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Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

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0 Introduction.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over the field $\mathbb{Q}$ of rational numbers, and let P be an integral weight lattice of $\mathfrak{g}$. In [L1] and [L2], Littelmann introduced the path model consisting of Lakshmibai-Seshadri paths (LS paths for short) for a representation of the symmetrizable Kac-Moody algebra $\mathfrak{g}$; for an integral weight $\lambda \in P$, an LS path of shape λ is, by definition, a path $\pi : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ (i.e., piecewise linear, continuous maps such that $\pi(0) = 0$ and $\pi(1) \in P$) determined by a pair of a sequence of elements in $W\lambda$, where W is the Weyl group of $\mathfrak{g}$, and a sequence of rational numbers satisfying a certain combinatorial condition (see §1.2 below). We denote by $\mathbb{B}(\lambda)$ the set of all LS paths of shape λ. Littelmann showed that the set $\mathbb{B}(\lambda)$ together with root operators (see §1.3 below) and the weight map $\mathrm{wt}(\pi) := \pi(1)$, $\pi \in \mathbb{B}(\lambda)$, is a crystal with weight lattice P. Then he proved that if $\lambda \in P$ is a dominant integral weight, then the crystal graph of the crystal $\mathbb{B}(\lambda)$ is connected, and the formal sum $\sum_{\pi \in \mathbb{B}(\lambda)} e(\pi(1))$ is equal to the character $\mathrm{ch}\, L(\lambda)$ of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ. Moreover, it was proved independently by Kashiwara [Kas3] and Joseph [J] that the $\mathbb{B}(\lambda)$ for dominant λ is, as a crystal, isomorphic to the crystal base of the highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of highest weight λ, where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q. Now, quite a natural question arises: Is there any $U_q(\mathfrak{g})$-module whose crystal base is isomorphic to the crystal $\mathbb{B}(\lambda)$ for general $\lambda \in P$? In a series of papers [NS1] $\sim$ [NS3], we gave a kind of answer to this question in the case where $\mathfrak{g}$ is an affine Lie algebra.

For a more precise description, we need some notation. Let $\mathfrak{g}$ be an affine Lie algebra over $\mathbb{Q}$ with Cartan subalgebra $\mathfrak{h}$, simple roots $\{\alpha_j\}_{j\in I} \subset \mathfrak{h}^*$, simple coroots $\{h_j\}_{j\in I} \subset \mathfrak{h}$, and Weyl group $W = \langle r_j \mid j \in I\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$, where r_j, $j \in I$, are the simple reflections. We denote by $\delta = \sum_{j\in I} a_j\alpha_j \in \mathfrak{h}^*$ the null root, and by $c = \sum_{j\in I} a_j^\vee h_j \in \mathfrak{h}$ the canonical central element. An integral weight $\lambda \in P$ is said to be of positive (resp., negative) level if $\lambda(c) > 0$ (resp., $\lambda(c) < 0$), and to be of level zero if $\lambda(c) = 0$:

$$P = \underbrace{\{\lambda \in P \mid \lambda(c) > 0\}}_{\text{positive level}} \sqcup \underbrace{\{\lambda \in P \mid \lambda(c) = 0\}}_{\text{level-zero}} \sqcup \underbrace{\{\lambda \in P \mid \lambda(c) < 0\}}_{\text{negative level}}.$$

If $\lambda \in P$ is of positive (resp., negative) level, then there exists a unique dominant (resp., anti-dominant) integral weight in $W\lambda$. Denote it by μ. Because $\mathbb{B}(\lambda) = \mathbb{B}(w\lambda)$ for all $w \in W$, we have that the set $\mathbb{B}(\lambda)$ is the same as the set $\mathbb{B}(\mu)$ of all LS paths of shape μ; accordingly, it follows from the result due to Kashiwara [Kas3] and Joseph [J] that $\mathbb{B}(\lambda)$ is, as a crystal, isomorphic to the crystal base of the highest (resp., lowest) weight module $V(\mu)$ of highest (resp., lowest) weight μ over the quantum affine algebra $U_q(\mathfrak{g})$.

Now we are left with the case where $\lambda \in P$ is of level zero. We take (and fix) a special vertex $0 \in I$ such that $a_0^\vee = 1$, and set $I_0 := I \setminus \{0\}$. Let Λ_i, $i \in I$, be the fundamental weights for $\mathfrak{g}$, and set $\varpi_i := \Lambda_i - a_i^\vee \Lambda_0$ for $i \in I_0$ (note that ϖ_i, $i \in I$, is a level-zero integral weight). In the case where $\lambda = m\varpi_i$ for some $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we proved in [NS1] and [NS2] that the LS path crystal is isomorphic to the crystal base of the extremal weight module over $U_q(\mathfrak{g})$ (Theorem 1). Here the extremal weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ with λ as an extremal weight is an integrable module over $U_q(\mathfrak{g})$ generated by a single element v_λ with the defining relations that the v_λ is an extremal weight vector of weight λ (see §1.4 below); we know from [Kas1, Proposition 8.2.2] that the extremal weight module $V(\lambda)$ admits a crystal base, denoted by $\mathcal{B}(\lambda)$.

Theorem 1. *For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, the crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal, isomorphic to the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ with $m\varpi_i$ as an extremal weight.*

We know from [NS1, Remark 5.2] and [NS3, §3.1] that for a general integral weight $\lambda \in P$ of level zero, there is no isomorphism of crystals between the set

$\mathbb{B}(\lambda)$ of all LS paths of shape λ and the crystal base $\mathcal{B}(\lambda)$ of the extremal weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of extremal weight λ. We do not know whether or not there exists a $U_q(\mathfrak{g})$-module having a crystal base isomorphic to $\mathbb{B}(\lambda)$, except for the case mentioned in Theorem 1.

Now we turn to a fundamental module of level zero (see §1.5 below). Let $\mathrm{cl}: \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/\mathbb{Q}\delta$ be the canonical projection. Denote by $U_q'(\mathfrak{g})$ the quantized universal enveloping algebra with $P_{\mathrm{cl}} := \mathrm{cl}(P)$ the integral weight lattice. In [Kas4, §5.2], Kashiwara introduced a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module $W(\varpi_i)$, called a fundamental module of level zero, and proved that it has a global basis with a simple crystal (see [Kas4, Theorem 5.17]). The fundamental module $W(\varpi_i)$ of level zero seems to be isomorphic to the Kirillov-Reshetikhin module $W_1^{(i)}$ in the notation of [HKOTT, §2.3] for $i \in I_0$ (see [HKOTT, Remark 2.3]). In [NS1] and [NS2], we gave a path model for $W(\varpi_i) \cong W_1^{(i)}$ as follows. Let $\lambda \in P$ be a level-zero integral weight. For an LS path $\pi \in \mathbb{B}(\lambda)$ of shape λ, we define a path $\mathrm{cl}(\pi) : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}$ by: $(\mathrm{cl}(\pi))(t) = \mathrm{cl}(\pi(t))$ for $t \in [0,1]$, and set $\mathbb{B}(\lambda)_{\mathrm{cl}} := \mathrm{cl}(\mathbb{B}(\lambda))$. Then the set $\mathbb{B}(\lambda)_{\mathrm{cl}}$ has a crystal structure with weight lattice P_{cl}, which is naturally induced from that of $\mathbb{B}(\lambda)$.

Theorem 2. *The crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic to the crystal base of the fundamental module $W(\varpi_i)$ of level zero.*

In [NS3], we studied the crystal structure of $\mathbb{B}(\lambda)_{\mathrm{cl}} = \mathrm{cl}(\mathbb{B}(\lambda))$ for a general integral weight $\lambda \in P$ of level zero. Before stating our main result of [NS3], we make some comments. If $\lambda' = \lambda + R\delta$ for some $R \in \mathbb{Q}$, then it follows from the definition of LS paths that $\mathbb{B}(\lambda') = \{\pi + \pi_{R\delta} \mid \pi \in \mathbb{B}(\lambda)\}$, where $(\pi + \pi_{R\delta})(t) := \pi(t) + tR\delta$, $t \in [0,1]$, and from the definition of the root operators that the crystal graph of $\mathbb{B}(\lambda + R\delta)$ is the same shape as that of $\mathbb{B}(\lambda)$, up to $R\delta$-shift of weight. In addition, we have that $\mathbb{B}(\lambda) = \mathbb{B}(w\lambda)$ for all $w \in W$. Therefore we may assume that the $\lambda \in P$ is of the form $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ from the beginning. Now we are ready to state our main result in [NS3].

Theorem 3. *Let $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, there exists a unique isomorphism $\mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \bigotimes_{i\in I_0} (\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m_i}$ of crystals (with weight lattice P_{cl}) between the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ and the tensor product $\bigotimes_{i\in I_0} (\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m_i}$.*

By combining Theorems 2 and 3, we can get the following corollary.

Corollary. *Let $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is, as a crystal with weight lattice P_{cl}, isomorphic to the crystal base of the tensor product $U_q'(\mathfrak{g})$-module $\bigotimes_{i\in_0} W(\varpi_i)^{\otimes m_i}$.*

1 Preliminaries.

1.1 Affine Lie algebras and quantum affine algebras. Let $\mathfrak{g}$ be an affine Lie algebra over the field $\mathbb{Q}$ of rational numbers with Cartan subalgebra $\mathfrak{h}$. Denote by $\Pi := \{\alpha_j\}_{j\in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$ the set of simple roots, and by $\Pi^\vee := \{h_j\}_{j\in I} \subset \mathfrak{h}$ the set of simple coroots, where $I = \{0, 1, 2, \ldots, \ell\}$ is an index set for the simple roots Π. Throughout this article, we use the numbering of the simple roots as in [Kac, §4.8 and §6]. Let $\delta \in \mathfrak{h}^*$ and

$$c = \sum_{j\in I} a_j^\vee h_j \in \mathfrak{h} \tag{1.1.1}$$

be the null root and the canonical central element of $\mathfrak{g}$, respectively. Denote by $W = \langle r_j \mid j \in I\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of the affine Lie algebra $\mathfrak{g}$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ is the simple reflection in α_j for $j \in I$. We call an element of the set $\Delta^{\mathrm{re}} := W\Pi$ a real root, and denote by Δ_+^{re} the set of positive real roots. Let Λ_j, $j \in I$, be the fundamental weights for the affine Lie algebra $\mathfrak{g}$. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ that contains all the simple roots α_j, $j \in I$, and fundamental weights Λ_j, $j \in I$. For each $i \in I_0 := I \setminus \{0\}$, we define a level-zero fundamental weight $\varpi_i \in P$ by

$$\varpi_i := \Lambda_i - a_i^\vee \Lambda_0. \tag{1.1.2}$$

Note that $\varpi_i(c) = 0$; an integral weight $\lambda \in P$ is said to be level-zero if $\lambda(c) = 0$. An integral weight $\lambda \in P$ of level zero is said to be dominant if $\lambda(h_i) \geq 0$ for all $i \in I_0$. Let

$$\mathrm{cl} : \mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/\mathbb{Q}\delta \tag{1.1.3}$$

be the canonical projection, and set $P_{\mathrm{cl}} := \mathrm{cl}(P)$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra (with weight lattice P) of the affine Lie algebra $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q. We denote by E_j, F_j, $j \in I$, and q^h, $h \in P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ the Chevalley generators of $U_q(\mathfrak{g})$, where E_j (resp., F_j) corresponds to the simple root α_j (resp., $-\alpha_j$). Denote by $U_q'(\mathfrak{g})$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by E_j, F_j, $j \in I$, and q^h, $h \in (P_{\mathrm{cl}})^\vee := \mathrm{Hom}_{\mathbb{Z}}(P_{\mathrm{cl}}, \mathbb{Z})$, which is the quantized universal enveloping algebra of $\mathfrak{g}$ with weight lattice P_{cl}.

1.2 Lakshmibai–Seshadri paths. A path (with weight in P) is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ from $[0,1] := \{t \in \mathbb{Q} \mid 0 \le t \le 1\}$ to $\mathbb{Q} \otimes_{\mathbb{Z}} P$ such that $\pi(0) = 0$ and $\pi(1) \in P$. In this subsection, we recall the definition of a Lakshmibai–Seshadri path (an LS path for short) from [L2, §4] (see also [NS2, §1.4] and [NS3, §2.1]).

We first recall some auxiliary notations. Let $\lambda \in P$ be an integral weight. For $\mu, \nu \in W\lambda$, we write $\mu \ge \nu$ if there exist a sequence $\mu = \xi_0, \xi_1, \dots, \xi_n = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \dots, \beta_n \in \Delta_+^{\mathrm{re}}$ of positive real roots such that $\xi_k = r_{\beta_k}(\xi_{k-1})$ and $\xi_{k-1}(\beta_k^\vee) < 0$ for $k = 1, 2, \dots, n$, where for a positive real root $\beta \in \Delta_+^{\mathrm{re}}$, r_β denotes the reflection with respect to β, and $\beta^\vee$ denotes the dual real root of β. If $\mu \ge \nu$, then we define $\mathrm{dist}(\mu, \nu)$ to be the maximal length n of all possible such sequences $\xi_0, \xi_1, \dots, \xi_n$ for the pair (μ, ν). Then, for $\mu, \nu \in W\lambda$ with $\mu > \nu$ and a rational number $0 < a < 1$, an a-chain for (μ, ν) is, by definition, a sequence $\mu = \xi_0 > \xi_1 > \cdots > \xi_n = \nu$ of elements in $W\lambda$ such that $\mathrm{dist}(\xi_{k-1}, \xi_k) = 1$ and $a\xi_{k-1}(\beta_k^\vee) \in \mathbb{Z}_{<0}$ for all $k = 1, 2, \dots, n$, where β_k is the positive real root corresponding to (ξ_{k-1}, ξ_k) with $\xi_{k-1} > \xi_k$.

Now we are ready for the definition of an LS path. Let $\lambda \in P$ be an integral weight. An LS path of shape λ is a path $\pi : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ associated to a pair $(\underline{\nu}; \underline{a})$ of a sequence $\underline{\nu} : \nu_1, \nu_2, \dots, \nu_s$ of elements in $W\lambda$ and a sequence $\underline{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists an a_k-chain for (ν_k, ν_{k+1}) for all $k = 1, 2, \dots, s-1$; to such a pair $(\underline{\nu}; \underline{a}) = (\nu_1, \nu_2, \dots, \nu_s; a_0, a_1, \dots, a_s)$, we associate the following path $\pi : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$:

$$\pi(t) = \sum_{l=1}^{k-1} (a_l - a_{l-1})\nu_l + (t - a_{k-1})\nu_k \quad \text{for} \quad a_{k-1} \le t \le a_k, \ 1 \le k \le s.$$

Note that $\pi(0)$ is obviously equal to $0 \in P$, and it follows from [L2, Lemma 4.5 a)] that $\pi(1) \in P$; namely, the π above is, in fact, a path for all such pairs $(\underline{\nu}\,;\underline{a}) = (\nu_1, \nu_2, \ldots, \nu_s\,; a_0, a_1, \ldots, a_s)$. Denote by $\mathbb{B}(\lambda)$ the set of LS paths of shape λ.

Remark 1.2.1. (1) The straight line $\pi_\nu(t) := t\nu$, $t \in [0,1]$, is contained in $\mathbb{B}(\lambda)$ for all $\nu \in W\lambda$ (put $s = 1$ and $\nu_1 = \nu$).

(2) It follows from the definition that $\mathbb{B}(w\lambda) = \mathbb{B}(\lambda)$ for all $w \in W$.

1.3 Root operators. In this subsection, we give a description of root operators e_j and f_j, $j \in I$, which was introduced in [L2, §1], on the set $\mathbb{B}(\lambda)$ of all LS paths of shape $\lambda \in P$ (see also [NS2, §1.2] and [NS4, §2.1]).

Let $\lambda \in P$ be an integral weight. For an LS path $\pi \in \mathbb{B}(\lambda)$ and $j \in I$, we define $e_j\pi$ as follows: First, we set

$$\begin{aligned} H_j^\pi(t) &:= (\pi(t))(h_j) \quad \text{for } t \in [0,1], \\ m_j^\pi &:= \min\{H_j^\pi(t) \mid t \in [0,1]\}. \end{aligned} \tag{1.3.1}$$

If $m_j^\pi > -1$, then we define $e_j\pi := \theta$. Here, θ is an extra element, which corresponds to the 0 in the theory of crystals (by convention, we put $e_j\theta = f_j\theta := \theta$). If $m_j^\pi \le -1$, then

$$(e_j\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \le t \le t_1, \\ \pi(t) + \alpha_j & \text{if } t_1 \le t \le 1, \end{cases} \tag{1.3.2}$$

where we set

$$\begin{aligned} t_1 &:= \min\{t \in [0,1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_0 &:= \max\{t' \in [0,t_1] \mid H_j^\pi(t) \ge m_j^\pi + 1 \text{ for all } t \in [0,t']\}. \end{aligned}$$

Similarly, $f_j\pi$ is given as follows: If $H_j^\pi(1) - m_j^\pi < 1$, then we set $f_j\pi := \theta$. If $H_j^\pi(1) - m_j^\pi \ge 1$, then

$$(f_j\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0, \\ \pi(t_0) + r_j(\pi(t) - \pi(t_0)) & \text{if } t_0 \le t \le t_1, \\ \pi(t) - \alpha_j & \text{if } t_1 \le t \le 1, \end{cases} \tag{1.3.3}$$

where we set

$$\begin{aligned} t_0 &:= \max\{t \in [0,1] \mid H_j^\pi(t) = m_j^\pi\}, \\ t_1 &:= \min\{t' \in [t_0,1] \mid H_j^\pi(t) \ge m_j^\pi + 1 \text{ for all } t \in [t',1]\}. \end{aligned}$$

Theorem 1.3.1 ([L2]). *For every integral weight $\lambda \in P$, the set $\mathbb{B}(\lambda) \cup \{\theta\}$ is stable under the action of the root operators e_j and f_j for $j \in I$. We define*

$$\begin{cases} \mathrm{wt}(\pi) := \pi(1) & \text{for } \pi \in \mathbb{B}(\lambda), \\ \varepsilon_j(\pi) := \max\{n \geq 0 \mid e_j^n \pi \neq \theta\} & \text{for } \pi \in \mathbb{B}(\lambda) \text{ and } j \in I, \\ \varphi_j(\pi) := \max\{n \geq 0 \mid f_j^n \pi \neq \theta\} & \text{for } \pi \in \mathbb{B}(\lambda) \text{ and } j \in I. \end{cases}$$

Then, the set $\mathbb{B}(\lambda)$ together with the root operators and the maps above is a crystal with weight lattice P.

1.4 Extremal weight modules.

Definition 1.4.1 (cf. [Kas1, §8] and [Kas4, §3.1]). Let M be an integrable $U_q(\mathfrak{g})$-module. A vector $v \in M$ of weight $\lambda \in P$ is said to be extremal, if there exists a family $\{v_w\}_{w \in W}$ of weight vectors of M satisfying the following conditions: for $w \in W$ and $j \in I$,

a) $v_w = v$ if $w = 1$;

b) if $n := (w(\lambda))(h_j) \geq 0$, then $E_j v_w = 0$ and $F_j^{(n)} v_w = v_{r_j w}$;

c) if $n := (w(\lambda))(h_j) \leq 0$, then $F_j v_w = 0$ and $E_j^{(-n)} v_w = v_{r_j w}$.

Here, $E_j^{(n)}$ and $F_j^{(n)}$ are the n-th q-divided powers of the Chevalley generators E_j and F_j of $U_q(\mathfrak{g})$, respectively.

Definition 1.4.2 (cf. [Kas1, §8] and [Kas4, §3.1]). Let $\lambda \in P$ be an integral weight. The extremal weight module $V(\lambda)$ over $U_q(\mathfrak{g})$ with λ as an extremal weight is, by definition, the integrable $U_q(\mathfrak{g})$-module generated by a single elemenet v_λ with the defining relations that v_λ is an extremal vector of weight λ.

We know the following theorem from [Kas1, Proposition 8.2.2].

Theorem 1.4.3. *For every $\lambda \in P$, the extremal weight module $V(\lambda)$ has a crystal base, which we denote by $\mathcal{B}(\lambda)$.*

Remark 1.4.4. The extremal weight module is a natural generalization of an integrable highest and lowest weight module; in fact, we know from [Kas1, §8] that if $\lambda \in P$ is dominant (resp. anti-dominant), then the extremal weight module $V(\lambda)$ is isomorphic to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight λ, and the crystal base $\mathcal{B}(\lambda)$ of $V(\lambda)$ is isomorphic to the crystal base of the integrable highest (resp., lowest) weight module as a crystal.

1.5 Fundamental module of level zero. We define a positive integer $d_i \in \mathbb{Z}_{\geq 1}$ by

$$\{n \in \mathbb{Z} \mid \varpi_i + n\delta \in W\varpi_i\} = \mathbb{Z}d_i. \tag{1.5.1}$$

Because $V(\varpi_i) \cong V(w\varpi_i)$ as $U_q(\mathfrak{g})$-modules for all $w \in W$ (see [Kas1, Proposition 8.2.2 iv)]), we see that there exists a $U_q(\mathfrak{g})$-module isomorphism $V(\varpi_i + d_i\delta) \xrightarrow{\sim} V(\varpi_i)$. In addition, there exists a $U_q'(\mathfrak{g})$-module isomorphism $V(\varpi_i) \xrightarrow{\sim} V(\varpi_i + d_i\delta)$, which maps the ϖ_i-weight space $V(\varpi_i)_{\varpi_i}$ of $V(\varpi_i)$ to the $(\varpi_i + d_i\delta)$-weight space $V(\varpi_i + d_i\delta)_{\varpi_i + d_i\delta}$ of $V(\varpi_i + d_i\delta)$ (by [Kas4, Proposition 5.16], these weight spaces are 1-dimensional). Thus we get a $U_q'(\mathfrak{g})$-module automorphism $z_i : V(\varpi_i) \xrightarrow{\sim} V(\varpi_i)$ of weight $d_i\delta$ (see [Kas4, §5.2]) as the composition of these maps. We now define a $U_q'(\mathfrak{g})$-module $W(\varpi_i)$ by

$$W(\varpi_i) := V(\varpi_i)/(z_i - 1)V(\varpi_i), \tag{1.5.2}$$

which is called a fundamental module of level zero. We know from [Kas4, Theorem 5.17] that $W(\varpi_i)$ is a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module, and has a simple crystal base, which is denoted by $\mathcal{B}(\varpi_i)_{\mathrm{cl}}$.

2 Our results.

2.1 Isomorphism theorems. Our main result in [NS1] and [NS2] is the following theorem (see [NS1, Theorem 5.1] and [NS2, Corollaries 2.2.1 and 3.3.8]).

Theorem 2.1.1. *For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, the crystal $\mathbb{B}(m\varpi_i)$ of all LS paths of shape $m\varpi_i$ is, as a crystal with weight lattice P, isomorphic to the crystal base $\mathcal{B}(m\varpi_i)$ of the extremal weight module $V(m\varpi_i)$ over $U_q(\mathfrak{g})$ with $m\varpi_i$ as an extremal weight.*

Here, let us give a sketch of our proof of Theorem 2.1.1. First we show the theorem for the case where $m = 1$. In [NS2, Theorem 2.1.1], we proved the following.

Theorem 2.1.2. *For every $i \in I_0$, the crystal graph of the crystal $\mathbb{B}(\varpi_i)$ is connected.*

We know from [Kas4, Proposition 5.4 (ii)] that the crystal graph of the crystal base $\mathcal{B}(\varpi_i)$ is also connected, and from [Kas4, Proposition 5.16 (ii)] that the cardinality of the subset $\mathcal{B}(\varpi_i)_{w\varpi_i}$ is equal to 1 for all $w \in W$, where $\mathcal{B}(\varpi_i)_\mu$ is the subset of $\mathcal{B}(\varpi_i)$ consisting of all elements of weight $\mu \in P$. In addition, we see from [BN, Theorem 4.16 (i)] that there exists a canonical embedding $\mathcal{B}_0(N\varpi_i) \hookrightarrow \mathcal{B}(\varpi_i)^{\otimes N}$ of crystals that sends $u_{N\varpi_i}$ to $u_{\varpi_i}^{\otimes N}$, where for each $\lambda \in P$, u_λ denotes the element of the crystal base $\mathcal{B}(\lambda)$ corresponding to the generator v_λ of the extremal weight module $V(\lambda)$, and $\mathcal{B}_0(\lambda)$ denotes the connected component of $\mathcal{B}(\lambda)$ containing the element u_λ. Further we showed the following proposition.

Proposition 2.1.3 ([NS1, Theorem 3.1]). *For every $N \in \mathbb{Z}_{>0}$ and $i \in I_0$, there exists an injective map $S_N : \mathcal{B}(\varpi_i) \hookrightarrow \mathcal{B}_0(N\varpi_i)$, which we call an N-multiple map, satisfying the following condition:*

(1) $S_N(u_{\varpi_i}) = u_{N\varpi_i}$,

(2) $\mathrm{wt}(S_N(b)) = N\,\mathrm{wt}(b)$ *for each* $b \in \mathcal{B}(\varpi_i)$,

(3) $S_N(e_j b) = e_j^N S_N(b)$, $S_N(f_j b) = f_j^N S_N(b)$ *for* $b \in \mathcal{B}(\varpi_i)$ *and* $i \in I$.

By using these facts, we can show that $\mathcal{B}(\varpi_i) \cong \mathbb{B}(\varpi_i)$ as crystals in exactly the same way as [Kas2, Theorem 4.1] (see [NS1, Theorem 5.1]).

As a consequence of Theorem 2.1.1 for the case where $m = 1$, we obtained the following corollary (cf. [NS1, Corollary 5.3]).

Corollary 2.1.4. *For every $m \geq 1$ and $i \in I_0$, we have*

$$\mathbb{B}_0(m\varpi_i) \cong \mathcal{B}_0(m\varpi_i) \quad \textit{as crystals},$$

where $\mathbb{B}_0(m\varpi_i)$ is the connected component of the crystal $\mathbb{B}(m\varpi_i)$ containing the straight line $\pi_{m\varpi_i}(t) = t(m\varpi_i)$, $t \in [0,1]$.

Next we prove Theorem 2.1.1 for the case where $m \geq 2$ (as seen below, the crystal graph of $\mathbb{B}(m\varpi_i)$ is not connected when $m \geq 2$). Let $\mathrm{Par}_{<m}$ be the set of partitions of length (i.e., the number of parts) strictly less than m. For each $\sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \mathrm{Par}_{<m}$, we denote by $|\sigma|$ the weight of σ, i.e.,

$|\sigma| := k_1 + k_2 + \cdots + k_{m-1}$. We can define a crystal structure on $\mathrm{Par}_{<m}$ as follows:

$$\begin{cases} e_j\sigma = f_j\sigma = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \varepsilon_j(\sigma) = \varphi_j(\sigma) = 0 & \text{for all } \sigma \in \mathrm{Par}_{<m} \text{ and } j \in I, \\ \mathrm{wt}(\sigma) := -|\sigma| d_i\delta & \text{for } \sigma \in \mathrm{Par}_{<m}. \end{cases}$$

In [NS2, §§3.2 ~ 3.6], we showed the following.

Lemma 2.1.5. (1) *For every $\sigma = (k_1 \geq k_2 \geq \cdots \geq k_{m-1}) \in \mathrm{Par}_{<m}$,*

$$\pi_\sigma := \Big(m(\varpi_i - k_1 d_i\delta), \ldots, m(\varpi_i - k_{m-1}d_i\delta), m\varpi_i\,;\, 0, \tfrac{1}{m}, \ldots, \tfrac{m-1}{m}, 1\Big).$$

is contained in $\mathbb{B}(m\varpi_i)$.

(2) *For each $\pi \in \mathbb{B}(m\varpi_i)$, there exists a unique $\sigma \in \mathrm{Par}_{<m}$ such that the π is connected to π_σ in the crystal graph of $\mathbb{B}(m\varpi_i)$.*

For $\sigma \in \mathrm{Par}_{<m}$, we denote by $\mathbb{B}_\sigma(m\varpi_i)$ the connected component of $\mathbb{B}(m\varpi_i)$ containing the path π_σ. Then it follows from the lemma above that

$$\mathbb{B}(m\varpi_i) = \bigsqcup_{\sigma \in \mathrm{Par}_{<m}} \mathbb{B}_\sigma(m\varpi_i).$$

Here recall from §1.3 that the root operators e_j, f_j are defined in terms of the function given by the pairing of a path and the simple coroot h_j. Because the path $\pi_\sigma(t)$ is the same as the straight line $\pi_{m\varpi_i}(t) = t(m\varpi_i)$, up to some δ-shift, and because $\delta(h_j) = 0$ for all $j \in I$, we deduce that the crystal graph of $\mathbb{B}_0(m\varpi_i)$ is isomorphic to the crystal graph of $\mathbb{B}_\sigma(m\varpi_i)$, up to some δ-shift of weight. More precisely, we have

$$\mathbb{B}_\sigma(m\varpi_i) \cong \{\sigma\} \otimes \mathbb{B}_0(m\varpi_i) \hookrightarrow \mathrm{Par}_{<m} \otimes \mathbb{B}_0(\varpi_i) \quad \text{as crystals,}$$

which sends π_σ to $\sigma \otimes \pi_{m\varpi_i}$. Thus we obtain

Theorem 2.1.6. *For $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we have*

$$\mathbb{B}(m\varpi_i) \cong \mathrm{Par}_{<m} \otimes \mathbb{B}_0(m\varpi_i) \quad \textit{as crystals.}$$

On the other hand, we know the following theorem from [BN, Theorem 4.16 (i)].

Theorem 2.1.7. *For each $m \in \mathbb{Z}_{\geq 1}$ and $i \in I_0$, we have*

$$\mathcal{B}(m\varpi_i) \cong \mathrm{Par}_{<m} \otimes \mathcal{B}_0(m\varpi_i) \quad \textit{as crystals.}$$

By combining Theorems 2.1.6 and 2.1.7 with Corollary 2.1.4, we can get our isomorphism theorem (Theorem 2.1.1). □

Now, for an integral weight $\lambda \in P$, we set

$$\mathbb{B}(\lambda)_{\mathrm{cl}} := \{\mathrm{cl}(\pi) \mid \pi \in \mathbb{B}(\lambda)\},$$

where for a path π, we define $\mathrm{cl}(\pi) : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}} \cong \mathfrak{h}^*/\mathbb{Q}\delta$ by: $(\mathrm{cl}(\pi))(t) := \mathrm{cl}(\pi(t))$ for $t \in [0,1]$. We can endow $\mathbb{B}(\lambda)_{\mathrm{cl}}$ with a structure of crystal with weight lattice P_{cl} in such a way that

$$\begin{cases} e_j\,\mathrm{cl}(\pi) := \mathrm{cl}(e_j\pi), & f_j\,\mathrm{cl}(\pi) := \mathrm{cl}(f_j\pi), \\ \varepsilon_j(\mathrm{cl}(\pi)) := \varepsilon_j(\pi), & \varphi_j(\mathrm{cl}(\pi)) := \varphi_j(\pi), \\ \mathrm{wt}(\mathrm{cl}(\pi)) := \mathrm{cl}(\mathrm{wt}(\pi)). \end{cases}$$

for $\pi \in \mathbb{B}(\lambda)$ and $j \in I$ (see [NS2, §3.3] and [NS3, §§1.3 and 1.4]). The following is a consequence of Theorem 2.1.1 (see [NS1, Proposition 5.8] and [NS2, Proposition 3.2]).

Theorem 2.1.8. *For each $i \in I_0$, the crystal $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$ is isomorphic to the crystal base $\mathcal{B}(\varpi_i)_{\mathrm{cl}}$ of the fundamental module $W(\varpi_i)$ of level zero as a crystal with weight lattice P_{cl}.*

2.2 Tensor product decomposition theorem. In [NS3], we studied the crystal structure of $\mathbb{B}(\lambda)_{\mathrm{cl}} = \mathrm{cl}(\mathbb{B}(\lambda))$ for a general integral weight $\lambda \in P$ of level zero. Before stating our main result in [NS3], we make some comments. Let $\lambda \in P$ be an integral weight of level zero. We can write the $\lambda \in P$ in the form $\lambda = \sum_{i \in I_0} m_i' \varpi_i + R\delta$ for some $m_i' \in \mathbb{Z}$, $i \in I_0$, and $R \in \mathbb{Q}$ (cf. [Kac, Chap. 6]). Then it follows from the definition of LS paths that

$$\mathbb{B}(\lambda) = \{\pi + \pi_{R\delta} \mid \pi \in \mathbb{B}(\textstyle\sum_{i \in I_0} m_i' \varpi_i)\},$$

where we set $(\pi + \pi_{R\delta})(t) := \pi(t) + tR\delta$, $t \in [0,1]$, and from the definition of the root operators that the crystal graph of $\mathbb{B}(\lambda)$ is the same shape as that of $\mathbb{B}(\sum_{i\in I_0} m'_i\varpi_i)$, up to $R\delta$-shift of weight. Therefore we have that $\mathbb{B}(\lambda)_{\mathrm{cl}} = \mathbb{B}(\sum_{i\in I_0} m'_i\varpi_i)_{\mathrm{cl}}$. In addition, the integral weight $\sum_{i\in I_0} m'_i\varpi_i \in P$ is equivalent to the one that is dominant with respect to the simple coroots $\{h_j\}_{j\in I_0}$ under the Weyl group $\mathring{W} := \langle r_j \mid j \in I_0\rangle \subset W$ (of finite type). Hence there exist nonnegative integers $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I_0$, such that $\mathbb{B}(\sum_{i\in I_0} m'_i\varpi_i)_{\mathrm{cl}} = \mathbb{B}(\sum_{i\in I_0} m_i\varpi_i)_{\mathrm{cl}}$ by Remark 1.2.1 (2). To sum up, for an integral weight $\lambda \in P$ of level zero, there exists $m_i \in \mathbb{Z}_{\geq 0}$, $i \in I_0$, such that $\mathbb{B}(\lambda)_{\mathrm{cl}} = \mathbb{B}(\sum_{i\in I_0} m_i\varpi_i)_{\mathrm{cl}}$. Thus, when we study the crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ for an integral weight $\lambda \in P$ of level zero, we may assume that the $\lambda \in P$ is of the form: $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ from the beginning. Now we are ready to state our main result in [NS3].

Theorem 2.2.1 ([NS3, Theorem 2.2.1]). *Let $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. Then, there exists an isomorphism $\mathbb{B}(\lambda)_{\mathrm{cl}} \xrightarrow{\sim} \bigotimes_{i\in I_0}(\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m_i}$ of crystals (with weight lattice P_{cl}) between $\mathbb{B}(\lambda)_{\mathrm{cl}}$ and the tensor product $\bigotimes_{i\in I_0}(\mathbb{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m_i}$ of the crystals $\mathbb{B}(\varpi_i)_{\mathrm{cl}}$, $i \in I_0$.*

By combining Theorems 2.1.8 and 2.2.1, we obtain the next corollary.

Corollary 2.2.2. *Let $\lambda = \sum_{i\in I_0} m_i\varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. The crystal $\mathbb{B}(\lambda)_{\mathrm{cl}}$ is, as a crystal (with weight lattice P_{cl}), isomorphic to the crystal base $\bigotimes_{i\in I_0}(\mathcal{B}(\varpi_i)_{\mathrm{cl}})^{\otimes m_i}$ of the tensor product $\bigotimes_{i\in I_0} W(\varpi_i)^{\otimes m_i}$ of fundamental $U'_q(\mathfrak{g})$-modules $W(\varpi_i)$, $i \in I_0$, of level zero.*

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