数理解析研究所講究録 1437 巻 2005 年 119-121

ON SOME INTEGRAL MONOID RINGS

YASUYUKI HIRANO

Department of Mathematics, Okayama University,
Okayama 700-8530, Japan
e-mail: yhiranomath.okayama-u.ac.jp

1. Introduction

Let **Rng** denote the category of rings with identity and let **Mon** denote the category of monoids. Let $\mathbf{Z}$ denote the ring of rational integers and define a functor $F : \mathbf{Mon} \to \mathbf{Rng}$ by $F(M) = \mathbf{Z}[M]$, the integral monoid ring of M. Then F is a left adjoint functor of the forgetful functor $U : \mathbf{Rng} \to \mathbf{Mon}$. We consider the functor $FU : \mathbf{Rng} \to \mathbf{Rng}$ and we study which properties of rings are preserved by the functor FU.

2. Examples

Let $\mathbf{Z}$ denote the ring of rational integers and let $\mathbf{Q}$ denote the field of rational numbers. Let $(R, \cdot, +)$ be a ring and consider the monoid rings $\mathbf{Z}[(R, \cdot)]$ and $\mathbf{Q}[(R, \cdot)]$. We briefly denote these rings by $\mathbf{Z}[R]$ and $\mathbf{Q}[R]$. In this section we consider some examples.

Example 1 Consider the polynomial ring $GF(3)[x]$ over the Galois field $GF(3)$ of three elements. We can easily see the monoid $(GF(3)[x], \cdot)$ is isomorphic to the monoid $(\mathbf{Z}, \cdot)$. Hence $\mathbf{Z}[(GF(3)[x]]$ is isomorphic to $\mathbf{Z}[\mathbf{Z}]$.

The monoid ring $\mathbf{Z}[R]$ of a ring R is determined by the monoid structure of the ring R. So we consider some properties of rings which depend only on the monoid structure of rings and we ask whether those properties are perserved by the the functor $FU : \mathbf{Rng} \to \mathbf{Rng}$ or not. More generally we can consider the following problem.

Problem. Let P be some property on (monoids of) rings. If a ring R has property P, then what can be said about the structure of $\mathbf{Z}[R]$ and $\mathbf{Q}[R]$?

A ring R is said to be *prime* if $aRb \neq 0$ for all nonzero $a, b \in R$. The following example shows that primeness does not preserved by the functor FU.

Example 2 Let $\mathbf{Q}$ denote the field of rational numbers. Then we can easily see that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}[(GF(3)] \cong \mathbf{Q}[(GF(3)]$ is isomorphic to $\mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}$.

A ring R is called a *von Neumann regular* ring if for each $a \in R$ there exists $x \in R$ such that $a = axa$. The following example shows that the von Neumann regularity does not preserved by the functor $\mathbf{Rng} \to \mathbf{Rng}$; $R \to \mathbf{Q}[R]$.

Example 3 Let D be a division ring. Then $\mathbf{Q}[D]$ is isomorphic to $\mathbf{Q} \oplus \mathbf{Q}[D^*]$. It is well-known that, for a group G, the group ring $\mathbf{Q}[G]$ is von Neumann regular if and only if G is locally finite. Hence $\mathbf{Q}[D^*]$ is von Neumann regular if and only if D is an algebraic extension of a finite field.

When R is a noncommutative ring, it is not easy to see the structure of the ring $\mathbf{Z}[R]$.

Example 4 Let $M_2(GF(2))$ denote the ring of 2×2 matrices over the field $GF(2)$. Then we can prove that $\mathbf{Z}[M_2(GF(2))]$ is a semiprime ring. In fact we can see that $\mathbf{Q}[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $\mathbf{Q}\oplus\mathbf{Q}\oplus\mathbf{Q}\oplus M_2(\mathbf{Q}) \oplus M_3(\mathbf{Q})$.

Example 5 Let $T_2(GF(2))$ denote the ring of 2×2 upper triangular matrices over the field $GF(2)$. Then we can see that $\mathbf{Q}[T_2(GF(2))]$ is isomorphic to the ring $\mathbf{Q} \oplus \mathbf{Q} \oplus T_3(\mathbf{Q})$.

Conjecture 1. *Let K be a finite field and consider the ring $M_n(K)$ of $n \times n$ matrices over K. Then $\mathbf{Q}[M_n(K)]$ is a semisimple Artinian ring.*

3. Structure of $\mathbf{Z}[R]$

Let R be a ring. Then every element of the monoid ring $\mathbf{Z}[R]$ can be written as a finite sum of the form $\sum_{r\in R} a_r\hat{r}$ where $a_r \in \mathbf{Z}$. In this section we consider general structure of $\mathbf{Z}[R]$.

Proposition 1. *Let R be a ring and let I be an ideal of R.*

(1) Let A denote the ideal of $Z[R]$ generated by $\{\widehat{r+s} - \hat{r} - \hat{s} \mid r, s \in R\}$. Then $Z[R]/A \cong R$.

(2) Let B denote the ideal of $Z[R]$ generated by $\{\hat{r} - \hat{s} \mid r - s \in I\}$. Then $Z[R]/B \cong Z[R/I]$.

Consider the following condition on a ring R:

(*) For any $x, y \in R$, $xy = 1$ implies $yx = 1$.

If R is a left (or right) Noetherian ring, R satisfies condition (*).

Proposition 2. *Let R be a ring with condition (*) and let R^* denote the group of units in R. Let C denote the ideal of $Z[R]$ generated by $\{\hat{r} \mid r \in R - R^*\}$. Then $Z[R]/C \cong Z[R^*]$.*

4. Semiprimeness

In this section we consider the semiprimeness of $\mathbf{Z}[R]$.

Assume that a ring R has a nonzero ideal I with $I^2 = 0$. Let $\bar{I}$ denote the ideal of $\mathbf{Z}[R]$ generated by $\{\hat{r} - \hat{0} \mid r \in I\}$. Then we can easily see that $\bar{I}^2 = 0$. Therefore we have the following.

Proposition 3. *Let R be a ring. If $\mathbf{Z}[R]$ is semiprime then R is semiprime.*

A commutative ring R is semiprime if and only if it has no nonzero nilpotent elements.

Theorem 1. *Let R be a commutative ring. Then $\mathbf{Z}[R]$ is semiprime if and only if R is semiprime.*

Corollary 1. *Let S be a subsemigroup of a commutative semiprime ring R. Then the integral monoid ring $\mathbf{Z}[S]$ is semiprime.*

Proposition 4. *Let R be a PI domain. Then $\mathbf{Z}[R]$ is a semiprime ring.*

Conjecture 2. *Let R be a ring. Then $\mathbf{Z}[R]$ is semiprime if and only if R is semiprime.*