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A recipe for finding Stokes geometry in the quantized Hénon map

Akira Shudo

Department of Physics, Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
shudo@phys.metro-u.ac.jp

1. Quantum propagator of the Hénon map

A discrete analog of the Feynman-type path integral,

$$< q_n|U^n|q_0 >= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1 dq_2\cdots dq_{n-1}\exp\Big[\frac{i}{\hbar}S(q_0,q_1,\cdots,q_n)\Big] \quad (1.1)$$

gives a standard recipe to formulate quantum mechanics of the symplectic map. Here, the function $S(q_0, q_1, \cdots, q_n)$ represents the discretized Lagrangian or the action functional given as

$$S(q_0,\cdots,q_n)=\sum_{j=1}^{n}\frac{1}{2}(q_j-q_{j-1})^2-\sum_{j=1}^{n-1}V(q_j). \quad (1.2)$$

The action functional is derived so that applying the variational principle generates the symplectic map. In fact, we can easily see that the condition $\partial S(q_0,\cdots,q_n)/\partial q_j = 0, \ \ (1\le j\le n-1)$ yields the classical map in the Lagrangian form:

$$(q_{j+1}-q_j)-(q_j-q_{j-1})=-V'(q_j). \quad (1.3)$$

The simplest possible choice of the potential function would be,

$$V(q)=-\frac{q^3}{3}-cq, \quad (1.4)$$

where c denotes a real parameter controlling qualitative features of the underlying classical map, and possibly the corresponding quantum dynamics. This system is related to the so-called Hénon family via an appropriate affine transformation with a change of parameter [1]. A merit to employ the Hénon map is not only that it is a canonical and the simplest possible polynomial map [2], but also that the theory of complex dynamical systems is most well developed in the Hénon family[3, 4]. This is important because the saddle point solutions of the quantum propagator (1.1) are just the classical trajectories in the complex plane. If one wants to describes quantum phenomena in term of classical dynamics, it is crucial to know complex classical dynamics in details. In this respect, the Hénon map is most suitable. The aim of this short note is to describe a concrete recipe to find Stokes geometry for the integral (1.1). In particular, by deriving differential operators acting on our integral, we will adapt our problem to the prescription proposed and validated in several examples in [6], in which virtual turning points and new Stokes curves play essential new roles in constructing Stokes geometry in higher-order differential equations.

2. Bicharacteristic equations

Semiclassical approximation is a widely used technique to study quantum-classical correspondence in chaotic systems, that is, to see how classical chaos influences the corresponding quantum systems in the semiclassical limit. Here the semiclassical limit is achieved by taking $\hbar \to 0$. Such an issue is sometimes called *quantum chaology* in the literature [5], and have extensively been studied in the past two decades. However, even after a great deal of efforts, the validity of semiclassical approximation has not yet been clarified, and well-fomulated mathematical problems are only limited. Here, we mean semiclassical approximation as just taking the leading order contribution in evaluating the multiple integral $< q_n|U^n|q_0 >$ via the stationary phase(or saddle point) method. The resulting semiclassical formula is, in general, expressed as a sum over contributions of classical trajectories connecting the initial and final states since the number of saddles (=complex classical trajectories). In addition, the number of contributing saddles increases exponentially as a function of the time step n. This reflects the existence of chaos in our symplectic map.

An important fact we should here note is that not all the saddle point solutions appear as a final semiclassical expression because the Stokes phenomenon occurs and some of saddle point solutions should not contribute. Therefore, in order to establish semiclassical prescription describing especially tunneling processes, where complex trajectories play essential roles, we have to work out a rigid prescription to deal with Stokes phenomenon for the quantum propagator (1.1). This is essentially important for our understanding quantum tunneling in the presence of chaos [1].

To this end, the work by Aoki, Kawai and Takei [6] is essential. They have proposed a concrete recipe to analyze Stokes phenomena in higher-order differential equations, say $P(x, \eta^{-1}d/dx)\psi(x) = 0$, within the exact WKB framework [6]. Their work contains not only a mathematical justification of the preceding work [8] in which *new Stokes curves* should be introduced in order to recover the univaluedness around crossing points of ordinary Stokes curves in an ad-hoc way, but also claims that *virtial turning points* (they are originally called new turning points in [6]) should first be taken into account to construct complete Stokes geometry. They also clarified that new Stokes curves play essentially the same part in the Stokes geometry [7].

In order to apply the same recipe to our integral $< q_n|U^n|q_0 >$, we here derive differential equations that our multiple integral (1.1) satisfies, and discuss how virtual turning points and new Stokes curves for multiple integral could be introduced based on the same argument in [6]. It is interesting to note that the procedure to derive differential operator for our multiple integral is nothing but to solve the initial values as functions of final values for the Hénon map. Bicharacteristic equaitions for the Borel transform of the differential operator can also be derived in a similar way. The order of differential equations increase exponentially as n increases, and their explicit forms appear complicated. Nevertheless, since the resulting differential operators are linked to the underlying classical map, so our algorithm to derive them is straightforward and automatic.

In what follows, we shall fix one of the initial coordinate q_0 and regard the quantum propagator (1.1) as a function of the final coordinate q_n. We therefore use the notation $I(q_n) \equiv < q_n|U^n|q_0 >$ to represent our integral defined in eq. (1.1), and introduce the large parameter $\eta = i/\hbar$.

To derive differential equations for $I(q_n)$, Let us consider a set of equations for

$e^{\eta S(q_0,q_1,\cdots,q_n)}$:

$$\eta\Big[(q_{i+1}-q_i)-(q_i-q_{i-1})-q_i^2-c\Big]e^{\eta S}=\frac{\partial}{\partial q_i}e^{\eta S}, \qquad (i=1,2,\cdots n-1) \tag{2.1}$$

$$\eta(q_n-q_{n-1})e^{\eta S}=\frac{\partial}{\partial q_n}e^{\eta S}. \tag{2.2}$$

If we carry out an integration of both sides of eqs. (2.1) and (2.2) over each variable q_i, it is easy to see that the integrals in right-hand sides vanish by taking an appropaiate integral contour. A set of conditions, $\partial e^{\eta S}/\partial q_i = 0$ $(i=1,\cdots,n-1)$, yields simulateneous equations for the operators acting on $I(q_n)$. These are nothing but a set of equation (1.3), that is the Hénon map [9].

A single differential operator acting on $I(q_n)$, which should be written as functions of q_n and $\partial/\partial q_n$, can be derived by expressing the operator $\Big[(q_2-q_1)-(q_1-q_0)-q_1^2-c\Big]$ as functions of $\partial/\partial q_n$. This is achieved by solving the recursion relation (1.3) into the form

$$q_1=q_1(q_{n-1},q_n) \tag{2.3}$$

$$q_2=q_2(q_{n-1},q_n), \tag{2.4}$$

and using the relation $\partial/\partial q_n=\eta(q_n-q_{n-1})$, which is obtained by (2.2), into (2.3) and (2.4). Eqs. (2.3) and (2.4) are equivalent to solving the *final value problem* for the Hénon map. Explicit forms of differential operators for $n=3$ and $n=4$ cases have already been presented in another report [10].

Next, we will derive bicharacteristic equations for these differential equations. For this purpose, we first present some generic property concerning the Hamiltonian flow associated with the maps. Suppose a map $f:(x,y)\mapsto(X,Y)$ be differentiable and have a unique invserse, a Hamiltonian flow is induced naturally if we regard one of variable, say y, as the time of Hamiltonian flow. Here the map f may also be viewed as finite iterations of other maps, that is, $f(x,y)=g^n(x,y)$ where n is the number of itetion. More precise statement is given as,

Proposition[11]
Let det J *be the Jabobian of the map* $(x,y)\to(X,Y)$ *and* H *be a function of* (X,Y) *which is given by*

$$H(X,Y)=\int^{x(X,Y)}(\det J)dx, \tag{2.5}$$

and satisfies

$$\frac{\partial H}{\partial y}=0. \tag{2.6}$$

Then the following set of Hamiltonian equations holds:

$$\frac{dX}{dy}=\frac{\partial H}{\partial Y},\quad \frac{dY}{dy}=-\frac{\partial H}{\partial X}. \tag{2.7}$$

The Hamitonian H is determined if the Jacobian of the map f is given as a function of x. If x is expressed as functions of (X,Y), then H is explicitely expressed by (X,Y). In particular, in case of the area preseving map for which $\det J=1$, the Hamiltonian

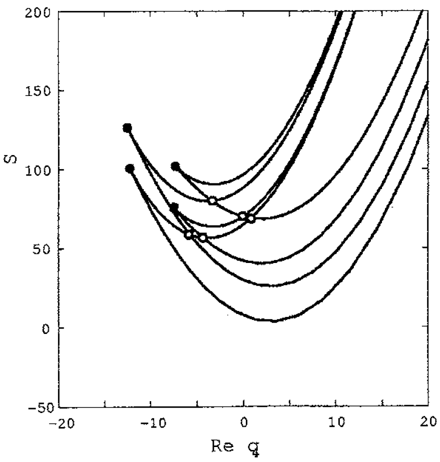

Figure 1. Bicharacteristic curve for $n = 4$. The parameters are chosen as $q_0 = 0$ and $c = 6$ where all the turning points are located on the real plane. Cuspidal points and transversally crossing points represent ordinally turning and virtual turning points respectively. The number of ordinary turning points totally amounts to 7, and that of virtual turning points to 21. The rest of turning points are outside the range of this figure.

is nothing but $x(X, Y)$. The proof of this proposition is straightforward and is given in Appendix A.

Since the Hénon map $(q_0, p_1) \mapsto (q_1(q_0, p_1), q_2(q_0, p_1))$ is polynomial diffemorphism and its inverse is also the Hénon map with different system parameters, the above proposition is straightforwardly applicable. Hamiltonian then is expressed as

$$H(q_{n-1}, p_n) = 2q_1(q_{n-1}, q_n) + q_1(q_{n-1}, q_n)^2 + c - q_2(q_{n-1}, q_n).$$

Here we introduce a new variable as

$$\xi \equiv= \eta(q_n - q_{n-1}), \tag{2.8}$$

and define

$$\sigma(\hat{\mathcal{H}})(q_n, S, \xi, \eta) \equiv -\eta \hat{H}(q_1, q_2) \Bigg|_{\substack{q_1 = q_1(q_{n-1}, q_n) \\ q_2 = q_2(q_{n-1}, q_n)}} \Bigg|_{Large q_{n-1} = q_n - \xi\eta^{-1}} \tag{2.9}$$

where

$$\hat{H}(q_1, q_2) \equiv \frac{\partial}{\partial q_1} S(q_0, q_1, \cdots, q_n)|_{q_0=0} = 2q_1 + q_1^2 + q_2 - c. \tag{2.10}$$

We can show that $\sigma(\hat{\mathcal{H}})(q_n, S, \xi, \eta)$ is just the principal symbol for our differential operator and thus gives bicharacteristic equations. Indeed we already know that the variables (q_n, ξ) form canonical conjugate variables and they satisfy Hamiltonian equations, because the above $H(q_{n-1}, p_n)$ can be regarded as Hamiltonian $H(q_n, \xi)$. Furthermore, as is shown in Appendix B, (η, S) also forms another canonical pair and satisfy a set of Hamiltonian equations:

$$\frac{dS}{dq_1} = \frac{\partial \sigma(\hat{H}(q_n, S, \xi, \eta))}{\partial \eta}, \tag{2.11}$$

$$\frac{d\eta}{dq_1} = -\frac{\partial\sigma(\hat{H}(q_n, S, \xi, \eta))}{\partial S}. \tag{2.12}$$

Note that time variable for bicharacteristic equations is q_1

Explicit forms of principal symbols $\sigma(\hat{H})(q_n, S, \xi, \eta)$ are given respectively as

$$\sigma(\hat{H}) = \eta^{-1}\xi^2 - 2(q_2+1)\xi + (q_2^2 + q_2 - q_0 + c)\eta \tag{2.13}$$

for $n = 2$, and

$$\begin{aligned}\sigma(\hat{H}) = {} & \eta^{-3}\xi^4 - 4(q_3+1)\eta^{-2}\xi^3 + (6q_3^2 + 10q_3 + 2c + 6)\eta^{-1}\xi^2 \\ & - (4q_3^3 + 8q_3^2 + 4cq_3 + 8q_3 + 4c + 3)\xi \\ & + (q_3^4 + 2q_3^3 + 2cq_3^2 + 3q_3^2 + 3cq_3 + q_3 + c^2 + 3c - q_0)\eta \end{aligned} \tag{2.14}$$

for $n = 3$.

3. A concrete recipe to draw Stokes geometry

Since we now have differential equations for our multiple integral (1.1), we can apply a general theory for higher-order differential equations developed in [6]. Concerning turning points, we say the point q_n^T is a turning point in the ordinary sense if the equation $\sigma(\tilde{H}) = 0$ for ξ has a double root. In this case, we have,

$$\frac{dq_n^T(q_0, q_1)}{dq_1} = 0, \tag{3.1}$$

$$\frac{dS(q_0, q_1, \cdots, q_n^T(q_0, q_1))}{dq_1} = 0. \tag{3.2}$$

Also we follow the definition of virtual turning points given in [6], that is, for $q_1^{(i)} \neq q_1^{(j)}$, q_n^T is a virtual turning point if

$$q_n^T(q_0, q_1^{(i)}) = q_n^T(q_0, q_1^{(j)}) \tag{3.3}$$

$$S(q_0, q_1^{(i)}, \cdots, q_n^T(q_0, q_1^{(i)})) = S(q_0, q_1^{(j)}, \cdots, q_n^T(q_0, q_1^{(j)})) \tag{3.4}$$

In the same way, we can apply the definition of Stokes curves given in the previous section. We say the curves emanating from the turning points q_n^T and satisfying the following relation Stokes curves;

$$\operatorname{Im} S(q_0, q_1^{(i)}, \cdots, q_{n-1}^{(i)}, q_n^T) = \operatorname{Im} S(q_0, q_1^{(j)}, \cdots, q_{n-1}^{(j)}, q_n^T). \tag{3.5}$$

Stokes curves emanating from ordinary turning points give the ones in the ordinary sense, and those from virtual turning points give new Stokes curves.

Since solving bicharacteristic equation is equivalent to expressing the initial value q_0 as a function of final values q_{n-1} and q_n, it is easy to draw bicharacteristic curves on $(q_n, S(q_0, \cdots, q_n))$ plane. Fig. 1 displays bicharacteristic curves for $n = 4$. We can see that ordinary turning points form cuspidal shape and virtual turning points are just crossing points on the plane as predicted.

A more concrete procedure to draw ordinary and new Stokes curves is to find curves satisfying the relation (3.5) on a circle C with a sufficiently large radius. The curves thus found include not only ordinary but also new Stokes curves. The number of ordinary Stokes curves is finite because the equation $q_n(q_0, q_1) = const$ is an algebraic equation and therefore the number of solutions is finite.

The ordinary turning points are obtained either by the algebraic equations (3.1) and (3.2), or by extending Stokes curves found on a circle C to locate the position at which three Stokes curves meet. Also the virtual turning points are found by extending Stokes curves found on a circle C to locate the position satisfying (3.5).

As mentioned, the ordinary turning points give folding points on the Lagrangian manifold, which is a manifold drawn on (q_n, q_{n-1}) plane. The ordinary turning poitns have such a clear geometrical meaning in chaotic time evolution, but one usually does not take into account the crossing points of Lagrangian manifolds on $(q_n, S(q_0, \cdots, q_n))$ plane in the description of chaotic dynamics. However, as discussed here, we cannot discuss Stokes geometry without those objects. Otherwise stated, one can say that self-intersection points of Lagrangian manifolds in $(q_n, S(q_0, \cdots, q_n))$ representation can only be interpreted quantum mechanically.

Appendix A. Proof of Proposition 1

Let us consider a differential map:

$$(x, y) \to (X(x, y), Y(x, y)), \tag{A.1}$$

and assume that it has an inverse

$$(X, Y) \to (x, y). \tag{A.2}$$

A small change of (x, y) cause a variation of (X, Y). They are governed by

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = J \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix}, \quad J \equiv \begin{pmatrix} J_{xX} & J_{xY} \\ J_{yX} & J_{yY} \end{pmatrix}, \tag{A.3}$$

where $\partial_x \equiv \partial/\partial x$, $J_{xX} \equiv \partial X/\partial x$, etc, or

$$\begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} = J^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}. \tag{A.4}$$

To be specific we consider the case in which y is fixed while x is changed. We introduce the notation

$$Q(x) \equiv X(x, y), \quad P(x) \equiv Y(x, y). \tag{A.5}$$

Then from (A.3) he variations of (Q, P) are given by

$$\frac{dQ}{dx} = J_{xX}, \quad \frac{dP}{dx} = J_{xY}. \tag{A.6}$$

The proof of the proposition is as follows: First applying (A.4) to H yields

$$\begin{pmatrix} \partial_Q H \\ \partial_P H \end{pmatrix} = J^{-1} \begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix}. \tag{A.7}$$

If we impose the condition that H satisfies

$$J^{-1} \begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \begin{pmatrix} -J_{xY} \\ J_{xX} \end{pmatrix}, \tag{A.8}$$

and compare with (A.5), the Hamiltonian equations (2.7) follow. To solve (A.8) for H, we multiply J from the left and obtain

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = J^{-1} \begin{pmatrix} -J_{xY} \\ J_{xX} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}, \end{pmatrix} \tag{A.9}$$

hence

$$\partial_x H = 0, \quad \partial_y H = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x} = \det J. \tag{A.10}$$

Therefore (2.5) is obtained.

Appendix B. Derivation of (2.11) and (2.12)

In this appendix we will derive the relation:

$$\frac{dS}{dq_1} = \frac{\partial\sigma(\hat{\mathcal{H}}(q_n, S, \xi, \eta))}{\partial\eta}, \tag{B.1}$$

First we give the right-hand side via the chain rule,

$$\begin{aligned}
&\hat{H}(q_1, q_2) + \eta\frac{\partial}{\partial\eta}\hat{H}(q_1(q_n - \eta^{-1}\xi, q_n), q_2(q_n - \eta^{-1}\xi, q_n)) \\
&= \hat{H}(q_1, q_2) + \eta\Big(\frac{\partial\hat{H}}{\partial q_1}\frac{dq_1}{dq_{n-1}}\frac{\partial q_{n-1}}{\partial\eta} + \frac{\partial\hat{H}}{\partial q_2}\frac{dq_2}{dq_{n-1}}\frac{\partial q_{n-1}}{\partial\eta}\Big) \\
&= \hat{H}(q_1, q_2) + \frac{\xi}{\eta}\Big(\frac{\partial\hat{H}}{\partial q_1}\big(\frac{\partial q_1}{\partial q_2}\frac{dq_2}{dq_{n-1}} + \frac{\partial q_1}{\partial q_3}\frac{dq_3}{dq_{n-1}}\big) + \frac{\partial\hat{H}}{\partial q_2}\big(\frac{\partial q_2}{\partial q_3}\frac{dq_3}{dq_{n-1}} + \frac{\partial q_2}{\partial q_4}\frac{dq_4}{dq_{n-1}}\big)\Big) \\
&= \hat{H}(q_1, q_2) + \cdots.
\end{aligned} \tag{B.2}$$

Putting

$$F_i \equiv \frac{dq_i}{dq_1}, \tag{B.3}$$

then we have the recursion relation,

$$F_i = \partial_{i,i-1}F_{i-1} + \partial_{i,i-2}F_{i-2}, \tag{B.4}$$

with initial conditions $F_1 = 1, F_2 = a_1 = dq_2/dq_1$. Here, $\partial_{i,i-1} \equiv \partial q_i/\partial q_{i-1}$.

For the left-hand side of eq. (B.2), using the condition $\partial S/\partial q_i = 0$ ($i = 1, 2, \cdots, n-1$), we have,

$$\begin{aligned}
&\frac{\partial S}{\partial q_1} + \Big(\frac{\partial S}{\partial q_3}\frac{dq_3}{dq_1} + \frac{\partial S}{\partial q_4}\frac{dq_4}{dq_1} + \cdots + \frac{\partial S}{\partial q_{n-1}}\frac{dq_{n-1}}{dq_1}\Big) + \frac{\partial S}{\partial q_n}\frac{dq_n}{dq_1} \\
&= \hat{H}(q_1, q_2) + \frac{\partial S}{\partial q_n}\frac{dq_n}{dq_1} \\
&= \hat{H}(q_1, q_2) + \frac{\xi}{\eta}\Big(\frac{\partial q_n}{\partial q_{n-1}}\frac{dq_{n-1}}{dq_1} + \frac{\partial q_n}{\partial q_{n-2}}\frac{dq_{n-2}}{dq_1}\Big) \\
&= \hat{H}(q_1, q_2) + \frac{\xi}{\eta}\Big\{\frac{\partial q_n}{\partial q_{n-1}}\Big(\frac{\partial q_{n-1}}{\partial q_{n-2}}\frac{dq_{n-2}}{dq_1} + \frac{\partial q_{n-1}}{\partial q_{n-3}}\frac{dq_{n-3}}{dq_1}\Big) + \frac{\partial q_n}{\partial q_{n-2}}\Big(\frac{\partial q_{n-2}}{\partial q_{n-3}}\frac{dq_{n-3}}{dq_1} + \frac{\partial q_{n-2}}{\partial q_{n-4}}\frac{dq_{n-4}}{dq_1}\Big)\Big\} \\
&= \hat{H}(q_1, q_2) + \cdots
\end{aligned} \tag{B.5}$$

Similarly, this gives the recursion relation,

$$G_i = \partial_{n-i,n-i+1}G_{i-1} + \partial_{n-i,n-i+2}G_{i-2}, \tag{B.6}$$

by putting

$$G_i \equiv \partial_{dq_0,dq_{i-1}}, \tag{B.7}$$

where the initial conditions are given as $G_0 = 1, G_1 = a_{n-1} = dq_n/dq_{n-1}$.

Using the relation,

$$\frac{\partial q_i(q_{i-1}, q_{i-2})}{\partial q_{i-1}} = \frac{\partial q_{i-2}(q_i, q_{i-1})}{\partial q_{i-1}} = 2(1 + q_{i-1}) \tag{B.8}$$

$$\frac{\partial q_{i+1}(q_i, q_{i-1})}{\partial q_{i-1}} = \frac{\partial q_{i-2}(q_i, q_{i-1})}{\partial q_i} = -1 \tag{B.9}$$

in order (B.1) holds, it is sufficient to show $F_n = G_n$, each of which is determined by the recursion relations,

$$F_i = a_{i-1}F_{i-1} + a_0 F_{i-2} \quad \text{where} \quad F_0 = 1, F_1 = a_1, \tag{B.10}$$

$$G_j = a_{n-i+1}G_{i-1} + a_0 G_{i-2} \quad \text{where} \quad G_0 = 1, G_1 = a_{n-1}. \tag{B.11}$$

Here note that (B.10) and (B.11) is identical by replacing a_{n-i+1} by a_{i-1} in (B.11). This means that what we have to prove is that F_n is invariant under the exchange between a_{n-i+1} and a_{i-1}.

Since a_0 is a constant, we express F_n as a power series with respect to a_0,

$$F_n = F_n^{(0)} + a_0 F_{n-2}^{(1)} + a_0^2 F_{n-4}^{(2)} + a_0^3 F_{n-6}^{(3)} + \cdots, \tag{B.12}$$

and each coefficient $F_n^{(j)}$ $(j = 1, 2, \cdots)$ is determined by the recursion relations:

$$F_{n+1}^{(0)} = a_n F_n^{(0)} \qquad F_1^{(0)} = 1 \tag{B.13}$$

$$F_{n+1}^{(1)} = a_{n+2} F_n^{(1)} + F_{n+1}^{(0)} \qquad F_1^{(1)} = 1 \tag{B.14}$$

$$F_{n+1}^{(2)} = a_{n+4} F_n^{(2)} + F_{n+1}^{(1)} \qquad F_1^{(2)} = 1 \tag{B.15}$$

$$F_{n+1}^{(3)} = a_{n+6} F_n^{(3)} + F_{n+1}^{(2)} \qquad F_1^{(3)} = 1 \tag{B.16}$$

$$\vdots$$

From eq. (B.13), we immediately have

$$F_n^{(0)} = \prod_{\nu=1}^{n+1} a_\nu, \tag{B.17}$$

and this term is invariant under the exchange between a_i and a_{n-i} in itself.

Next, from the recursion relation (B.14), we obtain

$$F_n^{(1)} = \sum_{i=1}^{n} \left(\prod_{\nu=i+2}^{n+2} a_\nu \right) F_i^{(0)} \quad (a_{n+2} = 1) \tag{B.18}$$

$$= \sum_{i=1}^{n} F_n^{(1)}(i). \tag{B.19}$$

where an explicit form (B.17) gives

$$F_n^{(1)}(i) = (a_{n+1} \cdots a_{i+2})(a_{i-1} \cdots a_1). \tag{B.20}$$

It is easy to check that the term $F_n^{(1)}(i)$ is mapped into the term $F_n^{(1)}(n - i + 1)$ by the exchange between a_m and a_{n+2-m} $(m = 1, 2, \cdots, n+1)$.

In the same way, the recusion relation (B.13) gives,

$$F_n^{(2)} = \sum_{i=1}^{n} \left(\prod_{\nu=i+4}^{n+4} a_\nu \right) F_i^{(1)} \quad (a_{n+4} = 1) \tag{B.21}$$

$$= \sum_{i=1}^{n} \sum_{j \le i} F_n^{(2)}(i, j). \tag{B.22}$$

Due to the form (B.20), we have

$$F_n^{(2)}(i,j) = (a_{n+3}\cdots a_{i+4})(a_{i+1}\cdots a_{j+2})(a_{j-1}\cdots a_1). \tag{B.23}$$

This form again allows us to see $F_n^{(1)}(i,j)$ and $F_n^{(1)}(n-j+1, n-i+1)$ are mapped each other under the exchange between a_m and a_{n+4-m} $(m = 1, 2, \cdots, n+3)$.

The same calculation in general yields,

$$F_n^{(N)} = \sum_{i=1}^{n}\Big(\prod_{\nu=i+2N}^{n+2N} a_\nu\Big) F_i^{(N-1)} \quad (a_{n+2N} = 1) \tag{B.24}$$

$$= \sum_{i_1=1}^{n}\sum_{i_2\le i_1}\cdots\sum_{i_n\le i_{N-1}} F_n^{(N)}(i_1, i_2, \cdots, i_N), \tag{B.25}$$

and an explicit form of each term is given as,

$$F_n^{(N)}(i_1, i_2, \cdots, i_N) = (a_{n+2N-1}\cdots a_{i_1+2N})(a_{i_1+2(N-1)-1}\cdots a_{i_2+2(N-1)}) \\ \cdots (a_{i_{N-1}+1}\cdots a_{i_N+2})(a_{i_N-1}\cdots a_1). \tag{B.26}$$

From this form, we can see $F_n^{(N)}(i_1, i_2, \cdots, i_N)$ is mapped into $F_n^{(N)}(n - i_N + 1, n - i_{N-1} + 1, \cdots, n - i_2 + 1, n - i_1 + 1)$ under the exchange between a_m and a_{n+2N-m} $(m = 1, 2, \cdots, n + 2N - 1)$.

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