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Generation of an interface of competition -diffusion systems with large interaction [1]

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1 Introduction

In this paper we are concerned with the following reaction-diffusion system:

$$
\text{(P)} \qquad \begin{cases} u_t = \Delta u + f(u,v) - \dfrac{buv}{\epsilon^3} & \text{in } \Omega \times (0,T), \\ v_t = D\Delta v + g(u,v) - \dfrac{cuv}{\epsilon^3} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) > 0, \ v(x,0) = v_0(x) > 0 & \text{in } \Omega. \end{cases}
$$

Here b, c, D, and ϵ are positive constants, Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, and ν stand for the outward unit normal vector on $\partial\Omega$. We suppose that f and g are C^1-functions in $\mathbf{R}^2_+$. Unknown functions are $u = u(x,t)$, $v = v(x,t)$, and u_0, v_0 are supposed to belong to $C^2(\Omega) \cap C^2(\overline{\Omega})$.

Reaction-diffusion systems are often studied to explain some pattern formation arising in physics, chemistry, and mathematical biology. If two reaction-terms in (P) satisfy

$$
\frac{\partial}{\partial v}\Big(f(u,v) - \frac{buv}{\epsilon^3}\Big) \le 0, \quad \frac{\partial}{\partial u}\Big(g(u,v) - \frac{cuv}{\epsilon^3}\Big) \le 0 \quad \text{for } (u,v) \in \mathbf{R}^2_+,
$$

then (P) is called a **competition-diffusion system**. As is seen in the following section, competition-diffusion system has an order preserving property: If an initial data satisfy

$$
u_1(x,0) \le u_2(x,0) \quad \text{and} \quad v_1(x,0) \ge v_2(x,0) \quad \text{for } x \in \Omega,
$$

[1] This is a joint work with Kimie Nakashima (Tokyo university of Marine Science and Technology).

then it holds that

$$u_1(x,t) < u_2(x,t) \quad \text{and} \quad v_1(x,t) < v_2(x,t) \quad \text{for } (x,t) \in \Omega \times (0,T).$$

Hence comparison argument can be applied to the above systems similarly as to single reaction-diffusion equations. We assume that

$$f(0,v) = 0,\ g(u,0) = 0,\ f_v(u,v) \le 0, \text{ and } g_u(u,v) \le 0, \quad \text{for } (u,v) \in \mathbf{R}_+^2. \quad \text{(A1)}$$

Assumption (A1) implies that (P) is a competition-diffusion system, and it assures that, if $u_0(x)$ and $v_0(x) > 0$, then $u(x,t) > 0$ and $v(x,t) > 0$ for $t > 0$ especially.

In this paper we will study the behavior of classical solutions of (P) when ϵ is sufficiently small (the strong interaction between u and v). For simplicity, let f and g be given by the following two logistic nonlinearities

$$f(u,v) = u(a-u),\ g(u,v) = v(d-v) \quad (a,\ d > 0).$$

This pair of functions is a typical example of (f,g) which satisfies (A1), and in this case (P) is called Lotka-Volterra competition-diffusion system. This system describes the population dynamics in mathematical biology. Here u and v stand for population densities of two competitive species in a region Ω. In this case Dancer-Hilhorst-Mimura-Peletier [1] have studied (P) with weak topology in $H^1(\Omega)$, and obtained that (P) converges, in a certain sense, to the following **singular limit problem**

$$\text{(FBP)} \quad \begin{cases} U_t = \Delta U + f(U,V) & \text{in } \Omega_U(t) \times (0,T), \\ V_t = D\Delta V + g(U,V) & \text{in } \Omega_V(t) \times (0,T), \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ U = V = 0,\ c\dfrac{\partial U}{\partial \nu} = Db\dfrac{\partial V}{\partial \nu} & \text{on } \Gamma(t) \times (0,T), \\ U(x,0) = U_0(x) > 0 & \text{in } \Omega_U(0), \\ V(x,0) = V_0(x) > 0 & \text{in } \Omega_V(0), \end{cases}$$

as $\epsilon \downarrow 0$. Here $\Gamma(t)$ is a 1-parameter family of (smooth) hyper-surface with codimension 1, and is called **free boundary**. Furthermore $\Gamma(t)$ gives a situation that Ω is separated into two region $\Omega_U(t)$ and $\Omega_V(t)$ at each time t. $U(x,t)$ and $V(x,t)$ are defined in for $(x,t) \in \Omega_U(t) \times [0,T]$, and $(x,t) \in \Omega_V(t) \times [0,T]$. If necessary, we extend these two functions as those in $\Omega \times [0,T]$ with zero extension outside $\Omega_U(t)$ and $\Omega_V(t)$. Hence we can expect that if ϵ is sufficiently small, then behaviors of solutions to (P) are very close to those of (FBP). More precisely, formal analysis shows that (u,v) generate an **interface** for a short time-period, and after that, the interface moves like $\Gamma(t)$, which is also an unknown function of (FBP). In view of biology, these precesses lead us to an idea that two species begin to form each "habitat" in a short time period, each habitat moves with interface by population pressure. Both phenomena, formation of interfaces at the first stage and motion of those interfaces in the second stage, are quite different, it is necessary to consider each phenomena by a different approach. For the motion of interfaces in the second stage, Iida-Karali-Mimura-Nakashima-Yanagida [2] have recently given a mathematically rigorous analysis by using an approximated solution obtained through asymptotic analysis. They constructed upper- and lower-solutions which have interfaces close to $\Gamma(t)$, the solution of (FBP). Their results are summarized as follows: If an interfaces once appears, it begins to move like a solution of (FBP). They also point out the length of interfaces are $O(\epsilon)$ (so we take $O(\epsilon^{-3})$ for interaction rates in (P)).

In this paper we deal with the formation of interfaces in the first stage and show that any solution of (P) for a large class of initial data develops an interface in a very short time $O(\epsilon^2)$. In Section 2 we introduce approximate solutions of (P) defined by solutions of a system of ordinary differential equations, and formally discuss the generation of interfaces. In Section 3 and 4 we give our main result, and describe the sketch of proofs.

2 Formal Analysis

In this section we show the generation of interface formally: Roughly speaking, we discuss behaviors of approximate solutions corresponding to (P). First we introduce an important quantity $A(\xi,\eta)$ by

$$A(\xi,\eta) := c\xi - b\eta \quad (\xi,\eta > 0),$$

and we use following notations

$$\begin{aligned}
\omega(x) &:= A(u_0(x), v_0(x)),\\
\Omega_u &:= \{x \in \Omega |\ \omega(x) > 0\},\\
\Omega_v &:= \{x \in \Omega |\ \omega(x) < 0\},\\
\Gamma &:= \{x \in \Omega |\ \omega(x) = 0\}.
\end{aligned}$$

Here we assume that $\Gamma \neq \emptyset$ and

$$\inf_{x\in\Gamma} |c\nabla u_0(x) - b\nabla v_0(x)| > 0. \tag{A2}$$

Assumption (A2) assures that $\partial\Omega_0$ is an $N-1$ dimensional hypersurface with bounded mean curvature. Furthermore we can observe that

$$|\omega(x)| > C\mathrm{dist}(x,\Gamma)$$

(C is positive constant) in a neighborhood of Γ.

Let us study the behavior of solution of (P). Generally the first stage where interface develops, disappears in singular limit problem as $\epsilon \to 0$. In fact the initial free boundary which determine a partition of Ω must be given at $t = 0$ in (FBP). Hence it is natural that time period for the first stage goes to 0 as $\epsilon \to 0$. So we introduce a rescaled time variable

$$\tau := \frac{t}{\epsilon^3},$$

instead of t, to see behaviors of solutions of (P) in short time period (in proportion to ϵ). Additionally let $\tilde{u}(x,\tau) := u(x, \epsilon^3\tau)$ and $\tilde{v}(x,\tau) := v(x,\epsilon^3\tau)$. Then $\tilde{u}$ and $\tilde{v}$ satisfy

$$\begin{cases} \tilde{u}_t = \epsilon^3(\Delta\tilde{u} + f(\tilde{u},\tilde{v})) - b\tilde{u}\tilde{v}, \\ \tilde{v}_t = \epsilon^3(D\Delta\tilde{v} + g(\tilde{u},\tilde{v})) - c\tilde{u}\tilde{v}. \end{cases}$$

It follows from the above system that $\tilde{u}$ and $\tilde{v}$ are essentially determined by interaction terms. Hence we introduce two functions $\phi(\tau;\xi,\eta)$ and $\psi(\tau;\xi,\eta)$ defined by

$$\begin{cases} \dot{\phi} = -b\phi\psi, \quad \phi(0) = \xi > 0, \\ \dot{\psi} = -c\phi\psi, \quad \psi(0) = \eta > 0. \end{cases} \tag{ODEs}$$

Then we can expect that two functions would be good approximations of u and v for short time period:

$$\Phi_0(x,t) := \phi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big), \quad \Psi_0(x,t) := \psi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big)$$

In this paper Φ_0 and Ψ_0 are called the first approximate solutions of u and v. The behavior of Φ_0 and Ψ_0 is understood by that of ϕ and ψ at each $x \in \Omega$. Here we give basic properties of ϕ and ψ. Observe that $A(\phi(\tau),\psi(\tau))$ is preserved for any $\tau > 0$; so that

$$\phi(\tau;\xi,\eta) = \frac{\xi A e^{A\tau}}{A + c\xi(e^{A\tau} - 1)}, \quad \psi(\tau;\xi,\eta) = \frac{\eta A e^{-A\tau}}{A + b\eta(1 - e^{-A\tau})},$$

and

$$\lim_{\tau\to+\infty} \phi(\tau;\xi,\eta) = \max\Big\{\frac{A(\xi,\eta)}{c}, 0\Big\}, \quad \lim_{\tau\to+\infty} \psi(\tau;\xi,\eta) = \max\Big\{0, -\frac{A(\xi,\eta)}{b}\Big\}.$$

We note that ϕ and ψ are decreasing functions with respect to τ. An orbit for (ϕ,ψ) lies on a line $A(\xi,\eta) =$ Constant in (ξ,η)-phase plane. Finally we have

$$\Big|\phi(\tau;\xi,\eta) - \max\Big\{\frac{A(\xi,\eta)}{c}, 0\Big\}\Big| < \frac{1}{c\tau}, \quad \Big|\psi(\tau;\xi,\eta) - \max\Big\{0, -\frac{A(\xi,\eta)}{b}\Big\}\Big| < \frac{1}{b\tau}.$$

Therefore the following estimates for Φ_0 and Ψ_0 can be derived by using the above estimates and putting $t = \epsilon^2 \left(\tau = \frac{1}{\epsilon}\right)$:

$$\left|\Phi_0(x,\epsilon^2) - \max\left\{\frac{\omega(x)}{c}, 0\right\}\right| < \frac{\epsilon}{c}, \quad \left|\Psi_0(x,\epsilon^2) - \max\left\{0, -\frac{\omega(x)}{b}\right\}\right| < \frac{\epsilon}{b} \tag{1}$$

for $x \in \Omega$. Estimate (1) implies that the first approximate solution (Φ_0, Ψ_0) becomes close to the continuous function

$$\begin{cases} (\omega(x)/c,\ 0) & \text{in } \Omega_u, \\ (0,\ -\omega(x)/b) & \text{in } \Omega_v \end{cases}$$

at $t = \epsilon^2$. Therefore, if an initial data (u_0, v_0) satisfies (A2), then we can observe generation of interface for the first approximate solutions. Furthermore, such interface arises near Γ.

3 Main results

In this section we give our main results in this paper. In addition to (A1) and (A2), we assume the following conditions

$$u_0(x) > 0, \text{ and } v_0(x) > 0 \quad \text{in } \overline{\Omega}, \tag{A3}$$

and

$$\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{A4}$$

We will construct upper-lower functions (see Section 4), which are close to first approximate solution (Φ_0, Ψ_0), to show the following theorem.

Theorem 1. *Assume u_0, $v_0 \in C^2(\Omega)$, and (A1)-(A4). Then there exist positive numbers $\epsilon_0 > 0$ and $C_i > 0$ $(i = 1 \cdots 4)$ such that*

(i) $|u(x,t) - \Phi_0(x,t)| < C_1\epsilon, \quad |v(x,t) - \Psi_0(x,t)| < C_2\epsilon, \quad$ *for* $(x,t) \in \Omega \times (0, \epsilon^2)$.

$$\text{(ii)} \begin{cases} \left|u(x,\epsilon^2) - \max\left\{\frac{\omega(x)}{c}, 0\right\}\right| < C_3\epsilon, & \text{for } x \in \Omega, \\ \left|v(x,\epsilon^2) - \max\left\{0, -\frac{\omega(x)}{b}\right\}\right| < C_4\epsilon, & \text{for } x \in \Omega. \end{cases}$$

for $\epsilon \in (0, \epsilon_0)$.

Remark 3.1. We note that (A4) is a technical condition and we can obtain Theorem 1 without assuming (A4). In this case, some modification is required to construct our comparison functions in Section 4 (see [3]).

Theorem 1 gives a rigorous justification of generation of interfaces with a large class of initial data.

Finally we recall the result by Iida et.al [2], which have studied with the motion of interfaces for Lotka-Volterra competition diffusion system. In this case we can understand the behavior of solution to (P) for $t \in [0, T]$: An interface develops at the time $O(\epsilon^2)$, and motion of the interface can be characterized by free boundary problem, for $t \in [\epsilon^2, T]$.

Remark 3.2. We note that in [3], we obtain sharper estimates than those of Theorem 1 (ii) as follows:

$$|u(x, \epsilon^2)| < C_5 \exp\Big(- \frac{\mathrm{dist}(x, \Gamma)}{\epsilon}\Big)$$

in $\{x \in \Omega_v |\ \mathrm{dist}(x, \Gamma) > d_1 \epsilon |\log \epsilon|\}$,

$$|v(x, \epsilon^2)| < C_6 \exp\Big(- \frac{\mathrm{dist}(x, \Gamma)}{\epsilon}\Big)$$

in $\{x \in \Omega_u |\ \mathrm{dist}(x, \Gamma) > d_2 \epsilon |\log \epsilon|\}$ for some $d_1, d_2, C_5, C_6 > 0$. Above estimates assures that our comparison functions for the first stage, are between comparison functions in [2] for the second stage, at $t = \epsilon^2$.

4 Sketch of Proof

In this section we describe a sketch of the proof of Theorem 1. First we introduce some notations below:

$$\mathcal{L}_1(u, v) := u_t - \Delta u - f(u, v) + \frac{buv}{\epsilon^3},$$
$$\mathcal{L}_2(u, v) := v_t - D\Delta v - g(u, v) + \frac{cuv}{\epsilon^3}.$$

Our argument is based on the order-preserving property for competition diffusion systems. we begin with the definitions of upper- and lower-solutions.

Definition 1 (upper-lower solution). We say $(\overline{u}, \overline{v})$ is an upper-solution of (P) if it satisfies

$$\begin{cases} \mathcal{L}_1(\overline{u},\ \overline{v}) \geq 0,\ \mathcal{L}_2(\overline{u},\ \overline{v}) \leq 0, \\ \dfrac{\partial \overline{u}}{\partial \nu} \geq 0,\ \dfrac{\partial \overline{v}}{\partial \nu} \leq 0. \end{cases}$$

A lower-solution of (P) is defined by reversing the above inequality signs.

The order preserving property for (P) is given by following proposition.

Proposition 4.1 (Order preserving property). *Under* (A1) *let* $(\overline{u},\overline{v})$ *and* $(\underline{u},\underline{v})$ *be a upper-solution and a lower-solution of (P), respectively. If*

$$\underline{u}(x,0) \le u_0(x) \le \overline{u}(x,0), \ \ \textit{and}\ \underline{v}(x,0) \ge v_0(x) \ge \overline{v}(x,0),$$

then

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t), \ \ \textit{and}\ \underline{v}(x,t) \ge v(x,t)\overline{v}(x,t).$$

Proposition 4.1 is derived by the comparison principle for single parabolic equations.

Hence our method of proof is reduced to construct upper- and lower-solutions which approximate (u,v) in a short time period $O(\epsilon^2)$. Such comparison functions are constructed by modifying the first approximate solutions. We introduce the following function with use of perturbed terms $s_1(t,\epsilon)>0$ and $s_2(t,\epsilon)>0$:

$$\begin{cases} \Phi_\pm(x,t) := \phi\Big(\dfrac{t}{\epsilon^3}; u_0(x) \pm s_1(t,\epsilon), v_0(x) \mp s_2(t,\epsilon)\Big), \\ \Psi_\pm(x,t) := \psi\Big(\dfrac{t}{\epsilon^3}; u_0(x) \pm s_1(t,\epsilon), v_0(x) \mp s_2(t,\epsilon)\Big). \end{cases}$$

It should be noted that

$$\frac{\partial \Phi_\pm}{\partial \nu} = \frac{\partial \Psi_\pm}{\partial \nu} = 0$$

since (A4) assures that u_0 and v_0 satisfy the homogeneous Neumann boundary conditions. If we do not assume (A4), we have to modify $\Phi_\pm$ and $\Psi_\mp$ near the boundary again, so that these functions satisfy the boundary conditions for upper- and lower-solutions (for more details, see [3]).

Next we derive $\mathcal{L}_1(\Phi_\pm,\Psi_\pm)$ and $\mathcal{L}_2(\Phi_\pm,\Psi_\pm)$ as follows:

$$\begin{aligned} \mathcal{L}_1(\Phi_\pm,\Psi_\pm) =& \Big(\frac{1}{\epsilon^3}\phi_t \pm \phi_\xi \dot{s_1}(t,\epsilon) \mp \phi_\eta \dot{s_2}(t,\epsilon)\Big) - f(\phi,\psi) + \frac{b\phi\psi}{\epsilon^3} \\ & - (\phi_{\xi\xi}|\nabla u_0|^2 + 2\phi_{\xi\eta}\nabla u_0 \nabla v_0 + \phi_{\eta\eta}|\nabla v_0|^2 + \phi_\xi \Delta u_0 + \phi_\eta \Delta v_0) \\ =& \pm \phi_\xi\Big(\dot{s_1}(t,\epsilon) - \frac{\phi}{\phi_\xi}\frac{f(\phi,\psi)}{\phi} - \Delta u_0 \\ & - \frac{\phi_{\xi\xi}}{\phi_\xi}|\nabla u_0|^2 - 2\frac{\phi_{\xi\eta}}{\phi_\xi}\nabla u_0 \nabla v_0 - \frac{\phi_{\eta\eta}}{\phi_\xi}|\nabla v_0|^2\Big) \\ & \pm (-\phi_\eta)\Big(\dot{s_2}(t,\epsilon) + \Delta v_0\Big), \end{aligned} \tag{2}$$

and

$$
\begin{aligned}
\mathcal{L}_2(\Phi_\pm, \Psi_\pm) =& \Big(\frac{1}{\epsilon^3}\psi_t \pm \psi_\xi \dot{s_1}(t,\epsilon) \mp \psi_\eta \dot{s_2}(t,\epsilon)\Big) - g(\phi,\psi) + \frac{c\phi\psi}{\epsilon^3} \\
& - D(\psi_{\xi\xi}|\nabla u_0|^2 + 2\psi_{\xi\eta}\nabla u_0 \nabla v_0 + \psi_{\eta\eta}|\nabla v_0|^2 + \psi_\xi \Delta u_0 + \psi_\eta \Delta v_0) \\
=& \mp D\psi_\eta \Big(\frac{1}{D}\dot{s_2}(t,\epsilon) - \frac{\psi}{\psi_\eta}\frac{g(\phi,\psi)}{\psi} - \Delta v_0 \\
& - \frac{\psi_{\xi\xi}}{\psi_\eta}|\nabla u_0|^2 - 2\frac{\psi_{\xi\eta}}{\psi_\eta}\nabla u_0 \nabla v_0 - \frac{\psi_{\eta\eta}}{\psi_\eta}|\nabla v_0|^2\Big) \\
& \mp (-D\psi_\xi)\Big(\frac{1}{D}\dot{s_1}(t,\epsilon) - \Delta v_0\Big).
\end{aligned}
\tag{3}
$$

In (2) and (3) we have used

$$
\phi_\xi = \phi_\xi\Big(\frac{t}{\epsilon^3}; u_0(x) \pm s_1(t,\epsilon), v_0(x) \mp s_2(t,\epsilon)\Big), \text{ etc.}
$$

We will use the following two lemmas in order to determine s_1 and s_2 such that $(\Phi_\pm, \Psi_\pm)$ becomes a suitable comparison function.

Lemma 4.1. *For all $\tau > 0, \xi > 0, \eta > 0$,*

$$
\begin{aligned}
0 < \phi_\xi(\tau;\xi,\eta) < 1, \quad & -\frac{b}{c} < \phi_\eta(\tau;\xi,\eta) < 0, \\
-\frac{c}{b} < \psi_\xi(\tau;\xi,\eta) < 0, \quad & 0 < \psi_\eta(\tau;\xi,\eta) < 1.
\end{aligned}
$$

Lemma 4.2. (i) $\left|\dfrac{\phi(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| < 2\xi$ *and* $\left|\dfrac{\psi(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| < 2\eta$*, for all $\tau > 0,\ \xi > 0,\ \eta > 0$.*

(ii) *There exist $M_{ij} > 0$ $(i = 1,2,3,\ j = 1,2)$ such that*

$$
\begin{aligned}
&\left|\frac{\phi_{\xi\xi}}{\phi_\xi}\right| \le \frac{M_{11}}{\xi} + M_{12}\tau, \quad \left|\frac{\phi_{\xi\eta}}{\phi_\xi}\right| \le \frac{M_{21}}{\xi} + M_{22}\tau, \quad \left|\frac{\phi_{\eta\eta}}{\phi_\xi}\right| \le \frac{M_{31}}{\xi} + M_{32}\tau, \\
&\left|\frac{\psi_{\xi\xi}}{\psi_\eta}\right| \le \frac{M_{31}}{\eta} + M_{32}\tau, \quad \left|\frac{\psi_{\xi\eta}}{\psi_\eta}\right| \le \frac{M_{21}}{\eta} + M_{22}\tau, \quad \left|\frac{\psi_{\eta\eta}}{\psi_\eta}\right| \le \frac{M_{11}}{\eta} + M_{12}\tau
\end{aligned}
$$

for all $\tau > 0,\ \xi > 0,\ \eta > 0$. Here we denote $\dfrac{\phi_{\xi\xi}}{\phi_\xi} = \dfrac{\phi_{\xi\xi}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}$, etc.

Lemmas 4.1 and 4.2 are obtained by direct calculations, so we omit their proofs (for precise proofs, see [3]).

Here assume that ϵ is sufficiently small. Additionally s_1 and s_2 are assumed to be small in proportion to ϵ. Applying lemmas 4.1, 4.2 and (A3) to (2) and (3) we see that, if s_1 and s_2 satisfy

$$\dot{s_1}(t,\epsilon) \geq \frac{M_1}{\inf_{x\in\bar{\Omega}} u_0(x)} \cdot \left(1+\frac{t}{\epsilon^3}\right)$$

and

$$\dot{s_2}(t,\epsilon) \geq \frac{M_2}{\inf_{x\in\bar{\Omega}} v_0(x)} \cdot \left(1+\frac{t}{\epsilon^3}\right)$$

for sufficiently large $M_1, M_2 > 0$, then $\mathcal{L}_1(\Phi_+,\Psi_+) \geq 0$, $\mathcal{L}_2(\Phi_+,\Psi_+) \leq 0$, and $\mathcal{L}_1(\Phi_-,\Psi_-) \leq 0$, $\mathcal{L}_2(\Phi_-,\Psi_-) \geq 0$. Additionally Lemma 4.1 and the mean value theorem imply

$$\begin{aligned}\Big|\Phi_\pm(x,t)-\Phi_0(x,t)\Big| &\leq s_1(t,\epsilon)\cdot\int_0^1 \Big|\phi_\xi\Big(\frac{t}{\epsilon^3};u_0\pm\rho s_1(t,\epsilon),v_0\mp s_2(t,\epsilon)\Big)\Big|d\rho \\ &\quad + s_2(t,\epsilon)\cdot\int_0^1 \Big|\phi_\eta\Big(\frac{t}{\epsilon^3};u_0,v_0\mp\rho s_2(t,\epsilon)\Big)\Big|d\rho \\ &\leq s_1(t,\epsilon)+\frac{b}{c}s_2(t,\epsilon).\end{aligned} \tag{4}$$

and

$$\begin{aligned}\Big|\Psi_\pm(x,t)-\Psi_0(x,t)\Big| &\leq s_1(t,\epsilon)\cdot\int_0^1 \Big|\psi_\xi\Big(\frac{t}{\epsilon^3};u_0\pm\rho s_1(t,\epsilon),v_0\mp s_2(t,\epsilon)\Big)\Big|d\rho \\ &\quad + s_2(t,\epsilon)\cdot\int_0^1 \Big|\psi_\eta\Big(\frac{t}{\epsilon^3};u_0,v_0\mp\rho s_2(t,\epsilon)\Big)\Big|d\rho \\ &\leq \frac{c}{b}s_1(t,\epsilon)+s_2(t,\epsilon).\end{aligned} \tag{5}$$

Hence s_1 and s_2 are supposed to be $O(\epsilon)$ for $t\in(0,\epsilon^2)$. More precisely we obtain the following lemma.

Lemma 4.3. *Assume* (A1)-(A4) *and set*

$$s_1(t,\epsilon) := \gamma_1\epsilon\exp\frac{t^2}{\epsilon^3} \quad s_2(t,\epsilon) := \gamma_2\epsilon\exp\frac{t^2}{\epsilon^3}$$

with positive constants γ_1 and γ_2. Then there exist $\epsilon_0 > 0$ and $\gamma_1,\gamma_2 > 0$ such that, for any $\epsilon\in(0,\epsilon_0)$, (Φ_+,Ψ_+) and (Φ_-,Ψ_-) are an upper-solution of and a lower-solution (P) *for $t\in(0,\epsilon^2)$, respectively.*

Finally we will accomplish the proof of Theorem 1. Proposition 4.1 and Lemma 4.3 yield that

$$\Phi_-(x,t) \le u(x,t) \le \Phi_+(x,t), \text{ and } \Psi_+(x,t) \ge v(x,t)\Psi_-(x,t),$$

and

$$|u(x,t) - \Phi_0(x,t)| \le \max\{|\Phi_+(x,t) - \Phi_0(x,t)|, |\Phi_0(x,t) - \Phi_-(x,t)|\},$$
$$|v(x,t) - \Psi_0(x,t)| \le \max\{|\Psi_+(x,t) - \Psi_0(x,t)|, |\Psi_0(x,t) - \Psi_-(x,t)|\},$$

especially. Estimates (4), (5), and above two inequalities enable us to complete the proof of (i) of Theorem 1. Furthermore combining Theorem 1 (i) and (1) we obtain (ii) of Theorem 1. Thus the proof is complete.

References

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