A SIMPLE INTRODUCTION TO CRYSTALS $B^{2,s}$ FOR KIRILLOV-RESHETIKHIN MODULES OF TYPE $D^{(1)}_n$

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Abstract. The Kirillov–Reshetikhin modules $W^{r,s}$ are finite-dimensional representations of quantum affine algebras $U_q'(g)$, labeled by a Dynkin node $r$ of the affine Kac–Moody algebra $g$ and a positive integer $s$. In this paper we explain the combinatorial structure of the crystal basis $B^{2,s}$ corresponding to $W^{2,s}$ for the algebra of type $D^{(1)}_n$.

Proofs of all claims, as well as more specific details of all constructions, may be found in [16].

1. Introduction

At the workshop on the Combinatorial Aspect of Integrable Systems held at RIMS Kyoto, one of the recurring themes was the $X = M$ conjecture of [1, 2]. Briefly, this conjecture states that the one-dimensional configuration sums $X$ of a certain class of lattice models can be expressed as fermionic formulas $M$, reflecting the corner transfer matrix method and the Bethe ansatz as methods for solving these lattice models. The combinatorial tools of these methods are Young tableaux/crystal bases and rigged configurations, respectively. The following table summarizes the three regimes of this conjecture.

<table>
<thead>
<tr>
<th>formulas</th>
<th>$X : 1$-D sum</th>
<th>$M :$ fermionic formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>stat. mech. methods</td>
<td>CTM</td>
<td>Bethe ansatz</td>
</tr>
<tr>
<td>comb. objects</td>
<td>tableaux/crystals</td>
<td>rigged configurations</td>
</tr>
</tbody>
</table>

More specifically, the theory of crystal bases is used to label the highest weight vectors of irreducible representations (i.e., Bethe vectors) of a certain algebra by crystal basis elements. Since each Bethe vector corresponds to a solution of the Bethe equations and these solutions are indexed by rigged configurations, there should be a natural bijection between highest weight crystal elements and rigged configurations. Such bijections have been found by Kirillov and Reshetikhin [7] for type $A^{(1)}_n$ (see also [8]), and later for all nonexceptional types for the vector representation [10] and symmetric powers [15]. For type $D^{(1)}_n$, the bijection was given in [14] for the fundamental representations.

The $X = M$ conjecture depends upon the existence of the crystals $B^{r,s}$ for the Kirillov–Reshetikhin modules $W^{r,s}$. The Kirillov–Reshetikhin (KR) modules are finite-dimensional irreducible representations of quantum affine algebras $U_q'(g)$. In general, it is not known yet whether the $B^{r,s}$ exist and what their combinatorial structure is. It is the purpose of this note to give the combinatorial structure of $B^{2,s}$ of type $D^{(1)}_n$. The KR crystals of type $A^{(1)}_n$ have been explicitly described [4, 13], as well as $B^{r,1}$ and $B^{2,s}$ for most types [4, 6]. Furthermore, according to the theory of virtual crystals [11, 12], the following algebra

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embeddings have been explicitly extended to the crystals of their KR modules:

\[
\begin{align*}
C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)} & \leftrightarrow A_{2n-1}^{(1)} \\
A_{2n-1}^{(2)}, B_n^{(1)} & \leftrightarrow D_{n+1}^{(1)} \\
E_6^{(2)}, F_4^{(1)} & \leftrightarrow E_8^{(1)} \\
D_4^{(3)}, G_2^{(1)} & \leftrightarrow D_4^{(1)}.
\end{align*}
\]

The next case to explore is therefore $B^{2,s}$ for type $D_n^{(1)}$, which is the focus of this paper. Here, we present the combinatorial construction of $B^{2,s}$ assuming existence as recently given in [16]. The combinatorial crystal is denoted by $\tilde{B}^{2,s}$; we illustrate our main definition with examples. Proofs and further details can be found in [16]. The main result of [16] is:

**Theorem 1.1.** If $B^{2,s}$ exists with the properties as in Conjecture 2.1, then $\tilde{B}^{2,s} \cong B^{2,s}$.

2. Review

For background on quantum groups, crystal bases, perfect crystals, and other well-understood concepts, please refer to [16] or any of the standard references on these topics.

The fermionic formulas suggest not only the existence of the crystals $B^{n,s}$, but also several conjectures about the structure of these crystals as well [1]. In the case of $B^{2,s}$, this specializes to

**Conjecture 2.1 ([1]).** The crystal $B^{2,s}$ of type $D_n^{(1)}$ exists and has the following properties:

1. As a classical crystal $B^{2,s}$ decomposes as $B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\Lambda_2)$.
2. $B^{2,s}$ is perfect of level $s$.
3. $B^{2,s}$ is equipped with an energy function $D_{B^{2,s}}$ such that $D_{B^{2,s}}(b) = k - s$ if $b$ is in the component of $B(k\Lambda_2)$ (in accordance with the energy $D$ as in [16]).

To construct $\tilde{B}^{2,s}$ so that it satisfies these properties, we first find a way to label the vertices of the crystal. Our approach is to define a set of rules for what a legal “affine tableau” is, and then show that this set is in bijection with the direct sum $\bigoplus_{b=0}^{s} B(k\Lambda_2)$.

This bijection provides the action of the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \leq i \leq n$, but we still need to know the action of $\tilde{e}_0$ and $\tilde{f}_0$. To define these crystal operators, we use an auxiliary construction called the branching component graph. It can be shown that the resulting affine crystal $\tilde{B}^{2,s}$ is perfect of level $s$. In fact it was proved in [16] that this is the unique perfect level $s$ crystal for which the energy function is as stated in Conjecture 2.1.

3. Affine Tableaux

We briefly recall the labelling by tableaux of the vertices of classical highest weight crystals $B(k\Lambda_2)$ of highest weight $k\Lambda_2$, following the construction by Kashiwara and Nakashima [5]. Each crystal element can be represented by a tableau of shape $\lambda = (k,k)$ on the partially ordered alphabet

\[1 < 2 < \cdots < n - 1 < \frac{n}{\lambda_1} < \frac{n - 1}{\lambda_2} < \cdots < \frac{2}{\lambda_{\lambda_1}}< 1\]

such that the following conditions hold [3, page 202]:

**Criterion 3.1.**

1. If $ab$ is in the filling, then $a \leq b$;
2. If $a \not\leq b$ in the filling, then $b \not\leq a$;
(3) No configuration of the form \( \frac{a}{a} \) or \( \frac{a}{a} \overline{a} \) appears;
(4) No configuration of the form \( \frac{n-1}{n} \frac{n}{n-1} \) or \( \frac{n-1}{n} \frac{n}{n-1} \) appears;
(5) No configuration of the form \( \frac{1}{1} \) appears.

Note that for \( k \geq 2 \), condition 5 follows from conditions 1 and 3.

We define the set of affine tableau in \( \tilde{B}^{2,s} \) by removing parts 3 and 5 from Criterion 3.1. The bijection between \( \tilde{B}^{2,s} \) and \( \bigoplus_{k=0}^{s} B(k\Lambda_{2}) \) is as follows. Given an affine tableau \( T \) which is not a classical tableau (i.e., a tableau that satisfies parts 1, 2, and 4 of 3.1, but violates part 3 or 5) there must be a configuration of the form \( \frac{a}{a} \frac{a}{a} \overline{a} \) or \( \frac{1}{1} \). Remove columns of the form \( \frac{a}{a} \) (possibly with \( a = 1 \)) until the resulting tableau satisfies Criterion 3.1. It can be shown that this procedure gives a well-defined bijection between the two sets.

The following examples are taken from \( \tilde{B}^{2,5} \) for \( D_{4}^{(1)} \).

**Example 3.2.** The affine tableau
\[
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
4 & 2 & 2 & 2 \\
4 & 2 & 1 \\
\end{array}
\]
 corresponds to the classical tableau
\[
\begin{array}{cccc}
1 & 3 & 3 \\
4 & 2 & 1 \\
\end{array}
\]
by removing the second and third columns.

It is easy to see that for any affine tableau the removed columns must be adjacent, as they are in these examples.

**Example 3.3.** The affine tableau
\[
\begin{array}{cccc}
2 & 3 & 3 & 4 \\
4 & 3 & 3 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\]
 corresponds to the classical tableau
\[
\begin{array}{cccc}
2 & 3 & 3 \\
4 & 3 & 2 \\
\end{array}
\]
by removing either the second or the third column.

As the above example indicates, if there is a choice about which column to remove, it has no effect on the outcome.

**Example 3.4.** The classical tableau
\[
\begin{array}{cc}
1 & 2 \\
4 & 2 \\
\end{array}
\]
 corresponds to the affine tableau
\[
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
4 & 2 & 2 & 2 \\
\end{array}
\]

While we could choose to add columns of the form \( \frac{2}{a} \) either to the middle or to the right side of the first tableau, either choice results in the same affine tableau.

**Example 3.5.** The classical tableau
\[
\begin{array}{ccc}
2 & 3 & 3 \\
4 & 2 & 1 \\
\end{array}
\]
 corresponds to the affine tableau
\[
\begin{array}{ccc}
2 & 2 & 3 \\
4 & 2 & 1 \\
\end{array}
\]

By part 1 of Criterion 3.1, the only place that a column of the form \( \frac{a}{a} \) may be inserted is between the first and second columns of \( t \). However, we may choose between using this to create a configuration of either of the forms \( \frac{a}{a} \frac{a}{a} \) or \( \frac{a}{a} \). Once again, this “choice” does not affect the outcome.

4. The branching component graph

Since the Dynkin diagram for type \( D_{n}^{(1)} \) has a graph automorphism interchanging nodes 0 and 1, we know that interchanging the role of 1-arrows and 0-arrows in \( \tilde{B}^{2,s} \) will produce an affine crystal isomorphic to \( \tilde{B}^{2,s} \). We may use this fact to our advantage at a
larger scale by considering the $D_{n-1}$-crystals that result from removing the 1-arrows from $\bigoplus_{k=0}^{s} B(k\Lambda_2)$, since this direct sum is isomorphic to $\tilde{B}^{2,s}$ with the 0-arrows removed.

The branching component graph of $\tilde{B}^{2,s}$, denoted $BC(\tilde{B}^{2,s})$, is defined as follows. Its vertices correspond to the $D_{n-1}$-crystals that remain connected after removing all 0-arrows and 1-arrows from $\tilde{B}^{2,s}$; we label the vertices (non-uniquely) by the partition $\lambda$ indicating the classical highest weight of the corresponding $U_q(D_{n-1})$-crystal. The edges of $BC(\tilde{B}^{2,s})$ are defined by placing an edge from $v$ to $w$ if there is a tableau $b \in B(v)$ such that $\tilde{f}_1(b) \in B(w)$, where $B(v)$ denotes the set of tableaux contained in the $D_{n-1}$-crystal indexed by $v$.

It suffices to describe the effect of removing the 1-arrows from $B(k\Lambda_2)$ for arbitrary $k$. We denote this branching component graph by $BC(k\Lambda_2)$, and use $v_k$ to denote the "highest weight branching vertex", i.e., the branching vertex such that the highest weight tableaux $b_{k\Lambda_2} \in B(v_k)$.

An intuitive way to construct $BC(k\Lambda_2)$ is as follows. Begin with a $1 \times k$ rectangle, which labels $v_k$. For $1 \leq j \leq k$, the partitions labeling the vertices of rank $j$ are those which are contained in a $2 \times k$ rectangle and which are joined by an edge in Young's lattice to some partition labeling a vertex in rank $j - 1$. In each rank, the partitions appear with multiplicity one. For $k + 1 \leq j \leq 2k$, the partitions in rank $j$ are the same as those in rank $2k - j$, again with multiplicity one. Finally, there is an edge from a vertex $v$ of rank $j$ to a vertex $w$ of rank $j + 1$ precisely when the corresponding partitions are joined by an edge in Young's lattice.

**Example 4.1.** Figure 1 depicts $BC(3\Lambda_2)$.

There is a unique inclusion of $BC(k\Lambda_2)$ in $BC((k + 1)\Lambda_2)$ that agrees with the labelling of the vertices. We may define a rank function on all of $BC(\tilde{B}^{2,s})$ by setting the rank of
a vertex to the rank of its image in $BC(s\lambda_2)$ under the appropriate composition of these inclusions. For example, every vertex labelled by $\emptyset$ always has rank $s$ in $BC(\tilde{B}^{2,s})$.

**Example 4.2.** Figure 2 depicts $BC(\tilde{B}^{2,2})$, which is the union of $BC(0)$, $BC(\Lambda_2)$, and $BC(2\Lambda_2)$.

5. **Affine Kashiwara Operators**

In this section we describe how to "overlay" a set of arrows, called $F_0$ arrows, on $BC(\tilde{B}^{2,s})$ in a way that specifies $\bar{e}_0$ and $\tilde{f}_0$. Let $v \in BC(\tilde{B}^{2,s})$ be a vertex of global rank $j$ in $BC(\Lambda_2)$ associated with the partition $(\lambda_1, \lambda_2)$. Place an $F_0$ arrow from $v$ to the following vertices, if they exist:

- the vertex of global rank $j - 1$ in $BC((k - 1)\Lambda_2)$ with shape $(\lambda_1 - 1, \lambda_2)$;
- the vertex of global rank $j - 1$ in $BC(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 - 1)$;
- the vertex of global rank $j - 1$ in $BC((k + 1)\Lambda_2)$ with shape $(\lambda_1 + 1, \lambda_2)$;
- the vertex of global rank $j - 1$ in $BC(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 + 1)$.

The directed graph that consists of the vertices of $BC(\tilde{B}^{2,s})$ and the $F_0$ arrows is isomorphic to $BC(\tilde{B}^{2,s})$. Via this graph isomorphism, which we denote $\sigma$, we may define $\tilde{f}_0$ for $\tilde{B}^{2,s}$. Let $b \in B(v)$ be a tableau in $\tilde{B}^{2,s}$. Note that $B(v)$ is isomorphic to $B(\sigma(v))$ as a $D_{n-1}$-crystal; let $b' \in B(\sigma(v))$ denote the tableau corresponding to $b$ under this isomorphism. We may have $\tilde{f}_1(b') = c' \in B(w)$ for some branching vertex $w$, or we may have $\tilde{f}_1(b') = 0$. In the former case, we say that $\tilde{f}_0(b) = c$, where $c$ corresponds to $c'$ under the isomorphism between $B(w)$ and $B(\sigma(w))$; in the latter case, $\tilde{f}_0(b) = 0$. By the definition of crystals, this also determines $\varepsilon_0$.

**Example 5.1.** In Figure 3 we have $BC(\tilde{B}^{2,2})$ with the original arrows removed and the $F_0$ arrows superimposed.

Of course, we could have chosen to define the graph isomorphism in terms of the branching vertices, and let the definition of the $F_0$ arrows follow. In fact, we did exactly that in [16], where $\sigma$ is used to denote the automorphism of the vertices of $\tilde{B}^{2,s}$ corresponding to interchanging nodes 0 and 1 of the Dynkin diagram.

We now present some examples taken from $\tilde{B}^{2,2}$. 

![Figure 2. Branching component graph $BC(\tilde{B}^{2,2})$](image)
Example 5.2. Let \( b = \frac{1}{1} \frac{2}{1} \), so \( b \in B(v) \) where \( v \) is the branching vertex of shape \((1, 0)\) with global rank 3 in \( BC(\Lambda_2) \). We see from Figures 2 and 3 that \( \sigma(v) \) is the vertex with the same shape with rank 1 in \( BC(2\Lambda_2) \). The corresponding tableau in \( \sigma(v) \) is \( b' = \frac{1}{2} \frac{2}{2} \), and \( c' = \tilde{f}_1(b') = \frac{1}{2} \frac{2}{2} \). The branching vertex containing \( c' \) is the vertex of shape \((1, 1)\) with rank 2 in \( BC(2\Lambda_2) \), which is fixed under \( \sigma \), so \( c = c' \). Therefore, \( \overline{f}_0(b) = 2 \frac{2}{1} \).

Example 5.3. Let \( b = \frac{3}{1} \frac{3}{1} \), so \( b \in B(v) \) where \( v \) is the branching vertex of shape \((2, 0)\) with rank 4 in \( BC(2\Lambda_2) \). We see from Figures 2 and 3 that \( \sigma(v) \) is the vertex of the same shape with rank 0 in \( BC(2\Lambda_2) \). The corresponding tableau in \( \sigma(v) \) is \( b' = \frac{3}{3} \frac{1}{1} \), and \( c' = \tilde{f}_1(b') = \frac{1}{2} \frac{2}{2} \). The branching vertex containing \( c' \) is the vertex of shape \((2, 1)\) with rank 1 in \( BC(2\Lambda_2) \). Its image under \( \sigma \) is the vertex of the same shape with rank 3 in \( BC(2\Lambda_2) \), so \( \tilde{f}_0(b) = c = \frac{3}{3} \frac{1}{1} \).

Example 5.4. Let \( b_{kA_2} \) denote the classical highest weight tableau of \( B(kA_2) \subset \tilde{B}^{2,s} \). Then \( \tilde{f}_0(b_{kA_2}) = b_{(k+1)A_2} \) for \( 0 \leq k \leq s - 1 \).

6. Perfectness

Several conditions must be satisfied for a crystal \( B \) to be a perfect crystal of level \( \ell \), but the most significant challenge is in the condition that the maps \( \epsilon \) and \( \varphi \) from \( B_{\text{min}} \) to \((P_{\text{c}1}^+)_\ell \) are bijective. We briefly recall the definition of these sets and maps below; for more detail see [16] or [4].

For a crystal basis element \( b \in B \), define the weights

\[
\epsilon(b) = \sum_{i \in I} \epsilon_i(b) \Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i,
\]

where

\[
\epsilon_i(b) = \max \{ n \geq 0 \mid \tilde{e}_i^n(b) \neq \emptyset \} \quad \text{and} \quad \varphi_i(b) = \max \{ n \geq 0 \mid \overline{f}_i^n(b) \neq \emptyset \}.
\]

The level of a weight \( \Lambda \) is \( \langle c, \Lambda \rangle \), where \( c = h_0 + h_1 + h_{n-1} + h_n + \sum_{n=2}^{n-2} 2h_i \) is the canonical central element of the algebra of type \( D_n^{(1)} \). The set of minimal vertices, denoted
TYPE \( D^{(1)}_{n} \) KIRillov-RESHETIKHIN CRYSTALS

\( B_{\text{min}} \), is the set of crystal elements \( b \) for which \( \langle c, \epsilon(b) \rangle \) is minimal. Finally, define \( (P_{i}^{+})_{\ell} \) to be the set of level \( \ell \) weights \( \Lambda \) with no \( \delta \) component for which \( \langle h_{i}, \Lambda \rangle \geq 0 \) for all \( i \in I \).

We now outline the construction of a \( 2 \times s \) tableau \( T \) such that given any level \( s \) weight \( \Lambda \), we have \( \epsilon(T) = \varphi(T) = \Lambda \). It was shown in \([16] \) that these are precisely the tableaux in \( B_{\text{min}} \).

For \( i = 0, \ldots, n \), let \( k_{i} = (h_{i}, \lambda) \). We first construct a tableau \( T_{\lambda'} \) corresponding to the weight \( \lambda' = \sum_{i=2}^{n} k_{i} \Lambda_{i} \). We begin with the middle \( k_{n-1} + k_{n} \) columns of \( T_{\lambda'} \). If \( k_{n-1} + k_{n} \) is even and \( k_{n} \geq k_{n-1} \), these columns of \( T_{\lambda'} \) are

\[
\begin{array}{ccccccccc}
& n-2 & n-2 & n-1 & n-1 & \bar{n} & \bar{n} & \bar{n} & n-1 & n-1 & n-1 \\
& k_{n-1} & (k_{n-1}-k_{n})/2 & (k_{n-1}-k_{n})/2 & & & & & & & \\
\end{array}
\]

If \( k_{n-1} + k_{n} \) is odd and \( k_{n} \geq k_{n-1} \), we have

\[
\begin{array}{ccccccccc}
& n-2 & n-2 & n-1 & n-1 & \bar{n} & \bar{n} & \bar{n} & n-1 & n-1 & n-1 \\
& k_{n-1} & (k_{n-1}-k_{n})/2 & (k_{n-1}-k_{n})/2 & & & & & & & \\
\end{array}
\]

In either case, if \( k_{n} < k_{n-1} \), interchange \( n \) and \( \bar{n} \), and \( k_{n} \) and \( k_{n-1} \) in the above configurations.

Next we put a configuration of the form

\[
\begin{array}{ccccccccc}
1 & 1 & 2 & 2 & \cdots & n-3 & n-3 \\
2 & 3 & 3 & \cdots & n-2 & n-2 \\
\end{array}
\]

on the left, and a configuration of the form

\[
\begin{array}{ccccccccc}
\bar{n} & \bar{n} & \bar{n} & \bar{n} & \cdots & \bar{n} & \bar{n} \\
\bar{n} & \bar{n} & \bar{n} & \bar{n} & \cdots & \bar{n} & \bar{n} \\
\end{array}
\]

on the right.

We now use Lecouvey \( D \) equivalence as in \([9] \) or type \( D \) sliding as in \([16] \) to change this tableau into a skew tableau of shape \((s-k_{0}, s-k_{0} - k_{1})/(k_{1}) \). If \( k_{1} > s-k_{0} - k_{1} \) (i.e., \( k_{1} - (s-k_{0} - k_{1}) = 2k_{1} + k_{0} - s > 0 \)), place a configuration of the following form in the empty spaces in the middle of this skew tableau:

\[
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 \\
2 & 2 & 1 & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
\end{array}
\]

where the number of 1's equals the number of \( \bar{1} \)'s and the number of 2's equals the number of \( \bar{2} \)'s.

If \( s-k_{0} \) is odd, the middle column of the tableau constructed so far is \( \bar{a} \) for \( 1 \leq a \leq n \) or \( \bar{n} \). Whatever it is, simply insert \( k_{0} \) of this column into the tableau next to the middle column (cf. Section 3). If \( s-k_{0} \) is even, the middle two columns are of the form \( a \bar{b} \) for some letters \( a \) and \( b \) (it is possible that \( b \) is barred, in which case \( \bar{b} \) is the corresponding unbarred letter). In this case, simply add \( k_{0} \) columns of the form \( \bar{a} \) between these columns.
Example 6.1. Let $n = 5$, and consider the weight $\Lambda_0 + 2\Lambda_1 + \Lambda_2 + 2\Lambda_4 + \Lambda_5$. This weight has level $1 + 2 + 2 + 1 + 2 + 1 = 8$, so our procedure will result in a $2 \times 8$ tableau; i.e., a minimal tableau in $\tilde{B}^{2,8}$. Since $k_4 + k_5$ is odd and $k_5 < k_4$, we begin with

$$
\begin{array}{ccccccc}
3 & 5 & 4 & 2 \\
4 & 5 & 3 & 1
\end{array}
$$

To incorporate $\Lambda_2$, we amend this tableau to get

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 4 & 5 & 4 & 2 \\
2 & 4 & 5 & 4 & 1 & 1 & 1
\end{array}
$$

Applying the type $D$ sliding algorithm twice and inserting 1’s and $\overline{1}$’s gives us

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 4 & 5 & 4 & 2 \\
2 & 4 & 5 & 4 & 1 & 1 & 1
\end{array}
$$

Finally, we insert one column in the middle, which yields

$$
\begin{array}{ccccccc}
1 & 1 & 1 & 2 & 6 & 6 & 3 \\
3 & 6 & 6 & 2 & 2 & 1 & 1
\end{array}
$$

Example 6.2. Let $n = 6$, and consider the weight $\Lambda_0 + 2\Lambda_1 + \Lambda_2 + 2\Lambda_6$. This weight has level $1 + 2 + 2 + 2 + 1 = 7$, so we will have a $2 \times 7$ tableau at the end; i.e., a minimal tableau in $\tilde{B}^{2,7}$. We begin with the tableau corresponding to $2\Lambda_6$, which is

$$
\begin{array}{ccccccc}
5 & 6 \\
6 & 5
\end{array}
$$

and expand it thus an account of $\Lambda_3$:

$$
\begin{array}{ccccccc}
2 & 5 & 6 & 3 \\
3 & 6 & 5 & 2
\end{array}
$$

Type $D$ sliding turns it into

$$
\begin{array}{ccccccc}
1 & 1 & 2 & 6 & 6 & 3 \\
3 & 6 & 6 & 2 & 2 & 1 & 1
\end{array}
$$

and inserting one column gives us

Example 6.3. Table 1 shows several weights and the corresponding tableaux. The first 11 entries are all the level 2 weights for $n = 4$.

References


[6] Y. Koga, Level one perfect crystals for $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$, J. Algebra 217 (1999), no. 1, 312–334.


<table>
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<th>tableau</th>
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<td>2</td>
<td>$1 \ 1$</td>
</tr>
<tr>
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<td>2</td>
<td>$1 \ 2$</td>
</tr>
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<td>$\Lambda_2$</td>
<td>2</td>
<td>$2 \ 1$</td>
</tr>
<tr>
<td>7</td>
<td>$2\Lambda_3 + 3\Lambda_4$</td>
<td>10</td>
<td>$2 \ 3 \ 3 \ 3 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 4 \ 3 \ 3 \ 3 \ 2 \ 2$</td>
</tr>
<tr>
<td>7</td>
<td>$2\Lambda_0 + \Lambda_2 + 2\Lambda_4 + \Lambda_5$</td>
<td>10</td>
<td>$1 \ 3 \ 3 \ 4 \ 4 \ 4 \ 4 \ 4 \ 2 \ 2 \ 4 \ 4 \ 3 \ 4 \ 4 \ 3 \ 3$</td>
</tr>
<tr>
<td>7</td>
<td>$\Lambda_4 + 2\Lambda_6 + 7\Lambda_7$</td>
<td>11</td>
<td>$3 \ 5 \ 5 \ 6 \ 6 \ 7 \ 7 \ 7 \ 7 \ 6 \ 6 \ 4 \ 4 \ 6 \ 6 \ 7 \ 7 \ 6 \ 8 \ 8 \ 8 \ 8$</td>
</tr>
</tbody>
</table>

**Table I.** Weights and minimal tableaux


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