ON POWERS OF MATRICES
PRESERVING A SELF-DUAL CONE

岩手大・人文社会科学研究部
三浦 康秀 (YASUHIDE MIURA)

" "

石川洋一郎 (YOICHIRO ISHIKAWA)

" "

大西 良博 (YOSHIHIRO ŌISHI)

Faculty of Humanities and Social Sciences, Iwate University

1. Introduction

Let $\mathcal{H}$ be a Hilbert space with an inner product $(\cdot,\cdot)$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be self-dual if $\mathcal{H}^+ = \{v \in \mathcal{H} | (v, w) \geq 0 \ \forall w \in \mathcal{H}^+\}$. Let $A, B$ be bounded linear operators on $\mathcal{H}$. For a fixed self-dual cone $\mathcal{H}^+$, we denote

$$A \succeq B$$

if $(A - B)(\mathcal{H}^+) \subset \mathcal{H}^+$. The relation `$\succeq$' defines an ordered vector space on the set of all bounded linear operators on $\mathcal{H}$, since $\mathcal{H}^+$ is a total set in $\mathcal{H}$. In fact, an arbitrary element $v$ of $\mathcal{H}$ can be written uniquely in the form

$$v = v_1 - v_2 + i(v_3 - v_4)$$

for $v_1, \cdots, v_4 \in \mathcal{H}^+$ and $(v_1, v_2) = (v_3, v_4) = 0$. This order is compatible with operator multiplication. In fact, since

$$AA' - BB' = A(A' - B') + (A - B)B',$$

we obtain that if $A \succeq B \succeq O$ and $A' \succeq B' \succeq O$, then $AA' \succeq BB' \succeq O$. This yields immediately $A^m \succeq B^m \succeq O$ for $m = 1, 2, \cdots$. On the contrary, this property does not hold in the case of usual order `$\geq$' for operators. Namely, for two bounded hermitian operators $A, B$ on $\mathcal{H}$, when $A - B$ is a positive semi-definite operator, we write $A \succeq B$. It is well-known as the Löwner-Heinz inequality [1, bf 3] that if $A \succeq B \succeq O$, then $A^x \succeq B^x \succeq O$ for all $x \in [0, 1]$.
2. The case of general self-dual cones in a Hilbert space

The set of all bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $L(\mathcal{H})$.

**Theorem 2.1.** Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and $A, B$ in $L(\mathcal{H})$ with $A \geq O, B \geq O$ and $A \geq B \geq O$ satisfying the following conditions:

(i) $A$ and $B$ are compact.

(ii) $\overline{A(S)} \subset (\mathcal{H}^+)^o$ and $\overline{(A-B)(S)} \subset (\mathcal{H}^+)^o$, where $S$ denotes the set of all unit vectors in $\mathcal{H}^+$.

Then there exists a number $s > 0$ such that $A^x \geq B^x \geq O$ for all $x \in [s, \infty)$.

**Proof.** We shall prove the second inequality. Since $B$ is a compact operator, $B(S)$ is compact. By the condition (ii) there exists a number $\varepsilon \in (0,1)$ such that for every $v \in S$ an $\varepsilon$-neighborhood $U(Bv; \varepsilon)$ of $Bv$ is contained in $\mathcal{H}^+$. Indeed, if such a neighborhood is not contained in $\mathcal{H}^+$, then for every natural number $n$ there exists $v_n \in S$ such that $U(Bv_n; \frac{1}{n}) \cap (\mathcal{H}^+)^c \neq \emptyset$. Since $B$ is compact, there exists a subsequence $\{Bv_n\}$ converging to some $w_0 \in B(S)$. This implies $w_0 \in (\mathcal{H}^+)^c$, a contradiction.

Now, consider a map: $x \mapsto A^x, x \in \mathbb{R}$. By the norm continuity of the map, there exists a number $\mu \in (0,1)$ such that

$$\|A - A^x\| < \varepsilon$$

holds for all $x \in (1 - \mu, 1 + \mu)$, hence

$$\|Bv - B^xv\| \leq \|B - B^x\| \|v\| = \|B - B^x\| < \varepsilon.$$ 

This means $B^xv \in \mathcal{H}^+$, i.e., $B^x \geq O$. Hence $B^{mx} \geq O$ for $m = 1, 2, \cdots$. Setting $m_0 = \left[\frac{1}{\varepsilon}\right] + 1$, we have $B^x \geq O$ for all $x \geq m_0(1 - \varepsilon)$.

The first inequality in the statement can be proved similarly by considering a map: $x \mapsto A^x - B^x$.

**Theorem 2.2.** Let $\mathcal{H}^+$ be a selfdual cone in $\mathcal{H}$, and $A$ in $L(\mathcal{H})$ with $A \geq O$ and $A \geq O$ satisfying the following conditions:

(i) $A$ is invertible.

(ii) $A(\mathcal{H}^+) \subseteq \mathcal{H}^+$.

Then $A^{-\lambda} \notin O$ for all $\lambda > O$.

**Proof.** Suppose that $A^{-\lambda_0} \geq O$ for some $\lambda_0 > 0$. In the case where $\lambda_0$ is a rational number, we choose $m, n \in \mathbb{N}$ with $m - n\lambda_0 = -1$. It follows by assumption that

$$A^{-1} = A^m A^{-n\lambda_0} \geq O.$$ 

This means that $A$ is an order isomorphism, i.e., $A(\mathcal{H}^+) = \mathcal{H}^+$, a contradiction. It is known that if $\lambda_0$ is an irrational number then the set $\{m - n\lambda_0|m, n \in \mathbb{N}\}$
is dense in $\mathbb{R}$. We choose a sequence $\{r_n\}$ from the dense set converging to $-1$. Then
\[ A^{-1} = \lim_{n \to \infty} A^{r_n} \geq O. \]
Similarly, we get the contradiction.

**Remark.** (cf. [4]) For a facial homogeneous cone $\mathcal{H}^+$, $A(\mathcal{H}^+) = \mathcal{H}^+$ implies $A^x \geq O$ for all $x \in \mathbb{R}$.

### 3. The Case of Finitely Generated Self-Dual Cones

From the rest of this manuscript we deal with a finite dimensional Hilbert space. In this section we consider the case of finitely generated self-dual cones. We first prove the following lemma:

**Lemma 3.1.** Let $a_i$, be real numbers and $\lambda_i$ be positive numbers with $1 \leq i \leq n$. Put
\[ f(x) = a_1 \lambda_1^x + \cdots + a_n \lambda_n^x \]
for $x > 0$. Suppose that there exists an unbounded increasing sequence $\{x_m\}$ such that
\[ f(x_m) \geq 0, \quad m = 1, 2, \cdots. \]
Then $f(x)$ is identically 0, or there exists $s > 0$ such that
\[ f(x) > 0, \quad x \in [s, \infty). \]
If, in particular,
\[ f(x_m) = 0, \quad m = 1, 2, \cdots, \]
then $f(x)$ is identically 0.

**Proof.** Let $f(x_m) \geq 0$ for all $m \in \mathbb{N}$. Suppose that $f(x)$ is not identically 0. We may assume $\lambda_1 > \cdots > \lambda_n > 0$ and $a_1 \neq 0$. Since
\[ \frac{f(x)}{\lambda_1^x} = a_1 + a_2 \left( \frac{\lambda_2}{\lambda_1} \right)^x + \cdots + a_n \left( \frac{\lambda_n}{\lambda_1} \right)^x, \]
it follows that
\[ a_1 = \lim_{x \to \infty} \frac{f(x)}{\lambda_1^x}. \]
Hence we have $a_1 \geq 0$, and so $a_1 > 0$ by the assumption. By the continuity of the function, we obtain the desired results. It is now immediate that the latter statement holds.
Theorem 3.2. Let $\mathcal{H}^+$ be a self-dual cone generated by a finite set in an $n$-dimensional Euclidean space $\mathcal{H}$. If $A, B \in L(\mathcal{H})$ satisfy $A \geq O, B \geq O$ and $A \triangleright B \triangleright O$, then there exists a number $s > 0$ such that $A^x \triangleright B^x \triangleright O$ for all $x \in [s, \infty)$.

Proof. Suppose that $\mathcal{H}^+$ is a self-dual cone and

$$\mathcal{H}^+ = \{c_1v_1 + \cdots + c_mv_m | c_1, \cdots, c_m \geq 0, v_1, \cdots, v_m \in \mathcal{H}^+\}$$

where $\{v_1, \cdots, v_m\}$ is linearly independent. By the assumption we have $A^k - B^k \triangleright O$ for $k = 1, 2, \cdots$. Then $((A^k - B^k)v_i, v_j) \geq 0$ for all $i, j$. Put

$$f_{ij}(x) = ((A^x - B^x)v_i, v_j).$$

We write

$$A^x = U \begin{pmatrix} \alpha_1^x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n^x \end{pmatrix} U^{-1}, B^x = V \begin{pmatrix} \alpha_{n+1}^x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{2n}^x \end{pmatrix} V^{-1},$$

for $\alpha_i \geq 0$ and unitaries $U, V$. Let $\{\beta_1, \cdots, \beta_\ell\}$ be the set of all distinct elements of $\{\alpha_1, \cdots, \alpha_{2n}\}$. Then we can write as $f_{ij}(x) = a_1\beta_1^x + \cdots + a_\ell\beta_\ell^x$. Since $f_{ij}(m) \geq 0$ for all $m = 1, 2, \cdots$, it follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s', \infty)$. Hence there exists a number $s > 0$ satisfying $f_{ij}(x) \geq 0$ for all $x \in [s, \infty)$ and all $i, j$. Choose any elements $v, v' \in \mathcal{H}^+$, which are expressed by

$$v = \sum_{i=1}^{m} c_i v_i, \quad v' = \sum_{i=1}^{m} c'_i v_i$$

for some $c_i, c'_i \geq 0$. It follows that

$$(A^x - B^x)v, v' = \sum_{i,j=1}^{m} ((A^x - B^x)c_i v_i, c'_j v_j) \geq 0$$

for all $x \in [s, \infty)$. This completes the proof.

4. The Case of $M_n(\mathbb{C})^+$

Let $M_n(\mathbb{R})$ (resp. $M_n(\mathbb{R})_s$) denote the set of all real (resp. real symmetric) $n \times n$-matrices. The set of all real positive semi-definite matrices is denoted by $M_n(\mathbb{R})^+$, which is one of the most important self-dual cones in the operator theory or in the theory of operator algebras. We know many operators preserving $M_n(\mathbb{R})^+$ such as

$$\hat{A} : X \mapsto \sum_{k=1}^{m} t_{Ak} X A_k$$
for $X, A_1, \cdots, A_m \in M_n(\mathbb{R})$, i.e., $\hat{A} \succeq O$.

We first introduce a notation. We identify $M_n(\mathbb{C})$ with an $n^2$-dimensional Euclidean space $\mathbb{C}^{n^2}$ by a bijective linear map

$$
\nu : \begin{pmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \vdots & \ddots & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{pmatrix} \in M_n(\mathbb{C}) \mapsto \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{n1} \\ \xi_{1n} \\ \vdots \\ \xi_{nn} \end{pmatrix} \in \mathbb{C}^{n^2}.
$$

Given a diagonal matrix $A$, we shall give a characterization for $A \succeq O$ with respect to the cone $M_n(\mathbb{C})^+$.  

**Lemma 4.1.** Let $A$ be an $n^2 \times n^2$ diagonal matrix with entries $\lambda = \{\lambda_{1,1}, \cdots, \lambda_{1,n}, \cdots, \lambda_{n,1}, \cdots, \lambda_{n,n}\}$ with $\lambda_{ij} \geq 0$, $i, j = 1, \cdots, n$. Then the following conditions are equivalent:

1. $A(\nu(M_n(\mathbb{C})^\perp)) \subseteq \nu(M_n(\mathbb{C})^+)$.  
2. $\iota/(\lambda) \in M_n(\mathbb{C})^+$.  

**Proof.** Let $A$ be a diagonal matrix in the assumption. Choose an arbitrary element $\Xi = (\xi_{ij}) \in M_n(\mathbb{C})^+$. Then

$$
A \nu(\Xi) = \begin{pmatrix} \lambda_{1,1} & & & \\ & \lambda_{1,2} & & \\ & & \ddots & \\ 0 & & & \lambda_{n,n} \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_{1,1}\xi_{11} \\ \lambda_{1,2}\xi_{12} \\ \vdots \\ \lambda_{n,n}\xi_{nn} \end{pmatrix}.
$$

If $A \nu(\Xi) \in \nu(M_n(\mathbb{C})^+)$ for all $\Xi \in M_n(\mathbb{C})^+$, that is,

$$
\nu^{-1}(\lambda) = \begin{pmatrix} \lambda_{1,1}\xi_{11} & \cdots & \lambda_{1,n}\xi_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1}\xi_{n1} & \cdots & \lambda_{n,n}\xi_{nn} \end{pmatrix} \in M_n(\mathbb{C})^+,
$$

then

$$
(\nu(\Xi), \lambda) = \sum_{i,j=1}^{n} \lambda_{ij}\xi_{ij} \geq 0.
$$

This yields $\lambda \in \nu(M_n(\mathbb{C})^+)$ from the self-duality of $\nu(M_n(\mathbb{C})^+)$. Conversely, let $\nu^{-1}(\lambda) \in M_n(\mathbb{C})^+$. Then for $\Xi \in M_n(\mathbb{C})^+$ the Shur product product

$$
\nu^{-1}(\lambda) \circ \Xi = \begin{pmatrix} \lambda_{1,1}\xi_{11} & \cdots & \lambda_{1,n}\xi_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1}\xi_{n1} & \cdots & \lambda_{n,n}\xi_{nn} \end{pmatrix}
$$

belongs to $M_n(\mathbb{C})^+$. Hence $\nu^{-1}(A \nu(\Xi)) \in M_n(\mathbb{C})^+$. This completes the proof.
Theorem 4.2. Under the order with respect to the cone $\nu(M_n(\mathbb{C})^+)$, let $A, B$ be $n^2 \times n^2$ matrices with $A \succeq O, B \succeq O$ and $A \succeq B \succeq O$. Suppose that both $A$ and $B$ are diagonalizable by a unitary $U$, and $U \nu(M_n(\mathbb{C})^+) = \nu(M_n(\mathbb{C})^+)$. Then there exists a number $s > 0$ such that $A^x \succeq B^x \succeq O$ for all $x \in [s, \infty)$.

Proof. Let $C, D$ be diagonal matrices with $A = UCU^{-1}, B = UDU^{-1}$. Since for any elements $v, w \in \nu(M_n(\mathbb{C})^+)$

$$(A - B)v, w) = (U(C - D)U^{-1}v, w) = ((C - D)v, w),$$

we may assume that $A, B$ are diagonal matrices. Put

$$A = \begin{pmatrix} \lambda_{11} & 0 & \cdots & 0 \\ \lambda_{12} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & \lambda_{n_{n}} \\ \end{pmatrix}, \quad B = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & 0 \\ \mu_{12} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots \\ \mu_{n_{n}} \\ \end{pmatrix},$$

with $\lambda_{ij} \geq 0, \mu_{ij} \geq 0, 1 \leq i, j \leq n$. Then

$$\nu^{-1}(A^x - B^x) = \begin{pmatrix} \lambda_{11}^x - \mu_{11}^x & \cdots & \lambda_{1n}^x - \mu_{1n}^x \\ \vdots & \ddots & \vdots \\ \lambda_{n1}^x - \mu_{n1}^x & \cdots & \lambda_{nn}^x - \mu_{nn}^x \\ \end{pmatrix}.$$ 

By Lemma 4.1 we obtain that $A^x - B^x \succeq O$ holds if and only if $\nu^{-1}(A^x - B^x)$ is positive semi-definite. We shall here denote $f(x)$ by an arbitrary principal minor of $\nu^{-1}(A^x - B^x)$. Then $f(x)$ is expressed by a finite linear combination of $x$-th powers of positive numbers. By the assumption $A \succeq B \succeq O$, we have $A^m \succeq B^m \succeq O$ for $m = 1, 2, \cdots$. This implies that $f(m) \geq 0$. It follows from Lemma 3.1 that there exists a number $s' > 0$ satisfying $f(x) \geq 0$ for all $x \in [s', \infty)$. Consequently, all principal minors of $\nu^{-1}(A^x - B^x)$ are non-negative for all $x$ more that a sufficiently large number. This completes the proof.

It is immediate in the above theorem that, when $B = O, A \succeq O$ implies $A^x \succeq O$ for all $x \geq 0$ in the case of $M_2(\mathbb{C})^+$. In the next remark we give the example that for a diagonal matrix $A \succeq O A^x \not\simeq O$ for $0 \leq x < 1$, and $A^x \succeq O$ for $x \geq 1$.

Remark. Put

$$A = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \end{pmatrix}.$$
Then

\[ \nu^{-1}(A) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \]

is positive semi-definite, so \( A \succeq O \). We have

\[ \begin{vmatrix} 1 & \left(\frac{1}{\sqrt{2}}\right)^x & \left(\frac{1}{\sqrt{2}}\right)^x \\ \left(\frac{1}{\sqrt{2}}\right)^x & 1 & 0 \\ \left(\frac{1}{\sqrt{2}}\right)^x & 0 & 1 \end{vmatrix} = 1 - 2 \left(\frac{1}{2}\right)^x. \]

This implies immediately that \( A^x \not\succeq O \) for \( 1 \leq x < 1 \), and \( A^x \succeq O \) for \( x \geq 1 \).

5. The Case of \( M_2(\mathbb{R})^+ \)

**Theorem 5.1.** (cf. [2]) Let \( A \) and \( B \) be matricial representations of linear transformations on \( \nu(M_2(\mathbb{R})_s) \) with a self-dual cone \( \nu(M_2(\mathbb{R})^+) \). If \( A \) and \( B \) are positive semi-definite and \( A \succeq B \succeq O \), then there exists a positive number \( s \) such that \( A^x \succeq B^x \succeq O \) for all \( x \in [s, \infty) \).

**Proof.** We may use in the proof the notation \( \succeq \) as follows: For \( A, B \in M_4(\mathbb{R}) \), \( A \succeq B \succeq O \) means \( (A - B)(\nu(M_2(\mathbb{R})^+)) \subseteq \nu(M_2(\mathbb{R})^+) \), though this relation in \( M_4(\mathbb{R}) \) does not satisfy the symmetric law of the axiom of an order. Suppose \( A, B \in M_4(\mathbb{R})^+ \) and \( A \succeq B \succeq O \). Let \( \{\alpha_1, \cdots, \alpha_4\} \) be the eigenvalues of \( A \) and \( \{\alpha_5, \cdots, \alpha_8\} \) the eigenvalues of \( B \). We then have

\[ A^x = U \begin{pmatrix} \alpha_1^x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_4^x \end{pmatrix} U^{-1}, \quad B^x = V \begin{pmatrix} \alpha_5^x & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_8^x \end{pmatrix} V^{-1}, \]

for some real orthogonal matrices \( U, V \) and \( x > 0 \). Put \( C(x) = A^x - B^x \) for \( x > 0 \). Any element of \( M_4(\mathbb{R})^+ \) can be expressed as a convex combination of elements of the boundary of \( M_4(\mathbb{R})^+ \) in the subspace \( M_2(\mathbb{R})^+ - M_2(\mathbb{R})^+ \) of all real symmetric matrices. Hence, in order to prove \( C(x)(\nu(M_2(\mathbb{R})^+)) \subseteq \nu(M_2(\mathbb{R})^+) \), it suffices to show that \( C(x)\xi \in \nu(M_2(\mathbb{R})^+) \) and \( C(x)\eta \in \nu(M_2(\mathbb{R})^+) \) for

\[ \xi = \begin{pmatrix} 1 \\ b \\ b \\ b^2 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

with \( b \in \mathbb{R} \).

**Step (i):** In this part, we shall show that \( \nu^{-1}(C(x)\xi) \) is symmetric for all \( x > 0 \). We choose distinct eigenvalues \( \{\beta_i\} \) of \( A \) and \( B \) such that \( \beta_1 > \cdots > \beta_t \geq 0 \)
$(1 \leq \ell \leq 8)$. Then

$$\nu^{-1}(C(x)\xi) = \begin{pmatrix}
\sum_{k=1}^{\ell} \mu_k^{(1,1)}(b)\beta_k^x \\
\sum_{k=1}^{\ell} \mu_k^{(1,2)}(b)\beta_k^x \\
\sum_{k=1}^{\ell} \mu_k^{(2,1)}(b)\beta_k^x \\
\sum_{k=1}^{\ell} \mu_k^{(2,2)}(b)\beta_k^x
\end{pmatrix}.$$ 

Since $C(1) \triangleright O$, we have $C(m) \triangleright O$ for all $m \in \mathbb{N}$. Hence $\nu^{-1}(C(m)\xi)$ is positive semi-definite. Hence a $(1,2)$-component and a $(2,1)$-component of this matrix are equal, i.e.,

$$\mu_1^{(1,2)}(b)\beta_1^m + \cdots + \mu_{\ell}^{(1,2)}(b)\beta_{\ell}^m = \mu_1^{(2,1)}(b)\beta_1^m + \cdots + \mu_{\ell}^{(2,1)}(b)\beta_{\ell}^m$$

for $m = 1, 2, \cdots$. Since $\beta_k$ are distinct, we have $\mu_k^{(1,2)} = \mu_k^{(2,1)}$ for all $k$. This yields immediately that the off-diagonal components of $\nu^{-1}(C(x)\xi)$ are equal.

Step (ii): Let $C(x) = [f_{ij}(x)]_{i,j=1}^{4}$. Here $f_{ij}(x)$ is expressed as a finite linear combination of $\beta_k^x$. Then

$$\nu^{-1}(C(x)\xi) = \begin{pmatrix}
f_{11}(x) + (f_{12}(x) + f_{13}(x))b + f_{14}(x)b^2 & f_{21}(x) + (f_{22}(x) + f_{23}(x))b + f_{24}(x)b^2 \\
f_{31}(x) + (f_{32}(x) + f_{33}(x))b + f_{34}(x)b^2 & f_{41}(x) + (f_{42}(x) + f_{43}(x))b + f_{44}(x)b^2
\end{pmatrix}.$$ 

In this part we first show that all diagonal components of $\nu^{-1}(C(x)\xi)$ are non-negative for all real numbers $b$, and all $x$ more than a sufficiently large number (which is independent on $b$). Since by the assumption every diagonal component of $\nu^{-1}(C(m)\xi))$, $m = 1, 2, \cdots$, is non-negative for all $b \in \mathbb{R}$, we have

$$f_{i4}(m) \geq 0, \ m = 1, 2, \cdots, i = 1, 4.$$ 

Suppose that $f_{i4}(x)$ is not identically 0. We then obtain from Lemma that $f_{i4}(x) > 0$ for all $x$ more than a sufficiently large number. Hence every diagonal component of $\nu^{-1}(C(x)\xi)$ is expressed by

$$f_{i4}(x) \left( b + \frac{f_{i2}(x) + f_{i3}(x)}{2f_{i4}(x)} \right)^2 + \frac{g_i(x)}{4f_{i4}(x)}.$$ 

Here

$$g_i(x) = 4f_{i4}(x)f_{i1}(x) - (f_{i2}(x) + f_{i3}(x))^2.$$ 

Similarly, since $g_i(m) \geq 0$ for $m = 1, 2, \cdots$, we obtain that $g_i(x) \geq 0$ for all $x$ more than a sufficiently large number. The above inequality is valid in the case where $f_{i4}(x)$ is identically 0. Indeed, if $f_{i4}(x)$ is identically 0, then $f_{i2}(m) + f_{i3}(m) = 0$ holds for $m \in \mathbb{N}$. For, if $f_{i02}(m_0) + f_{i03}(m_0) \neq 0$ for some $i_0$ and $m_0$, then the
infimum of a diagonal component of $\nu^{-1}(C(m_0)\xi)$ is $-\infty$. This contradicts the condition that $C(m) \geq O$ holds for all $m \in \mathbb{N}$. By Lemma, $f_{i2}(x) + f_{i3}(x)$ is identically 0. In this case, it suffices to consider the function $f_{i1}(x)$ in the same way.

Next, we examine the determinant of $\nu^{-1}(C(x)\xi)$. Put
\[
G = \det \nu^{-1}(C(x)\xi).
\]
Then $G$ is expressed as
\[
G = G(b, x) = a_0(x)b^4 + a_1(x)b^3 + \cdots + a_4(x).
\]
Here $a_i(x)$ is a finite linear combination of $x$-th powers of some positive numbers. By the assumption we have
\[
G(b, m) \geq 0, -\infty < b < \infty, m = 1, 2, \cdots.
\]
Suppose that $a_0(x)$ is not identically 0. Then $a_0(x) > 0$ for all $x$ for a sufficiently large number, since $a_0(x)$ satisfies the hypothesis in Lemma. Put
\[
L(x) = \inf_{b \in \mathbb{R}} G(b, x).
\]
Then $L(x)$ is given by the following formula:
\[
L(x) = \min_{1 \leq j \leq 3} L_j(x),
\]
where $L_j(x) = G(\text{Re } b_j(x), x)$ and $b_j(x)$ are all roots of the cubic equation $G_b(b, x) = 0$ of $b$. Note that $b_j(x)$ are algebraically expressed by $a_i(x)$, and are continuous for all $x$ more than a sufficiently large number. We must show the existence of a number $s > 0$ satisfying $L(x) \geq 0$ for all $x \in [s, \infty)$. Assume that there does not exist such a number $s$. Then for every natural number $m$ there exists $y_m$ with $y_m \geq m$ such that $L_{j_0}(y_m) < 0$ for some $j_0$. Since $L_{j_0}(m) \geq 0$ for all $m \in \mathbb{N}$, there exists by the intermediate value theorem a positive sequence $\{x_m\}$ with $\lim_{m \to \infty} x_m = \infty$ such that
\[
L_{j_0}(x_m) = 0, m = 1, 2, \cdots.
\]
Let $\hat{L}_{j_0}(x)$ be a polynomial of $a_i(x)$ such that the set of all zeros of $\hat{L}_{j_0}(x)$ includes the set of all zeros of $L_{j_0}(x)$. Hence there exists an unbounded increasing sequence of zeros of $\hat{L}_{j_0}(x)$. Since $\hat{L}_{j_0}(x)$ is a finite linear combination of $x$-th powers of some positive numbers, it follows from Lemma that $\hat{L}_{j_0}(x)$ is identically 0. This is a contradiction. On the other hand, if $a_0(x)$ is identically 0, then $a_1(x)$ is also identically 0 by the argument in the former part of Step (ii). In this case we also obtain the same result.

Similarly, we obtain the desired properties for $\eta$. This completes the proof.
The next remark shows that Theorem does not always hold for all positive number $x$. We shall give the example that there exists a $4 \times 4$ positive semi-definite matrix $A$ with $A \succeq O$ satisfying $A^x \not\preceq O$ for $x \in [0,1)$.

**Remark.** Note that $\nu(M_2(\mathbb{R})^+)$ is isometrically isomorphic to a circular cone

$$\mathcal{H}^+ = \left\{ \xi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid F(\xi) = x^2 + y^2 - z^2 \leq 0, z \geq 0 \right\}.$$

Consider the following positive semi-definite matrix $A$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Then $A \succeq O$. In fact, we have $A^\alpha \succeq O$ for all $\alpha \geq 1$. To see it, it suffices to examine that for $\eta(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$ we have $F(A^\alpha \eta(\theta)) \leq 0$ as follows:

$$6A^\alpha \eta(\theta) = \begin{pmatrix} 3^\alpha - 3 & 3^\alpha + 3 & 2 \cdot 3^\alpha \\ 3^\alpha - 3 & 3^\alpha + 3 & 2 \cdot 3^\alpha \\ 2 \cdot 3^\alpha & 2 \cdot 3^\alpha & 4 \cdot 3^\alpha \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} (3^\alpha + 3) \cos \theta + (3^\alpha - 3) \sin \theta + 2 \cdot 3^\alpha \\ (3^\alpha - 3) \cos \theta + (3^\alpha + 3) \sin \theta + 2 \cdot 3^\alpha \\ (2 \cdot 3^\alpha) \cos \theta + (2 \cdot 3^\alpha) \sin \theta + 4 \cdot 3^\alpha \end{pmatrix}$$

and so

$$F(6A^\alpha \eta(\theta)) = 18 - 10 \cdot 3^{2\alpha} - (4 \cdot 3^{2\alpha} + 36) \cos \theta \sin \theta - 8 \cdot 3^{2\alpha} (\cos \theta + \sin \theta)$$

$$= -(36 + 4 \cdot 3^{2\alpha})(\cos \theta + 1)(\sin \theta + 1) - (4 \cdot 3^{2\alpha} - 36)(2(\cos \theta + \sin \theta) + 3)$$

$$\leq 0$$

for $\alpha \geq 1$ and $0 \leq \theta \leq 2\pi$. On the other hand, we shall show that for every $\alpha \in [0,1)$ there exists a real number $\theta_0$ such that $F(6A^\alpha \eta(\theta_0)) > 0$. Indeed, one can choose $\theta_0$ satisfying

$$\cos \theta_0 + \sin \theta_0 = \sqrt{2} \sin \left( \theta_0 + \frac{\pi}{4} \right) = -\frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}},$$

since

$$0 < \frac{2 \cdot 3^{2\alpha}}{\sqrt{2}(9 + 3^{2\alpha})} \leq \frac{1}{\sqrt{2}}$$

for $0 \leq \alpha \leq 1$. Taking a square of both sides of the above equalities, we have

$$1 + 2 \cos \theta_0 \sin \theta_0 = \frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2}.$$
It follows that for $0 \leq \alpha < 1$

\[
F(6A^\alpha \eta(\theta_0)) = 18 - 10 \cdot 3^{2\alpha} - 2(9 + 3^{2\alpha}) \left( \frac{4 \cdot 3^{4\alpha}}{(9 + 3^{2\alpha})^2} - 1 \right) + 8 \cdot 3^{2\alpha} \cdot \frac{2 \cdot 3^{2\alpha}}{9 + 3^{2\alpha}}
\]

\[
= \frac{36}{9 + 3^{2\alpha}} (9 - 3^{2\alpha}) > 0,
\]

from which we have $A^\alpha \nsubseteq O$.

Finally, we obtain immediately the following theorem, which is understood to be a matrix version of Lemma, reviewing the proof of Theorem 5.1:

**Theorem 5.2.** Let $A_1, \cdots, A_n$ be positive semi-definite matricial representations of linear transformations on $\nu(M_2(\mathbb{R})_s)$ with a self-dual cone $\nu(M_2(\mathbb{R})^+)$, and $a_1, \cdots, a_n$ be real numbers. Suppose that

\[
a_1 A_1^m + \cdots + a_n A_n^m \succeq O
\]

holds for $m = 1, 2, \cdots$. Then there exists $s > 0$ such that

\[
a_1 A_1^s + \cdots + a_n A_n^s \succeq O
\]

for all $x \in [s, \infty)$.

**References**


