数理解析研究所講究録 1428 巻 2005 年 120-130

A deterministic-control-based approach to motion by curvature

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We analyze the continuum limit of a family of two-person games which reduces to motion by curvature, thus giving a new game-theoretic interpretation for this geometric evolution law.

Our deterministic-control-based approach to motion by curvature has precursors. A closely related idea was introduced 10 years ago as a numerical approximation scheme for motion by curvature [CDK, Gu]; more recently, similar approximation schemes have been developed for other geometric flows [C, Pa] and in higher dimensions [CL]. These authors' goals and results were however quite different from ours.

Our game-theoretic interpretation provides a new viewpoint on mean curvature flow, parallel to the theory of Hamilton-Jacobi equations. There are in fact two different ways of linking an optimal control problem with the associated Hamilton-Jacobi equation. One, known as a "verification argument," works best when the solution is smooth. The other, involving "viscosity solutions," is more general since it requires no smoothness. The analysis of our continuum limit can be done using either technique.

Our analysis combines two key ideas. One is the level-set approach to the analysis of motion by curvature and related geometric flows (see [ESp, CGG, Gi]). The other is the analysis of differential games via dynamic programming and Hamilton-Jacobi equations (see e.g. [ESo, Bardi, BC]). Ten years ago it seemed a happy accident that viscosity solutions – invented for applications in optimal control – were also the right tool for analyzing motion by curvature. Now we see that this is no accident; it is in fact quite natural, since motion by curvature can be viewed as an optimal control problem.

Ours is not the first control-theoretic characterization of motion by curvature. An interpretation involving stochastic control was developed in [BCQ, ST1]. There is a link between our deterministic viewpoint and the stochastic framework, however our approach is entirely deterministic.

1 The game

Let Ω be a bounded set in $\mathbb{R}^2$. There are two players, Paul and Carol. Paul starts at a point $x \in \Omega$, and his goal is to reach the boundary. Carol is trying to obstruct him. The rules of the game are simple. At each timestep:

1. Paul chooses a direction, i.e. a unit vector $v \in \mathbb{R}^2$ with $\|v\| = 1$.

2. Carol chooses whether to let Paul's choice stand or reverse it – i.e. she chooses $b = \pm 1$ and replaces v with bv.

3. Paul takes a step of size $\sqrt{2}\varepsilon$, moving from x to $x + \sqrt{2}\varepsilon bv$.

Here ε is a small parameter, fixed throughout the game, and we are interested in the continuum limit $\varepsilon \to 0$.

Can Paul reach the boundary? Yes indeed. The explanation – and the optimal strategy – are easily found using the method of dynamic programming. The key observation is that if Ω is a circle of radius R, then Paul can exit in a single step if his initial position satisfies $|x|^2 + 2\varepsilon^2 > R^2$, in other words if $|x| > R - \Delta R$ with $\Delta R \approx \varepsilon^2/R$. He has only to choose v *tangent* to the circle; Carol cannot stop him, since he exits whether she reverses him or not (see Figure 1a). For initial positions x lying farther from the boundary, we can find the minimum exit time – and the optimal strategies – by repeating this calculation as many times as necessary. For example, starting from the innermost circle shown in Figure 1b, Paul can exit in three steps. He should choose v at each step tangent to the circle on which he sits. (Notice that the optimal paths are not unique: Paul has two equally valid choices of direction at each timestep; moreover the one he actually uses is determined by Carol's whim.)

It is intuitively clear that for *any* convex domain in the plane, as $\varepsilon \to 0$, the sets from which Paul can exit in a fixed number of steps converge (after an appropriate scaling in time) to the trajectory of $\partial\Omega$ as it evolves under the curvature flow. The main goal of the present paper is to prove this statement and generalize it.

How many steps does Paul need to exit? A convex domain shrinks to a point under motion by curvature [GH, Gr]. Since the area changes at constant rate 2π, its disappearance time T is exactly $|\Omega|/2\pi$. Now, the point x_* to which $\partial\Omega$ shrinks is the location from which Paul needs the most steps to exit. Our results show that starting from x_* he needs approximately T/ε^2 steps to exit.

The preceding discussion – about Paul's exit time – is limited to convex domains. But the game can also be played in a nonconvex domain. In fact, the nonconvex case is interesting and different, because Paul can only exit at the convex part of $\partial\Omega$. We will focus on this topic in Section 3.1.

We first learned of this game from Joel Spencer, who introduced it in [Sp1] (Game 1). It is a variant of his "pusher-chooser" game (see [Sp2]), which is similar except the game is played in $\mathbb{R}^n$ and the number of steps is exactly equal to n. Paul's ability to exit is related to the question of discrepancy of two-color colorings of sets (see also [Sp3]).

2 The minimum exit time, for convex domains in the plane

Our analysis uses the method of dynamic programming. To explain the main ideas, we focus first on the minimum exit time problem, for a bounded convex domain in the plane. For any $\varepsilon > 0$, consider the *minimum exit time*

$$U^\varepsilon(x) = \begin{cases} \varepsilon^2 k \text{ if Paul needs } k \text{ steps to exit, starting} \\ \text{from } x \text{ and following an optimal strategy.} \end{cases} \tag{2.1}$$

Clearly U^ε satisfies the principle of dynamic programming

$$U^\varepsilon(x) = \min_{\|v\|=1} \max_{b=\pm 1} \{\varepsilon^2 + U^\varepsilon(x + \sqrt{2}\varepsilon b v)\}. \tag{2.2}$$

We shall show that as $\varepsilon \to 0$, U^ε converges to the solution of the PDE

$$\begin{cases} \Delta U - \langle D^2U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|}\rangle + 1 = 0 & \text{in } \Omega \\ U = 0 & \text{at } \partial\Omega. \end{cases} \tag{2.3}$$

This equation was first studied by Evans and Spruck in [ESp]. Its solution has the property that each level set $U = t$ is the image of $\partial\Omega$ under motion by curvature for time t. To see why, consider neighboring level sets $U = t$ and $U = t + \Delta t$. If the normal distance between them is Δx then $|\nabla U| \approx \Delta t/\Delta x$, while the curvature of the level set is $\kappa = -\text{div}(\nabla U/|\nabla U|)$. One verifies by elementary manipulation that the PDE (2.3) is equivalent to

$$|\nabla U|\text{div}\left(\frac{\nabla U}{|\nabla U|}\right) + 1 = 0$$

when $|\nabla U| \neq 0$. Thus the PDE says $\kappa = 1/|\nabla U| \approx \Delta x/\Delta t$, whence $\Delta x \approx \kappa\Delta t$ as asserted.

The analysis of (2.3) in [ESp] uses the framework of viscosity solutions. This is necessary because in its classical form the PDE (2.3) does not make sense where $|\nabla U| = 0$. However there is nothing wrong with the solution. Indeed, for a convex domain, $\partial\Omega$ remains smooth under motion by curvature, and it becomes asymptotically circular as it shrinks to a point [GH, Gr]; using this, we prove that U is C^3 in the entire domain.

The PDE (2.3) is, in essence, the Hamilton-Jacobi-Bellman equation associated with our exit-time problem. To explain, let us derive it heuristically, using the dynamic programming principle (2.2) and Taylor expansion. The former suggests that

$$U(x) \approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{\varepsilon^2 + U(x + \sqrt{2}\varepsilon b v)\right\}.$$

Expanding U we get

$$U(x) \approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{\varepsilon^2 + U(x) + \sqrt{2}\varepsilon b v \cdot \nabla U(x) + \varepsilon^2 \langle D^2U(x)v, v\rangle\right\},$$

which simplifies to

$$0 = 1 + \min_{\|v\|=1} \left\{ \frac{1}{\varepsilon}\sqrt{2}|v \cdot \nabla U(x)| + \langle D^2U(x)\, v, v\rangle \right\}. \tag{2.4}$$

As $\varepsilon \to 0$ the first term in the minimum requires $v \cdot \nabla U = 0$, in other words $v = \pm \nabla^\perp U / |\nabla U|$, and with this substitution (2.4) becomes

$$0 = 1 + \langle D^2U \frac{\nabla^\perp U}{|\nabla U|}, \frac{\nabla^\perp U}{|\nabla U|}\rangle. \tag{2.5}$$

Since we are in two space dimensions

$$\Delta U = \langle D^2U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|}\rangle + \langle D^2U \frac{\nabla^\perp U}{|\nabla U|}, \frac{\nabla^\perp U}{|\nabla U|}\rangle,$$

so (2.5) can be rewritten as

$$0 = 1 + \Delta U - \langle D^2U \frac{\nabla U}{|\nabla U|}, \frac{\nabla U}{|\nabla U|}\rangle,$$

which is precisely (2.3).

The preceding calculation, though formal, captures the essence of the matter. It shows, in particular, why the Hamilton-Jacobi-Bellman equation for this game is second rather than first order. The reason is that first-order Taylor expansion does not suffice to characterize U; rather, it tells us only that Paul should choose $v \perp \nabla U$. We need the second-order terms in the Taylor expansion to know how effective this strategy is. Notice that while Carol does not prevent Paul from reaching the boundary, she certainly slows him down. Indeed, Paul's local velocity (step size per time step) is $\sqrt{2}\varepsilon$, but his macroscopic velocity (distance travelled divided by number of time steps) is of order ε^2.

In optimal control, the Hamilton-Jacobi-Bellman equation can be used in two rather different ways. One, known as a "verification argument," uses a solution of the PDE to bound the minimum exit time. The other characterizes the value function of the optimal control problem as the unique viscosity solution of the PDE. The two approaches are complementary, and both are useful for the problem at hand. The viscosity-solution framework is extremely robust, since it requires no information about the PDE solution U. When U is smooth enough however the verification argument gives stronger result, namely convergence *with a rate*:

Theorem 1 *Let Ω be a smoothly bounded strictly convex domain in the plane, and let $U(x)$ be the time $\partial\Omega$ arrives at x as it shrinks under motion by curvature, i.e. the solution of (2.3). For $\varepsilon > 0$, let $U^\varepsilon(x)$ be Paul's scaled minimum exit time, defined by (2.1). Then there exists a constant C such that for all $x \in \Omega$*

$$\|U^\varepsilon(x) - U(x)\|_{L^\infty(\overline{\Omega})} \le C\varepsilon.$$

Moreover C depends only on the C^3 norm of U.

3 Motion by curvature

The curvature flow of $\partial\Omega$ is well-defined even if Ω is not convex. So it should have a game-theoretic interpretation that does not require convexity.

The idea is simple: Paul and Carol play the same game as before, but Paul's goal is different. He has an "objective function" u_0 and a "maturity time" T, and his goal is to optimize the value of the objective at maturity. More precisely: his goal is

$$\min u_0(y(T)), \tag{3.1}$$

where $y(s)$ is his piecewise linear path – determined by his choices and Carol's – starting from position x at time t. (His stepsize is $\sqrt{2}\varepsilon$ as before, and each step takes time ε^2.)

This is closely related to our previous discussion. The level sets of u_0 form a nested family of domains in the plane, and Paul's goal is to reach the outermost domain possible by time T. This is different from exiting a specific domain in minimum time – but not very different.

To explain our analysis heuristically, consider Paul's *value function*

$$u^\varepsilon(x,t) = \text{minimal value of } u_0(y(T)), \text{ starting from } x \text{ at time } t. \tag{3.2}$$

It satisfies the dynamic programming principle

$$u^\varepsilon(x,t) = \min_{\|v\|=1} \max_{b=\pm 1} u^\varepsilon(x+\sqrt{2}\varepsilon bv, t+\varepsilon^2). \tag{3.3}$$

We shall show that as $\varepsilon \to 0$, u^ε converges to the solution of

$$\begin{cases} u_t + \Delta u - \langle D^2u\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle = 0 & \text{for } t < T \\ u = u_0 & \text{at } t = T. \end{cases} \tag{3.4}$$

This PDE is familiar from the level-set approach to interface motion. With the time change $\tau = T - t$ it becomes

$$u_\tau - \Delta u + \langle D^2u\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\rangle = 0$$

for $\tau > 0$, with $u = u_0$ at $\tau = 0$. This is equivalent to

$$\frac{u_\tau}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

so the PDE says that each level set of u moves with normal velocity equal to its curvature. In the original time variable t the statement is this: as t decreases from T, each level set $u = c$ describes the curvature flow of the corresponding level set of u_0.

To link the game and the PDE, we argue as in the previous subsection. The dynamic programming principle (3.3) suggests that

$$\begin{aligned} u(x,t) &\approx \min_{\|v\|=1} \max_{b=\pm 1} u(x+\sqrt{2}\varepsilon bv, t+\varepsilon^2) \\ &\approx \min_{\|v\|=1} \max_{b=\pm 1} \left\{u(x,t) + \sqrt{2}\varepsilon bv \cdot \nabla u + \varepsilon^2(u_t + \langle D^2u\, v, v\rangle)\right\} \end{aligned}$$

using Taylor expansion in the second step. This simplifies to

$$0 = \min_{\|v\|=1} \left\{ \frac{1}{\varepsilon}\sqrt{2}|v \cdot \nabla u| + u_t + \langle D^2 u\, v, v\rangle \right\}. \tag{3.5}$$

As $\varepsilon \to 0$ the first term in the minimum requires $v \cdot \nabla u = 0$. Since we are in 2D the remaining terms give precisely (3.4).

Theorem 2 *Consider the game described above, with a continuous "objective function" $u_0 : \mathbb{R}^2 \to \mathbb{R}$ that is constant outside a compact set. Let $u^\varepsilon(x,t)$ be the associated value functions, defined by (3.2). Then the functions u^ε converge as $\varepsilon \to 0$, uniformly on compact sets, to the unique viscosity solution of (3.4).*

3.1 The minimum exit time, for nonconvex domains in the plane

Let's return now to the minimum exit time problem. What happens when the domain Ω is nonconvex? Qualitatively, the situation is pretty clear. Paul can still exit – but only from the convex part of $\partial\Omega$. If he starts near the concave part of the boundary he needs many steps to exit, because the convex part of the boundary is far away. But still the scaled minimum exit time U^ε, defined by (2.1), is bounded independently of ε.

As in the convex case, our goal is to characterize the limit of U^ε as $\varepsilon \to 0$. In fact we offer two distinct characterizations:

(a) it is the unique viscosity solution of the boundary value problem (2.3), interpreting the boundary condition in the viscosity sense; and

(b) its level sets trace the evolution of $\partial\Omega$ under the "positive curvature flow," i.e. the evolution with normal velocity $\kappa_+ = \max\{\kappa, 0\}$.

Our proof of (a) requires Ω to be star-shaped.

The first characterization is more or less expected. It is directly analogous to the situation for the first-order Hamilton-Jacobi equations associated with pursuit-evasion games [BC]. In that context, as in ours, the Dirichlet boundary condition should be imposed only on the part of the boundary from which Paul can exit. By interpreting the boundary condition in the "viscosity sense" we assure that it is only imposed on the appropriate part of the boundary.

The proper statement of assertion (a) requires a bit of care, because $\lim_{\varepsilon\to 0} U^\varepsilon$ may not exist, and the relevant viscosity solution can be discontinuous. Therefore it is natural to consider

$$\overline{U} = \limsup{}^* U^\varepsilon \quad \text{i.e. } \overline{U}(x) = \limsup_{y\to x,\, \varepsilon\to 0} U^\varepsilon(y) \tag{3.6}$$

$$\underline{U} = \liminf{}^* U^\varepsilon \quad \text{i.e. } \underline{U}(x) = \liminf_{y\to x,\, \varepsilon\to 0} U^\varepsilon(x). \tag{3.7}$$

We prove:

Theorem 3 *Let Ω be a bounded domain in the plane, possibly nonconvex. Let $U^\varepsilon(x)$ be Paul's minimum exit time, defined by (2.1). Then $\overline{U}$, defined by (3.6), is a viscosity subsolution of (2.3); similarly $\underline{U}$, defined by (3.7), is a viscosity supersolution of (2.3).*

Usually a convergence theorem is proved by combining a statement like Theorem 3 with a suitable comparison result. Unfortunately, very little is known about comparison theorems for viscosity solutions of second-order elliptic equations like (2.3) in nonconvex domains. So rather than apply a general comparison result, we must prove one from scratch. Barles and Da Lio showed:

Theorem 4 (Barles and Da Lio) *Assume Ω is a bounded, star-shaped domain in $\mathbb{R}^n$. Let u be a viscosity subsolution of (2.3), and let v be a viscosity supersolution. Then $u_* \leq v$ and $u \leq v^*$, where*

$$u_*(x) = \liminf_{y\to x} u(y), \quad v^*(x) = \limsup_{y\to x} v(y).$$

Taken together, Theorems 3 and 4 show that when Ω is star-shaped, $\overline{U}_* = \underline{U}$ and $\overline{U} = \underline{U}^*$. This characterizes the limiting behavior as the unique-up-to-envelope (possibly discontinuous) viscosity solution of (2.3).

We turn now to the second characterization of the limit. To explain the relevance of the positive curvature flow, consider the modified game in which Paul is no longer required to choose a unit vector – instead, he can choose any v such that $\|v\| \leq 1$. For the exit-time problem, we prove that Theorem 3 also holds for the modified game. Thus the $\|v\| = 1$ and $\|v\| \leq 1$ versions of the exit-time problem are equivalent: they give the *same* arrival times, at least if Ω is star-shaped. However the time-dependent versions of the two games – with "objective function" u_0, to be minimized at time T – are different. Let us see formally how. Repeating the discussion of Section 3 for $\|v\| \leq 1$ version of the game, we find that the time-dependent Hamilton-Jacobi-Bellman equation is

$$0 = \min_{\|v\|\leq 1} \left\{ \frac{1}{\varepsilon}\sqrt{2}|v \cdot \nabla u| + u_t + \langle D^2 u\, v, v\rangle \right\}.$$

rather than (3.5). As $\varepsilon \to 0$ the first term forces $v \perp \nabla u$. Since

$$\min_{\|v\|\leq 1, v\cdot\nabla u = 0} \langle D^2u(x)v, v\rangle = \left(\langle D^2u(x)\frac{\nabla^\perp u(x)}{|\nabla u(x)|}, \frac{\nabla^\perp u(x)}{|\nabla u(x)|}\rangle \right)_-, \tag{3.8}$$

using the notation $x_- = \min\{x, 0\}$, the associated PDE is

$$u_t + \left(\langle D^2 u, \frac{\nabla^\perp u}{|\nabla u|}, \frac{\nabla^\perp u}{|\nabla u|}\rangle \right)_- = 0.$$

This is equivalent (in 2D) to

$$u_t + \left(|\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \right)_- = 0. \tag{3.9}$$

Since $\kappa = curv(u) = -\text{div}(\nabla u/|\nabla u|)$ is the curvature of a level set of u, the level sets of u solve the "positive curvature flow" backward in time, in other words they flow with

$$\text{normal velocity} = \begin{cases} \kappa & \text{where } \kappa \geq 0, \text{ i.e. the curve is convex} \\ 0 & \text{where } \kappa \leq 0, \text{ i.e. the curve is concave.} \end{cases}$$

The existence and uniqueness of this "positive curvature flow" follows from the general framework of [CGG].

The preceding discussion was formal, but its conclusion is correct:

Theorem 5 *Consider the modified game where Paul's choices are restricted by $\|v\| \leq 1$ rather than $\|v\| = 1$. Assume the "objective function" $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and constant outside a compact set. Let $u^\varepsilon(x,t)$ be the associated value functions, defined by (3.2). Then the functions u^ε converge as $\varepsilon \to 0$, uniformly on compact sets, to the unique viscosity solution of (3.9) satisfying $u = u_0$ at $t = T$.*

The point of course is that the minimum-exit-time problem and the positive curvature flow are related. Indeed, we shall show that in the limit $\varepsilon \to 0$, the level sets of Paul's minimum exit time are precisely the images of $\partial\Omega$ as it evolves under the positive curvature flow. The proof uses the underlying game: we show, in essence, that the associated control problems have the same optimal strategy:

Theorem 6 *Let Ω be a bounded domain in the plane (possibly nonconvex), and let $U^\varepsilon(x)$ be Paul's minimum exit time, defined using the modified game in which $\|v\| \leq 1$ is permitted. Let $u^\varepsilon(x,t)$ be the value function for the time-dependent version of the modified game, with objective function u_0 and maturity T, and recall from Theorem 5 that the level sets of $u = \lim_{\varepsilon \to 0} u^\varepsilon$ execute the positive-curvature-flow backward in time. Finally, suppose $\Omega = \{u_0 > 0\}$. Then we have*

$$u(x, T - \overline{U}(x)) = u(x, T - \underline{U}(x)) = 0 \tag{3.10}$$

where $\overline{U}$ and $\underline{U}$ are defined in (3.6)–(3.7).

Not much is known about the positive curvature flow. Is the solution of (3.9) strictly monotone in time, in other words

$$\text{is } u_t > 0 \text{ for } x \in \Omega? \tag{3.11}$$

If so, then (3.10) would show $\overline{U} = \underline{U}$, proving convergence of U^ε and continuity of the limit without making use of a comparison theorem.

Speculating further: let $\partial\Omega^+$ and $\partial\Omega^-$ be the strictly convex and concave parts of $\partial\Omega$ respectively. We believe that

$$U(x) \to 0 \text{ as } x \to x_0 \in \partial\Omega^+ \tag{3.12}$$

$$\liminf U(x) > 0 \text{ as } x \to x_0 \in \partial\Omega^-. \tag{3.13}$$

This corresponds to a waiting-time result for positive curvature flow, i.e. the conjecture that the concave part of the boundary sits still for some nonzero time before it begins to move.

We are not yet quite done. Our assertion (b) was that the exit times of the *original* game, with $\|v\| = 1$, have level sets given by the positive curvature flow. But Theorem 6 links the positive curvature flow to the exit times of the *modified* game, with $\|v\| \leq 1$. To close the loop, we shall show that the two games' exit times yield viscosity sub and supersolutions of the *same* elliptic boundary value problem:

Theorem 7 *Let Ω be a bounded domain in the plane, possibly nonconvex. Let $U^\varepsilon(x)$ be Paul's minimum exit time for the $\|v\| \leq 1$ game. Then $\overline{U}$, defined by (3.6), is a viscosity subsolution of (2.3), and $\underline{U}$, defined by (3.7), is a viscosity supersolution of (2.3).*

This theorem closes the loop, provided we have uniqueness for viscosity solutions of (2.3). Such uniqueness is valid for star-shaped domains, as a consequence of Theorem 4.

4 Generalizations

We give analogue interpretations of curvature flows in higher dimensions and of flow by a function of the curvature, by defining respectively appropriate higher-dimensional versions of the game and appropriate two-dimensional modifications of it.

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