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Some remarks on ordered $*$-groupoids

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A non-empty set G with a partial product, a unary operation $*$ and a partial order $\leqq$ is called an *ordered $*$-groupoid* if it satisfies the following axioms:

(A1) $a(bc)$ exists if and only if $(ab)c$ exists, in which case they are equal.

(A2) $a(bc)$ exists if and only if ab and bc exist.

(A3) $(a^*)^* = a$.

(A4) If ab exists, then b^*a^* exists and $(ab)^* = b^*a^*$.

(A5) For any $a \in G$, a^*a exists and a^*a is the unique projection of G such that there exists $a(a^*a)$ and $a(a^*a) = a$. We denote a^*a by $d(a)$.

(A6) $a \leqq b$ implies $a^* \leqq b^*$.

(A7) For $a, b, c, d \in G$, if $a \leqq b$, $c \leqq d$ and there exist ac and bd, then $ac \leqq bd$.

(A8) Let $a \in G$ and $e\ (= e^2 = e^*)$ a projection such that $e \leqq d(a)$. Then there exists a unique element $(a|e)$, say, such that $(a|e) \leqq a$ and $d(a|e) = e$.

(A9) $E(G)$ is an order ideal.

Lemma 1. [3] *Let G be an ordered $*$-groupoid.*

(1) *For any $a \in G$, aa^* exists and aa^* is the unique element of $P(G)$, the set of all projections of G, such that there exists $(aa^*)a$ and $(aa^*)a = a$. We denote aa^* by $r(a)$.*

(2) *Let $a \in G$ and $e \in P(G)$ such that $e \leqq r(a)$. Then there exists a unique element $(e|a)$, say, such that $(e|a) \leqq a$ and $r(e|a) = e$.*

An ordered $*$-groupoid G is called a *locally inductive $*$-groupoid* if it satisfies

(LG) For any $e, f \in P(G)$, the set of projections of G, there exists the maximum element in $< e, f >= \{(g, h) \in P(G) \times P(G) : g \leqq e,\ h \leqq f \text{ and } \exists gh\}$.

A regular $*$-semigroup S is called a locally inverse $*$-semigroup if eSe is an inverse subsemigroup of S for any projection e in S. Let S be a locally inverse $*$-semigroup. The representation in [4] raise us a new partial product $\cdot$ on S, which is called a *restricted product*, as follows:

$$a \cdot b = \begin{cases} ab & ab \in R_a \cap L_b \\ \text{undefined} & \text{otherwise} \end{cases}$$

where R_a and L_a denote the $\mathcal{R}$-class and the $\mathcal{L}$-class containing a, respectively.

Lemma 2. [3] *Let S be a locally inverse $*$-semigroup with the natural order $\leqq$. Then $S(\cdot, *, \leqq)$ is a locally inductive $*$-groupoid, which is denoted by* $\mathbf{G}(S)$.

Conversely, let $G(\circ, *, \leqq)$ be a locally inductive $*$-groupoid. For any $a, b \in G$, there exists the maximun element (e, f) in $< d(a), r(b) > = \{(g, h) \in P(S) \times P(S) : g \leqq d(a),\ h \leqq r(b),\ \exists g \circ h\}$. We define a new product $\otimes$ on G as follows:

$$a \otimes b = (a|e) \circ (f|b),$$

and we call it a *pseudoproduct* of a and b.

Lemma 3. [3] *Let $G(\circ, *, \leqq)$ be a locally inductive $*$-groupoid. The $G(\otimes, *)$ is a locally inverse $*$-semigroup, which is denoted by* $\mathbf{S}(G)$.

Lemma 4. [3] (1) *For a locally inverse $*$-semigroup S, we have* $\mathbf{S}(\mathbf{G}(S)) = S$.

(2) *For a locally inductive $*$-groupoid $G(\circ, *, \leqq)$, we have* $\mathbf{G}(\mathbf{S}(G(\circ, *, \leqq))) = G(\circ, *, \leqq)$.

Let S and T be regular $*$-semigroups. A mapping $\phi : S \to T$ is called a *prehomomorphism* if it satisfies that $(ab)\phi \leq (a\phi)(b\phi)$ and $(a\phi)^* = a^*\phi$ for all $a, b \in S$.

Lemma 5. [2] *Let S and T be locally inverse $*$-semigroups and $\phi : S \to T$ a mapping.*

(1) *ϕ is a prehomomorphism if and only if it preserves the restricted product and the natural order.*

(2) *ϕ is a homomorphism if and only if it is a prehomomorphism which satisfies $(ef)\phi = (e\phi)(f\phi)$ for all $e, f \in E(S)$.*

(3) *The product of prehomomorphisms between locally inverse $*$-semigroups is also a prehomomorphism.*

A functor between two ordered $*$-groupoids is said to be *ordered* if it is order-preserving. An ordered functor between two locally inductive $*$-groupoids is said to be *inductive* if it preserves the pseudoproduct.

Now, we have the main result.

Theorem 6. (Compare with Theorem 4.1.8 [5]) *The category of locally inverse $*$-semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$-groupoids and ordered functors. Moreover, the category of locally inverse $*$-semigroups and homomorphisms is isomorphic to the category of locally inductive $*$-groupoids and inductive functors.*

Proof. Let $\mathbf{G}$ be a function of the category of locally inverse $*$-semigroups and prehomomorphisms to the category of locally inductive $*$-groupoids and ordered functors as follows: for any locally inverse $*$-semigroups S, T and any prehomomorphism $\theta : S \to T$,

(1) $\mathbf{G}(S) = S(\cdot, *, \leqq)$,

(2) $\mathbf{G}(\theta) : \mathbf{G}(S) \to \mathbf{G}(T)$ $(s \mapsto \theta(s))$.

Then it follows from Lemma 2 and Lemma 5 (1) that $\mathbf{G}$ is a functor.

Conversely, define a function $\mathbf{S}$ from the category of locally inductive $*$-groupoids and ordered functors to the category of locally inverse $*$-semigroups and prehomomorphisms as follws: for any locally inductive $*$-groupoids G, H and any ordered functor $\theta : G \to H$,

(1) $\mathbf{S}(G) = G(\otimes, *)$,

(2) $\mathbf{S}(\theta) : \mathbf{S}(G) \to \mathbf{S}(H)$ $(g \mapsto \theta(g))$.

By Lemma 3 and Lemma 5 (1), $\mathbf{S}$ is a functor. Moreover, it follows from Lemma 4 that $\mathbf{G}(\mathbf{S}(G)) = G$ and $\mathbf{S}(\mathbf{G}(S)) = S$. Thus we have that the category of locally inverse $*$-semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$-groupoids and ordered functors.

By Lemma 5 (2), we can easily obtain the second statement. □

References

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