beta_\ell^x$. Since $f_{ij}(m) \geq 0$ for all $m = 1, 2, \cdots$, it follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s', \infty)$. Hence there exists a number $s > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s, \infty)$ and all i, j. Choose any elements $v, v' \in \mathcal{H}^+$, which are expressed by

$$v = \sum_{i=1}^m c_i v_i, \ v' = \sum_{i=1}^m c_i' v_i$$

for some $c_i, c_i' \geq 0$. It follows that

$$\begin{aligned} ((A^x - B^x)v, v') &= \sum_{i,j=1}^m ((A^x - B^x)c_i v_i, c_j' v_j) \\ &= \sum_{i,j=1}^m c_i c_j' f_{ij}(x) \geq 0 \end{aligned}$$

for all $x \in [s, \infty)$. This completes the proof.

4. The case of $M_n(\mathbb{C})^+$

let $M_n(\mathbb{R})$(resp. $M_n(\mathbb{R})_s$) denote the set of all real(resp. real symmetric) $n \times n$-matrices. The set of all real positive semi-definite matrices is denoted by $M_n(\mathbb{R})^+$, which is one of the most important self-dual cones in the operator theory or in the theory of operator algebras. We know many operators preserving $M_n(\mathbb{R})^+$ such as

$$\hat{A}: \ X \mapsto \sum_{k=1}^m {}^t A_k X A_k$$

for $X, A_1, \cdots, A_m \in M_n(\mathbb{R})$, i.e., $\hat{A} \trianglerighteq O$.

We first introduce a notation. We identify $M_n(\mathbb{C})$ with an n^2-dimensional Euclidean space $\mathbb{C}^{n^2}$ by a bijective linear map

$$\nu : \begin{pmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{pmatrix} \in M_n(\mathbb{C}) \mapsto \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1n} \\ \vdots \\ \vdots \\ \xi_{n1} \\ \vdots \\ \xi_{nn} \end{pmatrix} \in \mathbb{C}^{n^2}.$$

Given a diagonal matrix A, we shall give a characterization for $A \trianglerighteq O$ with respect to the cone $M_n(\mathbb{C})^+$.

Lemma 4.1. *Let A be an $n^2 \times n^2$ diagonal matrix with entries $\lambda = \{\lambda_{1_1}, \cdots, \lambda_{1_n}, \cdots, \lambda_{n_1}, \cdots, \lambda_{n_n}\}$ with $\lambda_{i_j} \geq 0$, $i, j = 1, \cdots, n$. Then the following conditions are equivalent:*

(1) $A(\nu(M_n(\mathbb{C})^+)) \subset \nu(M_n(\mathbb{C})^+)$.
(2) $\nu^{-1}(\lambda) \in M_n(\mathbb{C})^+$.

Proof. Let A be a diagonal matrix in the assumption. Choose an arbitrary element $\Xi = (\xi_{ij}) \in M_n(\mathbb{C})^+$. Then

$$A\nu(\Xi) = \begin{pmatrix} \lambda_{1_1} & & & \mathbf{0} \\ & \lambda_{1_2} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_{n_n} \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{1_1}\xi_{11} \\ \lambda_{1_2}\xi_{12} \\ \vdots \\ \lambda_{n_n}\xi_{nn} \end{pmatrix}.$$

If $A\nu(\Xi) \in \nu(M_n(\mathbb{C})^+)$ for all $\Xi \in M_n(\mathbb{C})^+$, that is,

$$\nu^{-1}(A\nu(\Xi)) = \begin{pmatrix} \lambda_{1_1}\xi_{11} & \cdots & \lambda_{1_n}\xi_{1n} \\ \vdots & & \vdots \\ \lambda_{n_1}\xi_{n1} & \cdots & \lambda_{n_n}\xi_{nn} \end{pmatrix} \in M_n(\mathbb{C})^+,$$

then

$$(\nu(\Xi), \lambda) = \sum_{i,j=1}^{n} \lambda_{i_j}\xi_{ij} \geq 0.$$

This yields $\lambda \in \nu(M_n(\mathbb{C})^+)$ from the self-duality of $\nu(M_n(\mathbb{C})^+)$.

Conversely, let $\nu^{-1}(\lambda) \in M_n(\mathbb{C})^+$. Then for $\Xi \in M_n(\mathbb{C})^+$ the Shur product product

$$\nu^{-1}(\lambda) \circ \Xi = \begin{pmatrix} \lambda_{1_1}\xi_{11} & \cdots & \lambda_{1_n}\xi_{1n} \\ \vdots & & \vdots \\ \lambda_{n_1}\xi_{n1} & \cdots & \lambda_{n_n}\xi_{nn} \end{pmatrix}$$

belongs to $M_n(\mathbb{C})^+$. Hence $\nu^{-1}(A\nu(\Xi)) \in M_n(\mathbb{C})^+$. This completes the proof.

Theorem 4.2. *Under the order with respect to the cone $\nu(M_n(\mathbb{C})^+)$, let A, B be $n^2 \times n^2$ matrices with $A \geq O, B \geq O$ and $A \trianglerighteq B \trianglerighteq O$. Suppose that both A and B are diagonalizable by a unitary U, and $U\nu(M_n(\mathbb{C})^+) = \nu(M_n(\mathbb{C})^+$. Then there exists a number $s > 0$ such that $A^x \trianglerighteq B^x \trianglerighteq O$ for all $x \in [s, \infty)$.*

Proof. Let C, D be diagonal matrices with $A = UCU^{-1}, B = UDU^{-1}$. Since for any elements $v, w \in \nu(M_n(\mathbb{C})^+)$

$$((A - B)v, w) = (U(C - D)U^{-1}v, w) = ((C - D)v, w),$$

we may assume that A, B are diagonal matrices. Put

$$A = \begin{pmatrix} \lambda_{1_1} & & & \text{\huge 0} \\ & \lambda_{1_2} & & \\ & & \ddots & \\ \text{\huge 0} & & & \lambda_{n_n} \end{pmatrix}, \quad B = \begin{pmatrix} \mu_{1_1} & & & \text{\huge 0} \\ & \mu_{1_2} & & \\ & & \ddots & \\ \text{\huge 0} & & & \mu_{n_n} \end{pmatrix},$$

with $\lambda_{i_j} \geq 0, \mu_{i_j} \geq 0$, $1 \leq i, j \leq n$. Then

$$\nu^{-1}(A^x - B^x) = \begin{pmatrix} \lambda_{1_1}^x - \mu_{1_1}^x & \cdots & \lambda_{1_n}^x - \mu_{1_n}^x \\ \vdots & & \vdots \\ \lambda_{n_1}^x - \mu_{n_1}^x & \cdots & \lambda_{n_n}^x - \mu_{n_n}^x \end{pmatrix}.$$

By Lemma 4.1 we obtain that $A^x - B^x \trianglerighteq O$ holds if and only if $\nu^{-1}(A^x - B^x)$ is positive semi-definite. We shall here denote $f(x)$ by an arbitrary principal minor of $\nu^{-1}(A^x - B^x)$. Then $f(x)$ is expressed by a finite linear combination of x-th powers of positive numbers. By the assumption $A \trianglerighteq B \trianglerighteq O$, we have $A^m \trianglerighteq B^m \trianglerighteq O$ for $m = 1, 2, \cdots$. This implies that $f(m) \geq 0$. It follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f(x) \geq 0$ for all $x \in [s', \infty)$. Consequetly, all principal minors of $\nu^{-1}(A^x - B^x)$ are non-negative for all x more that a sufficiently large number. This completes the proof.

It is immediate in the above theorem that, when $B = O$, $A \trianglerighteq O$ implies $A^x \trianglerighteq O$ for all $x \geq 0$ in the case of $M_2(\mathbb{C})^+$. In the next remark we give the example that for a diagonal matrix $A \geq O$ $A^x \ntrianglerighteq O$ for $0 \leq x < 1$, and $A^x \trianglerighteq O$ for $x \geq 1$.

Remark. Put

$$A = \begin{pmatrix} 1 & & & & & & & & \text{\huge 0} \\ & \frac{1}{\sqrt{2}} & & & & & & & \\ & & \frac{1}{\sqrt{2}} & & & & & & \\ & & & \frac{1}{\sqrt{2}} & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & & & \\ & & & & & & \frac{1}{\sqrt{2}} & & \\ & & & & & & & 0 & \\ \text{\huge 0} & & & & & & & & 1 \end{pmatrix}.$$

Then

$$\nu^{-1}(A) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$

is positive semi-definite, so $A \trianglerighteq O$. We have

$$\begin{vmatrix} 1 & \left(\frac{1}{\sqrt{2}}\right)^x & \left(\frac{1}{\sqrt{2}}\right)^x \\ \left(\frac{1}{\sqrt{2}}\right)^x & 1 & 0 \\ \left(\frac{1}{\sqrt{2}}\right)^x & 0 & 1 \end{vmatrix} = 1 - 2\left(\frac{1}{2}\right)^x.$$

This implies immediately that $A^x \not\trianglerighteq O$ for $1 \leq x < 1$, and $A^x \trianglerighteq O$ for $x \geq 1$.

5. The case of $M_2(\mathbb{R})^+$

Theorem 5.1. (cf. [**2**]) *Let A and B be matricial representations of linear transformations on $\nu(M_2(\mathbb{R})_s)$ with a self-dual cone $\nu(M_2(\mathbb{R})^+)$. If A and B are positive semi-definite and $A \trianglerighteq B \trianglerighteq O$, then there exists a positive number s such that $A^x \trianglerighteq B^x \trianglerighteq O$ for all $x \in [s, \infty)$.*

Proof. We may use in the proof the notation '$\trianglerighteq$' as follows: For $A, B \in M_4(\mathbb{R})$, $A \trianglerighteq B \trianglerighteq O$ means $(A - B)(\nu(M_2(\mathbb{R})^+)) \subset \nu(M_2(\mathbb{R})^+$, though this relation in $M_4(\mathbb{R})$ does not satisfy the symmetric law of the axiom of an order. Suppose $A, B \in M_4(\mathbb{R})^+$ and $A \trianglerighteq B \trianglerighteq O$. Let $\{\alpha_1, \cdots, \alpha_4\}$ be the eigenvalues of A and $\{\alpha_5, \cdots, \alpha_8\}$ the eigenvalues of B. We then have

$$A^x = U \begin{pmatrix} \alpha_1^x & & 0 \\ & \ddots & \\ 0 & & \alpha_4^x \end{pmatrix} U^{-1}, B^x = V \begin{pmatrix} \alpha_5^x & & 0 \\ & \ddots & \\ 0 & & \alpha_8^x \end{pmatrix} V^{-1},$$

for some real orthogonal matrices U, V and $x > 0$. Put $C(x) = A^x - B^x$ for $x > 0$. Any element of $M_2(\mathbb{R})^+$ can be expressed as a convex combination of elements of the boundary of $M_2(\mathbb{R})^+$ in the subspace $M_2(\mathbb{R})^+ - M_2\mathbb{R})^+$ of all real symmetric matrices. Hence, in order to prove $C(x)(\nu(M_2(\mathbb{R})^+)) \subset \nu(M_2(\mathbb{R})^+)$, it suffices to show that $C(x)\xi \in \nu(M_2(\mathbb{R})^+)$ and $C(x)\eta \in \nu(M_2(\mathbb{R})^+)$ for

$$\xi = \begin{pmatrix} 1 \\ b \\ b \\ b^2 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $b \in \mathbb{R}$.

Step (i): In this part, we shall show that $\nu^{-1}(C(x)\xi)$ is symmetric for all $x > 0$. We choose distinct eigenvalues $\{\beta_i\}$ of A and B such that $\beta_1 > \cdots > \beta_\ell \geq 0$

$(1 \leq \ell \leq 8)$. Then

$$\nu^{-1}(C(x)\xi) = \begin{pmatrix} \sum_{k=1}^{\ell} \mu_k^{(1,1)}(b)\beta_k^x & \sum_{k=1}^{\ell} \mu_k^{(1,2)}(b)\beta_k^x \\ \sum_{k=1}^{\ell} \mu_k^{(2,1)}(b)\beta_k^x & \sum_{k=1}^{\ell} \mu_k^{(2,2)}(b)\beta_k^x \end{pmatrix}.$$

Since $C(1) \trianglerighteq O$, we have $C(m) \trianglerighteq O$ for all $m \in \mathbb{N}$. Hence $\nu^{-1}(C(m)\xi)$ is positive semi-definite. Hence a $(1,2)$-component and a $(2,1)$-component of this matrix are equal, i.e.,

$$\mu_1^{(1,2)}(b)\beta_1^m + \cdots + \mu_\ell^{(1,2)}(b)\beta_\ell^m = \mu_1^{(2,1)}(b)\beta_1^m + \cdots + \mu_\ell^{(2,1)}(b)\beta_\ell^m$$

for $m = 1, 2, \cdots$. Since β_k are distinct, we have $\mu_k^{(1,2)} = \mu_k^{(2,1)}$ for all k. This yields immediately that the off-diagonal components of $\nu^{-1}(C(x)\xi)$ are equal.

Step (ii): Let

$$C(x) = [f_{ij}(x)]_{i,j=1}^4.$$

Here $f_{ij}(x)$ is expressed as a finite linear combination of β_k^x. Then

$$\nu^{-1}(C(x)\xi) =$$
$$= \begin{pmatrix} f_{11}(x) + (f_{12}(x) + f_{13}(x))b + f_{14}(x)b^2 & f_{21}(x) + (f_{22}(x) + f_{23}(x))b + f_{24}(x)b^2 \\ f_{31}(x) + (f_{32}(x) + f_{33}(x))b + f_{34}(x)b^2 & f_{41}(x) + (f_{42}(x) + f_{43}(x))b + f_{44}(x)b^2 \end{pmatrix}.$$

In this part we first show that all diagonal components of $\nu^{-1}(C(x)\xi)$ are non-negative for all real numbers b, and all x more than a sufficiently large number (which is independent on b). Since by the assumption every diagonal component of $\nu^{-1}(C(m)\xi)), m = 1, 2, \cdots$, is non-negative for all $b \in \mathbb{R}$, we have

$$f_{i4}(m) \geq 0, \;\; m = 1, 2, \cdots, i = 1, 4.$$

Suppose that $f_{i4}(x)$ is not identically 0. We then obtain from Lemma that $f_{i4}(x) > 0$ for all x more than a sufficiently large number. Hence every diagonal component of $\nu^{-1}(C(x)\xi)$ is expressed by

$$f_{i4}(x)\left(b + \frac{f_{i_2}(x + f_{i3}(x))}{2f_{i4}(x)}\right)^2 + \frac{g_i(x)}{4f_{i4}(x)}.$$

Here

$$g_i(x) = 4f_{i4}(x)f_{i1}(x) - (f_{i2}(x) + f_{i3}(x))^2.$$

Similarly, since $g_i(m) \geq 0$ for $m = 1, 2, \cdots$, we obtaine that $g_i(x) \geq 0$ for all x more than a sufficiently large number. The above inequality is valid in the case where $f_{i4}(x)$ is identically 0. Indeed, if $f_{i4}(x)$ is identically 0, then $f_{i2}(m) + f_{i3}(m) = 0$ holds for $m \in \mathbb{N}$. For, if $f_{i_02}(m_0) + f_{i_03}(m_0) \neq 0$ for some i_0 and m_0, then the infimum of a diagonal component of $\nu^{-1}(C(m_0)\xi)$ is $-\infty$. This contradicts the condition that $C(m) \trianglerighteq O$ holds for all $m \in \mathbb{N}$. By Lemma, $f_{i2}(x) + f_{i3}(x)$ is identically 0. In this case, it suffices to consider the function $f_{i1}(x)$ in the same way.

Next, we examine the determinant of $\nu^{-1}(C(x)\xi)$. Put

$$G = \det \nu^{-1}(C(x)\xi).$$

Then G is expressed as

$$G = G(b, x) = a_0(x)b^4 + a_1(x)b^3 + \cdots + a_4(x).$$

Here $a_i(x)$ is a finite linear combination of x-th powers of some positive numbers. By the assumption we have

$$G(b, m) \geq 0, -\infty < b < \infty, m = 1, 2, \cdots .$$

Suppose that $a_0(x)$ is not identically 0. Then $a_0(x) > 0$ for all x for a sufficiently large number, since $a_0(x)$ satisfies the hypothesis in Lemma. Put

$$L(x) = \inf_{b\in\mathbb{R}} G(b, x).$$

Then $L(x)$ is given by the following formula:

$$L(x) = \min_{1\leq j\leq 3} L_j(x),$$

where $L_j(x) = G(\mathrm{Re}\ b_j(x), x)$ and $b_j(x)$ are all roots of the cubic equation $G_b(b, x) = 0$ of b. Note that $b_j(x)$ are algebraically expressed by $a_i(x)$, and are continuous for all x more than a sufficiently large number. We must show the existence of a number $s > 0$ satisfying $L(x) \geq 0$ for all $x \in [s, \infty)$. Assume that there does not exist such a number s. Then for every natural number m there exists y_m with $y_m \geq m$ such that $L_{j_0}(y_m) < 0$ for some j_0. Since $L_{j_0}(m) \geq 0$ for all $m \in \mathbb{N}$, there exists by the intermediate value theorem a positive sequence $\{x_m\}$ with $\lim_{m\to\infty} x_m = \infty$ such that

$$L_{j_0}(x_m) = 0, \ m = 1, 2, \cdots .$$

Let $\hat{L}_{j_0}(x)$ be a polynomial of $a_i(x)$ such that the set of all zeros of $\hat{L}_{j_0}(x)$ includes the set of al zeros of $L_{j_0}(x)$. Hence there exists an unbounded increasing sequence of zeros of $\hat{L}_{j_0}(x)$. Since $\hat{L}_{j_0}(x)$ is a finite linear combination of x-th powers of some positive numbers, it follows from Lemma that $\hat{L}_{j_0}(x)$ is identically 0. This is a contradiction. On the other hand, if $a_0(x)$ is identically 0, then $a_1(x)$ is also identically 0 by the argument in the former part of Step (ii). In this case we also obtain the same result.

Similarly, we obtain the desired properties for η. This completes the proof.

The next remark shows that Theorem does not always hold for all positive number x. We shall give the example that there exists a 4×4 positive semi-definite matrix A with $A \trianglerighteq O$ satisfying $A^x \ntrianglerighteq O$ for $x \in [0,1)$.

Remark. Note that $\nu(M_2(\mathbb{R})^+)$ is isometrically isomorphic to a circular cone

$$\mathcal{H}^+ = \left\{ \xi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| F(\xi) = x^2 + y^2 - z^2 \leq 0, z \geq 0 \right\}.$$

Consider the following positive semi-definite matrix A:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Then $A \trianglerighteq O$. In fact, we have $A^\alpha \trianglerighteq O$ for all $\alpha \geq 1$. To see it, it suffices to examine that for $\eta(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}$ we have $F(A^x\eta(\theta)) \leq 0$ as follows:

$$\begin{aligned} 6A^\alpha\eta(\theta) &= \begin{pmatrix} 3^\alpha + 3 & 3^\alpha - 3 & 2 \cdot 3^\alpha \\ 3^\alpha - 3 & 3^\alpha + 3 & 2 \cdot 3^\alpha \\ 2 \cdot 3^\alpha & 2 \cdot 3^\alpha & 4 \cdot 3^\alpha \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} (3^\alpha + 3)\cos\theta + (3^\alpha - 3)\sin\theta + 2 \cdot 3^\alpha \\ (3^\alpha - 3)\cos\theta + (3^\alpha + 3)\sin\theta + 2 \cdot 3^\alpha \\ (2 \cdot 3^\alpha)\cos\theta + (2 \cdot 3^\alpha)\sin\theta + 4 \cdot 3^\alpha \end{pmatrix} \end{aligned}$$

and so

$$\begin{aligned} F(6A^\alpha\eta(\theta)) &= 18 - 10 \cdot 3^{2\alpha} - (4 \cdot 3^{2\alpha} + 36)\cos\theta\sin\theta - 8 \cdot 3^{2\alpha}(\cos\theta + \sin\theta) \\ &= -(36 + 4 \cdot 3^{2\alpha})(\cos\theta + 1)(\sin\theta + 1) - (4 \cdot 3^{2\alpha} - 36)(2(\cos\theta + \sin\theta) + 3) \\ &\leq 0 \end{aligned}$$

for $\alpha \geq 1$ and $0 \leq \theta \leq 2\pi$. On the other hand, we shall show that for every $\alpha \in [0,1)$ there exists a real number θ_0 such that $F(6A^\alpha\eta(\theta_0)) > 0$. Indeed, one can choose θ_0 satisfying

$$\cos\theta_0 + \sin\theta_0 = \sqrt{2}\sin\left(\theta_0 + \frac{\pi}{4}\right) = -\frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}},$$

since

$$0 < \frac{2 \cdot 3^{2\alpha}}{\sqrt{2}(9 + 3^{2\alpha})} \leq \frac{1}{\sqrt{2}}$$

for $0 \leq \alpha \leq 1$. Taking a square of both sides of the above equalities, we have

$$1 + 2\cos\theta_0\sin\theta_0 = \frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2}.$$

It follows that for $0 \leq \alpha < 1$

$$F(6A^{\alpha}\eta(\theta_0)) = 18 - 10 \cdot 3^{2\alpha} - 2(9 + 3^{2\alpha})\left(\frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2} - 1\right) + 8 \cdot 3^{2\alpha} \cdot \frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}}$$
$$= \frac{36}{9 + 3^{2\alpha}}(9 - 3^{2\alpha}) > 0,$$

from which we have $A^{\alpha} \not\trianglerighteq O$.

Finally, we obtain immediately the following theorem, which is understood to be a matrix version of Lemma, reviewing the proof of Theorem 5.1:

Theorem 5.2. *Let $A_1, \cdots, A_n$ be positive semi-definite matricial representations of linear transformations on $\nu(M_2(\mathbb{R})_s)$ with a self-dual cone $\nu(M_2(\mathbb{R})^+)$, and $a_1, \cdots, a_n$ be real numbers. Suppose that*

$$a_1 A_1^m + \cdots + a_n A_n^m \trianglerighteq O$$

holds for $m = 1, 2, \cdots$. Then there exists $s > 0$ such that

$$a_1 A_1^x + \cdots + a_n A_n^x \trianglerighteq O$$

for all $x \in [s, \infty)$.

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