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A priori bounds for global solutions of nonlinear heat equations in general domain

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Abstract

We consider the initial-boundary value problem of some semilinear parabolic equations with superlinear and subcritical nonlinear terms. In this paper, we consider global solutions, which could be sign-changing, and estimate the dependence of upper bounds of global solutions on some norm of the initial data. Furthermore, we allow (possibly unbounded) general domain with boundary.

1 INTRODUCTION

In this note, we consider the following initial-boundary value problem of the following semi-linear heat equations.

$$(\mathbf{P})\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,u(x,t)), & (x,t)\in\Omega\times[0,\infty) \\ u(x,0) = u_0(x), & x\in\Omega \\ \frac{\partial u}{\partial n}(x,t) + \sigma u(x,t) = 0, & (x,t)\in\partial\Omega\times[0,\infty),\ \sigma>0. \end{cases}$$

Here Ω is a general domain, which could be unbounded, with uniformly $C^{2,\alpha}$-class smooth boundary $\partial\Omega$.
The nonlinearity to be considered here is given as follows:

$$(f)\begin{cases} \text{f}(\cdot,\cdot) \text{ is a continuous function from } \Omega\times\mathbb{R}^1 \text{ into } \mathbb{R}^1 \text{ and there exist} \\ \text{constants } K_i (i=0,1,2) \text{ and numbers } p\in(2,2^*),\ \delta>0 \text{ and } \epsilon>0 \\ \text{such that} \\ \text{(i) } |f(x,u)| \le K_0(|u|+|u|^{p-1}), \quad \forall(x,u)\in\Omega\times\mathbb{R}^1, \\ \text{(ii) } \int_0^u f(x,t)dt \ge K_1|u|^{2+\delta} - K_2, \quad \forall(x,u)\in\Omega\times\mathbb{R}^1, \\ \text{(iii) } uf(x,u) \ge (2+\varepsilon)\int_0^u f(x,t)dt,\ \forall(x,u)\in\Omega\times\mathbb{R}^1, \end{cases}$$

where 2^* is the Sobolev's critical exponent given by ∞ for the cases $N = 1$ or 2, and $2N/(N-2)$ for the cases $N \geq 3$.
The initial data u_0 is always taken from $L^2(\Omega) \cap L^\infty(\Omega)$ and is not necessarily assumed to be of definite sign.
We here recall that standard argument assures the existence of unique local solution. We denote by T_m the "maximal existence time" of the solution.
Then the first possible case is that T_m is finite, that is to say, T_m gives the blow up time of the solution. It is well known that the L^∞-norm of solution tends to infinity as t approaches the blow up time T_m.
Concerning this case, there are many studies on the blow up rate as well as the profile of blow up solutions.
The other case is that T_m is equal to infinity, in other words, the solution can be continued globally.
For example, if Ω is bounded and the initial data is taken small enough, then the problem admits a global solution.
Our main interest here is the asymptotic behavior of global solutions. More precisely, the problem admits a growing up solution or not? Here by the growing up solution, we mean a global solution whose norm (say L^∞-norm) tend to infinity as t goes to infinity. For this question, the anser is no and so we can show the boundedness of solution.
Furtheremore we may ask how is the dependence of the bound on the initial data. For this question, under additional assumptions, the answer is yes and the supremun of the norm of solution is some constant which depends on the norm of the initial data.
The boundedness of global solution was first reported by M.Otani[1] in 1980. He proved the H_0^1-norm boundedness of global solutions for superlinear and subcritical power nonlinearity.
Ni-sacks-Tavantzis[2] obtained the L^∞-norm boundedness, but they assumed that the solution is positive, the domain is convex and the power nonlinearity should be strictly less than $2 + 2/N$.
Cazenave-Lions[3] also derived the L^∞-norm boundedness for a fairly general nonlinearity of subcritical growth order.
As for the bound dependence on the initial data. Cazenave-Lions[3] also showed that the bound depends only on the L^∞ norm of initial data, if the growth order p is strictly less than 2_*, which is given such that $2_* = \infty$ for $N = 1$; $2_* = 2+12/(3N-4)$ for $N \geq 2$, (2_* is always less than 2^*).
Giga[4] excluded this restriction up to the subcritical case for positive solutions, and Quittner[5] extended Giga's result also for sign-changing solutions.

The results quoted above are all concerned with bounded domains and homogeneous Dirichlet boundary conditions.
Recently Otani and T[6] excluded the boundedness condition on domains for non-negative solutions.
The main purpose of this note is to discuss another type of boundary condition in general domains.

2 MAIN RESULTS

Our main results are stated as follows.
Theorem I Let (f) be satisfied and u be a global solution of (P) such that $u \in V \equiv W^{1,2}_{loc}([0,\infty);L^2(\Omega)) \bigcap L^2_{loc}([0,\infty);H^2(\Omega))$. Then there exists a positive constant $C_0 = C_0(|u_0|_{H^1}, K_0, K_1, K_2, \delta, \varepsilon)$ such that

(1) $\sup_{t\geq 0} |u(t)|_{L^2} \leq C_0$,

(2) $\sup_{t\geq 0} |u(t)|_{H^1} < +\infty$,

(3) There exists a number T_1 such that $\sup_{t\geq T_1} |u(t)|_{H^1} \leq C_0$,

(4) $\sup_{t\geq 0} |u(t)|_{H^1} \leq C_0$, provided that $p \in (2, 2_*)$,
where $2_* = \infty$ for $N = 1$ and $2_* = 2 + 12/(3N-4)$ for $N \geq 2$.

Theorem II Let (f) be satisfied and u be a global solution of (P) such that $u \in L^\infty_{loc}([0,\infty);L^\infty(\Omega)) \bigcap W^{1,2}_{loc}((0,\infty);L^2(\Omega)) \bigcap L^2_{loc}((0,\infty);H^2(\Omega))$. Then there exists a positive constant $C_1 = C_1(|u_0|_{L^2}, |u_0|_{L^\infty}, K_0, K_1, K_2, \delta, \varepsilon)$ such that

(5) $\sup_{t\geq 0} |u(t)|_{L^\infty} < +\infty$,

(6) There exists a number T_1 such that $\sup_{t\geq T_1} |u(t)|_{L^\infty} \leq C_1$,

(7) $\sup_{t\geq 0} |u(t)|_{L^\infty} \leq C_1$, provided that $p \in (2, 2_*)$.

3 ENERGY IDENTITY AND INEQUALITY

Our basic tools here are energy estimates and the phase plane argument. We first prepare one energy identity and another energy inequality. The first one is derived from the multiplication of the equation by u_t.

$$\int_\Omega u_t^2 dx - \int_\Omega \Delta u \cdot u_t = \int_\Omega f(x,u) \cdot u_t dx$$

As for the second term of the left-hand side, we apply the integration by parts to get two integrations on the boundary $\partial\Omega$ and in Ω.

$$\int_\Omega \Delta u \cdot u_t dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot u_t d\Gamma - \int_\Omega \nabla u \cdot \nabla u_t$$

On the boundary $\partial\Omega$, by virtue of the boundary condition, $\partial u/\partial n$ can be replaced by -σ·u,

$$= \int_{\partial\Omega} -\sigma \cdot u_t - \int_\Omega \nabla u \cdot (\nabla u)_t dx$$

Then, it is clear that these two terms can be expressed as the time-derivative of the functional given by,

$$= -\frac{d}{dt}\{\frac{\sigma}{2}\int_{\partial\Omega} u^2 d\Gamma + \frac{1}{2}\int_\Omega |\nabla u|^2 dx\}$$

Using this positive definite functional $(\frac{\sigma}{2}\int_{\partial\Omega} u^2 d\Gamma + \frac{1}{2}\int_\Omega |\nabla u|^2 dx)$ denoted by $A(u)$ and the primitive function $F(\cdot)$ of f(x,t) evaluated at u(x), we get

$$\begin{aligned}\int_\Omega u_t^2 dx &= -\frac{d}{dt}\{\frac{\sigma}{2}\int_{\partial\Omega} u^2 d\Gamma + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \int_0^{u(x)} f(x,t) dt dx\} \\ &= -\frac{d}{dt}\{A(u(t)) - F(u(t))\}\end{aligned}$$

Thus, by putting $J(u) = A(u) - F(u)$, we obtain the energy identity:

$$(8) \quad \frac{d}{dt}J(u) = -\int_\Omega u_t^2 dx$$

In particular, this identity implies that $J(u(t))$ is monotone decreasing in time t and this imformation plays an essential role in the following arguments.

To get the second energy inequality, we multiply the equation by u.

$$\int_\Omega u_t \cdot u dx - \int_\Omega \Delta u \cdot u dx = \int_\Omega f(x,u) \cdot u dx$$

Then, again by the integration by parts, we get the identity (here and henceforth we denote the L^2-norm by $\|\cdot\|$):

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= \int_{\partial\Omega} -\sigma u \cdot u - \int_\Omega \nabla u \cdot \nabla u dx + \int_\Omega f(x,u)\cdot u dx \\
&= -\|\nabla u\|^2 - \sigma\|u\|^2 + \int_\Omega f(x,u)\cdot u dx \\
&= -2A(u(t)) + \int_\Omega f(x,u)\cdot u dx
\end{aligned}
$$

Moreover, by using the third condition of the structure condition of f,

$$\geq \ (2+\varepsilon)F(u(t)) - 2A(u(t))$$

$$\equiv j(u(t)) \tag{9}$$

4 PHASE PLANE

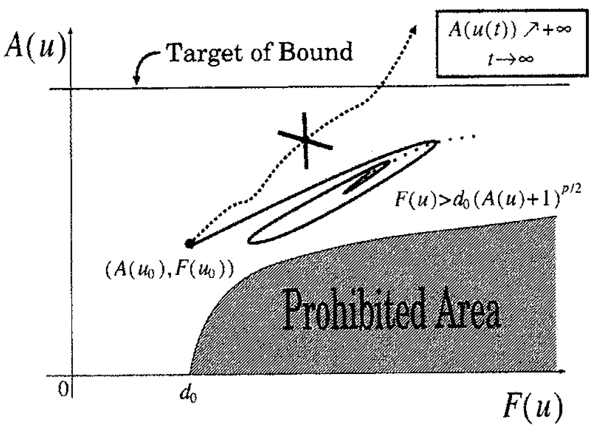

Figure 1:

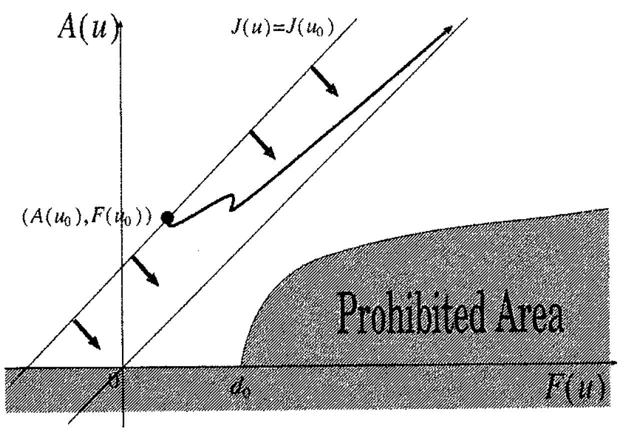

Figure 2:

We here introduce the phase-plane where we work. The vertical axis designates A(u), which is defined by $\frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\sigma\|u\|^2_{L^2(\partial\omega)}$. The horizontal axis designates F(u), which is defined by $\int_\Omega \int_0^{u(x)} f(x,t)dtdx$. By the subcriticality of nonlinear

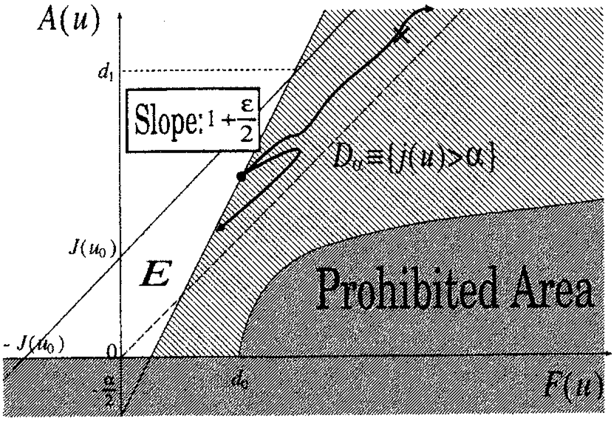

Figure 3:

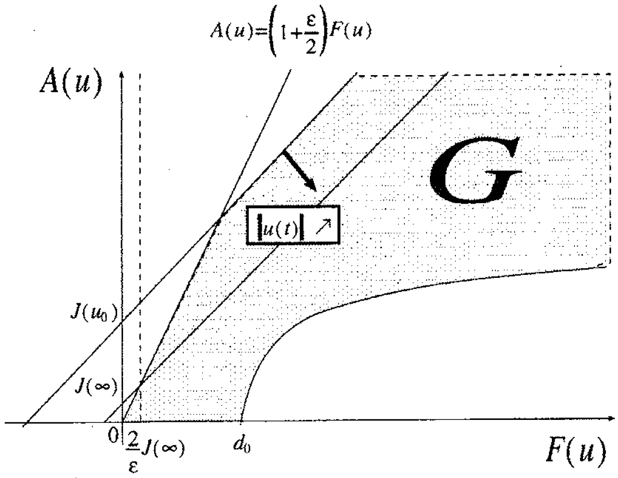

Figure 4:

term f and Sobolev's embedding theorem, F(u) is bounded by $d_0(A(u)+1)^{p/2}$, where d_0 depends only on k_0 and p. In fact, we have

$$
\begin{aligned}
F(u) &= \int_\Omega \int_0^{u(x)} f(x,t)dtdx \le \int_\Omega \int_0^{|u(x)|} |f(x,t)|dtdx \le \int_\Omega \int_0^{|u(x)|} K_0(|t|+|t|^{p-1})dtdx \\
&= K_0 \int_\Omega (\frac{1}{2}|u(x)|^2 + \frac{1}{p}|u(x)|^p)dx = \frac{K_0}{2}\|u\|^2 + \frac{K_0}{p}\|u(x)\|_p^p \\
&\le \frac{K_0}{2}\|u\|^2 + \frac{K_0}{p}(\|u\| + \|\nabla u\|)^p \le d_0'(\|u\| + \|\nabla u\| + 1)^p \le d_0(A(u)+1)^{p/2}.
\end{aligned}
$$

Therefore, there is no element of H_0^1 in the dotted region in Fig.1, which we call "prohibited area". Thus the initial data should be located outside of this region and the orbit of any sulotion can not enter into this prohibited area. If the orbit of solution grows up in such way along the dotted curve, A(u) may go to infinity.
On the other hand, if we can show that the orbit stays on the bounded curve, we can obtain the a proiri bound for A(u(t)).

In order to analyze the trajectory of the solution, we first introduce $J(u)$-line. By the definition of the energy functional $J(u) = A(u) - F(u)$, the slope of $J(u)$-line is one.
Furtheremore, the first energy identity (8) tells us that J(u) is monotone decreasing. Therefore, if the initial data is located on the $J(u_0)$-line, the orbit of the corresponding solution should be confined below the $J(u_0)$-line. However, there still exists a possibility that A(u) may grow up if the orbit goes far away along the curve drawn

in Fig.2.
In order to exclude this possibility, we need the additional arguments. To this end, we introduce a new region D_α characterized by the value of $j(u)$ as follows.

Our new shaded region D_α is defined as the area where $j(u) > \alpha$.(Fig.3)
It is clear that if u stays outside of D_α, say in E, then A(u) is bounded.
Furthermore, in the following arguments, we are going to show that u can not stay in the region D_α long enough to grow up to infinity.

5 KEY LEMMA

First of all, we derive the boundedness of the L^2-norm of solutions. To do this, we rely on the argument due to Phillip Souplet[7].
Lemma
Let $p > 1$ and u(x,t) be a global solution of this problem. Then for any positive number ε', there exists a large enough time t_1 such that

$$F(u(t_1)) \leq \frac{2}{\epsilon} J(\infty) + \varepsilon'$$

(Sketch of Proof)

$$\text{Set } \int_\Omega u^2(t)dx = f(t), \text{then}$$

$$\frac{f(t) - f(s)}{2} = \int_s^t \int_\Omega u \cdot u_t dxdt$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} &\leq (\int_s^t \int_\Omega u_t^2)^{1/2} (\int_1^t u^2)^{1/2} \\ &\leq (J(u(s)) - J(\infty))^{1/2} (\int_1^t f(s)ds)^{1/2} \end{aligned}$$

Here, by Levine's result[8], it is known that $J(u)$ is bounded below and $J(u(t))$ converges to $J(\infty)$. Hence it is easy to see that $f(t)$ should behave as small order of t as t tends to ∞. Multiplying the equation by u and integrating on Ω, we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 \geq \varepsilon F(u) - 2J(u),$$

Then

$$\frac{1}{2}(\|u(T+1)\|^2 - \|u(1)\|^2) \geq \varepsilon \int_1^{T+1} F(u)dt$$

Therefore,

$$\frac{1}{T}\int_1^{T+1} F(u)dt \leq \frac{1}{\varepsilon}(\frac{f(T+1)-f(1)}{2T} + \frac{2}{T}\int_1^{T+1} J(u)dt)$$

Hence, for any $\varepsilon' > 0$ we can find a large enough time t_1, such that

$$F(u(t_1)) \leq \frac{2}{\varepsilon}J(\infty) + \varepsilon' \qquad Q.E.D.$$

Since, in the region G(Fig.4), the L^2-norm of u is increasing, and this lemma says that u comes back around $J(u) = \frac{2}{\varepsilon}J(\infty)$ line at the time $t = t_1$, we can conclude that the L^2-norm of u should be bounded.

6 H^1 BOUND

Next, we are going to show that the growth rate of $A(u)$ can be controlled for a short time. To see this, we multiply the equation by $-\Delta u$, then we get

$$\frac{d}{dt}A(u(t)) + \|\Delta u(t)\|^2 \leq C \cdot \|F(u)\|| \cdot \|\Delta u\|$$

Here, by virtue of the subcritical growth condition on f, we can obtain

$$\|F(u)\|^2 \leq \frac{1}{2}\|\Delta u\|^2 + M(A(u)),$$

where $M(\cdot)$ is a monotone increasing function. Plugging this estimate to the previous inequality, we get

$$\frac{d}{dt}A(u(t)) \leq C \cdot M(A(u(t)))$$

Hence, it is rather easy to show that by putting $T(r) = 1/2M(r+1)$, we see

$$(10) \quad A(u(t)) \leq A(u(t_0)) + 1 \qquad \text{for all} t \in [t_0, t_0 + T(A(u(t_0)))]$$

We here claim that
Solution $u(t)$ cannot stay in Region D_α longer than $T(d_1)$.
(Sketch of Proof)

By the definition of $D_\alpha = \{A(u) < (1+\frac{\epsilon}{2})F(u) - \alpha\}$, we can easily find that if the solution remains in the domain D_α during $[t_0, t_1]$, then $j(u) > \alpha$ holds for all $t \in [t_0, t_1]$. Integrating the both side from t_0 to t_1, we get

$$\alpha(t_1 - t_0) \le \int_{t_0}^{t_1} j(u(t))dt$$

By the energy inequality (9),

$$\le \int_{t_0}^{t_1} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 dt \quad \le \int_{t_0}^{t_1} \|u_t(t)\| \cdot \|u(t)\| dt$$

Using the L^2-bound of the solution, we have

$$\le K_4(t_1 - t_0)^{1/2}(\int_{T_0}^{\infty} \|u_t(t)\|^2)^{1/2}, \quad \forall t_1 > t_0 \ge T_0$$

On the other hand, there exists T_0 such that

$$(t_1 - t_0) \le K_4^2 \int_{T_0}^{\infty} \|u_t(t)\|^2 dt \le T(d_1)$$

Hence by virtue of (10), we obtain

$$A(u(t)) \le d_1 + 1 \quad \forall t \in [t_0, t_1]$$

Thus a priori bound for $A(u(t))$ is derived.
Now the rest part of Theorem I and Theorem II can be proved by the same arguments in Ôtani[9].

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