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PATHS, TABLEAUX, AND q-CHARACTERS OF QUANTUM AFFINE ALGEBRAS

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1. Introduction

This paper is based on [15], joint work with T. Nakanishi.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\hat{\mathfrak{g}}$ be the corresponding non-twisted affine Lie algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is the quantized universal enveloping algebra of $\hat{\mathfrak{g}}$. The *q-character* of $U_q(\hat{\mathfrak{g}})$ was introduced in [9] to study the intricate structure of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$.

Earlier than the introduction of the q-character, the tableaux descriptions of the spectra of the transfer matrices of a vertex model associated to $U_q(\hat{\mathfrak{g}})$ was studied in [5, 11, 13] etc. for $\mathfrak{g}$ of classical type. Since the q-character is designed to be a "universalization" of the family of transfer matrices, one can interpret their results in the context of the q-character. Then, the Jacobi-Trudi determinant $\chi_{\lambda/\mu,a}$ is conjectured to be the q-character of certain finite dimensional representation associated to a skew diagram λ/μ and $a \in \mathbb{C}$ for A_n and B_n. For these cases, the determinant $\chi_{\lambda/\mu,a}$ is described by the tableaux which satisfy certain "horizontal" and "vertical" rules [5, 11]. In contrast, the tableaux description of $\chi_{\lambda/\mu,a}$ for C_n and D_n is known only for the cases when λ/μ is a one-row or one-column diagram.

In this paper, we conjecture that $\chi_{\lambda/\mu,a}$ is the q-character of a finite dimensional representation, and give a summary of the method to give a tableaux description for C_n, using the "paths" method of Gessel-Viennot [10]. For simplicity, the paths method is introduced by applying it for the A_n case. To apply it for the C_n case, some modifications are required. As a result, the tableaux are given by certain horizontal and vertical rules with an "extra" rule. The case for two-row diagrams are given, which is the simplest typical example of it. In conclusion, we give a conjecture of an implicit form of the extra rule in terms of paths.

2. Preliminaries

2.1. **Quantum affine algebras.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n, and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. Let $U_q(\hat{\mathfrak{g}})$ be the *quantum affine algebra*, namely, it is the associative algebra generated by $x_i^{\pm}$, $k_i^{\pm 1}$ $(i = 0, \dots, n)$ with relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i,$$

$$k_i x_j^{\pm} k_i^{-1} = q^{\pm B_{ij}} x_j^{\pm},$$

$$x_i^+ x_j^- - x_j^- x_i^+ = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-C_{ij}} \begin{bmatrix} 1 - C_{ij} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-C_{ij}-r} = 0, \quad i \neq j.$$

Here, $q \in \mathbb{C}^\times$, $C = (C_{ij})_{0 \le i,j \le n}$ is the Cartan matrix of $\hat{\mathfrak{g}}$, and $q_i := q^{r_i}$, where r_i's are relatively prime integers such that $B = (B_{ij}) = DC$ for $D = \operatorname{diag}(r_1, \dots, r_n)$. We also set

$$\begin{bmatrix} s \\ t \end{bmatrix}_q := \frac{[s]_q!}{[t]_q![s-t]_q!},$$

$$[s]_q! := [s]_q[s-1]_q \dots [1]_q, \qquad [s]_q := \frac{q^s - q^{-s}}{q - q^{-1}}.$$

If we let $q \to 1$, then $U_q(\hat{\mathfrak{g}})$ becomes the universal enveloping algebra $U(\hat{\mathfrak{g}}')$ of the subalgebra $\hat{\mathfrak{g}}' := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \subset \hat{\mathfrak{g}}$ (c is the center).

2.2. **Finite dimensional representations of $U_q(\hat{\mathfrak{g}})$.** There is a bijection between the set of the isomorphism classes of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ and the set of n-tuples polynomials $\mathbf{P}(u) = (P_i(u))_{i=1,\dots,n}$ with constant term 1, which are called the *Drinfel'd polynomials* [6, 7]. For any skew diagram λ/μ with its depth $d(\lambda/\mu) \le n$ ($d(\lambda/\mu) \le n-1$ for B_n and $d(\lambda/\mu) \le n-2$ for D_n) and $a \in \mathbb{C}$, let $V(\lambda/\mu, a)$ be the representation that corresponds to the Drinfel'd polynomial

$$\prod_{j=1}^{d(\lambda'/\mu')} \mathbf{P}_{\lambda'_j - \mu'_j, a(j)}(u)$$

where $\mathbf{P}_{i,a}(u) = (P_j(u))_{j=1,\dots,n}$ is defined as

$$P_j(u) = \begin{cases} 1 - uq^a, & \text{if } j = i, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$a(j) = a + (2j - \lambda'_j - \mu'_j - 1)\delta.$$

Here, λ' denotes the conjugate of λ. Then, the highest weight of $V(\lambda/\mu, a)$ considered as a $U_q(\mathfrak{g})$-module is $\sum_{j=1}^{d(\lambda'/\mu')} \omega_{\lambda'_j - \mu'_j}$, where ω_r is the rth fundamental weight.

For the definition of the representation $V(\lambda/\mu, a)$ associated to a skew diagram λ/μ of $d(\lambda/\mu) = n$ for B_n and $d(\lambda/\mu) = n-1, n$ for D_n, see [15].

2.3. **The q-characters of quantum affine algebras.** The *q-character* of $U_q(\hat{\mathfrak{g}})$, introduced in [9], is an injective ring homomorphism

$$\chi_q : \operatorname{Rep} U_q(\hat{\mathfrak{g}}) \to U_q(\tilde{\mathfrak{h}})[[u]], \tag{2.1}$$

where $\operatorname{Rep} U_q(\hat{\mathfrak{g}})$ be the Grothendieck ring of the category of the finite dimensional representations of $U_q(\hat{\mathfrak{g}})$, and $U_q(\tilde{\mathfrak{h}})$ is a certain subalgebra of $U_q(\hat{\mathfrak{g}})$. It is defined as a composition of two maps; the map that sends $V \in \operatorname{Rep} U_q(\hat{\mathfrak{g}})$ to the "universal" transfer matrix

$$t_V(u) := \operatorname{Tr}_V(\pi_{V(u)} \otimes \mathrm{id})(\mathcal{R}) \in U_q(\hat{\mathfrak{g}})[[u]],$$

and the projection $U_q(\hat{\mathfrak{g}})[[u]] \to U_q(\tilde{\mathfrak{h}})[[u]]$. The element $\mathcal{R} \in U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}})$, called the *universal R-matrix*, satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Sending the second component of $\mathcal{R}$ by a representation $(\pi^{\otimes p}, W^{\otimes p})$, the element $t_V(u)$ becomes the transfer matrix

$$t_V(u; u_1, \dots, u_p) := \mathrm{Tr}_V(R_{V,W}(u-u_1)\dots R_{V,W}(u-u_p)).$$

For the simplest example, the tableaux description (with spectral parameter $a \in \mathbb{C}$) of the q-character for the first fundamental representation $V((1),a)$ is given as follows:

$$\chi_q(V((1),a)) = \sum_{i\in I} z_{i,a} = \sum_{i \in I} \boxed{i}_a. \tag{2.2}$$

Here, the set I is

$$I = \begin{cases} \{1, 2, \dots, n, n+1\}, & (A_n) \\ \{1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}\}, & (B_n) \\ \{1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}\}, & (C_n) \\ \{1 \prec 2 \prec \cdots \prec \begin{smallmatrix} n \\ \overline{n} \end{smallmatrix} \prec \cdots \prec \overline{2} \prec \overline{1}\}. & (D_n) \end{cases} \tag{2.3}$$

See [9] for the monomials $z_{i,a}$ occurring in (2.2).

3. The Jacobi-Trudi formula for the q-characters

For any $a \in \mathbb{C}$, we define $E_a(z,X)$ and $H_a(z,X)$ as follows:

$$E_a(z,X) :=$$

$$\begin{cases} \overrightarrow{\prod}_{1\le k\le n+1}(1+z_{k,a}X) & (A_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1+z_{k,a}X)\}(1-z_{0,a}X)^{-1}\{\overleftarrow{\prod}_{1\le k\le n}(1+z_{\overline{k},a}X)\} & (B_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1+z_{k,a}X)\}(1-z_{n,a}Xz_{\overline{n},a}X)\{\overleftarrow{\prod}_{1\le k\le n}(1+z_{\overline{k},a}X)\} & (C_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1+z_{k,a}X)\}(1-z_{\overline{n},a}Xz_{n,a}X)^{-1}\{\overleftarrow{\prod}_{1\le k\le n}(1+z_{\overline{k},a}X)\} & (D_n) \end{cases} \tag{3.1}$$

(3.2)

$$H_a(z,X) :=$$

$$\begin{cases} \overleftarrow{\prod}_{1\le k\le n+1}(1-z_{k,a}X)^{-1} & (A_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1-z_{\overline{k},a}X)^{-1}\}(1+z_{0,a}X)\{\overleftarrow{\prod}_{1\le k\le n}(1-z_{k,a}X)^{-1}\} & (B_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1-z_{\overline{k},a}X)^{-1}\}(1-z_{n,a}Xz_{\overline{n},a}X)^{-1}\{\overleftarrow{\prod}_{1\le k\le n}(1-z_{k,a}X)^{-1}\} & (C_n) \\ \{\overrightarrow{\prod}_{1\le k\le n}(1-z_{\overline{k},a}X)^{-1}\}(1-z_{\overline{n},a}Xz_{n,a}X)\{\overleftarrow{\prod}_{1\le k\le n}(1-z_{k,a}X)^{-1}\} & (D_n) \end{cases}$$

where $\overleftarrow{\prod}_{1\le k\le n} A_k = A_1 \dots A_n$ and $\overleftarrow{\prod}_{1\le k\le n} A_k = A_n \dots A_1$, and

$$z_{i,a}z_{j,a'} = z_{j,a'}z_{i,a}, \qquad Xz_{i,a} = z_{i,a-2\delta}X, \qquad i,j \in I; a, a' \in \mathbb{C}, \tag{3.3}$$

where δ is

$$\delta = \begin{cases} 1, & (A_n, C_n, D_n) \\ 2. & (B_n) \end{cases} \tag{3.4}$$

Then we have

$$H_a(z,X)E_a(z,-X) = E_a(z,-X)H_a(z,X) = 1. \tag{3.5}$$

It has been observed in [8, 12] (see also [11, 13]) that $e_{i,a}$ is the q-character of the rth fundamental representation $V((1^r),a)$ of $U_q(\hat{\mathfrak{g}})$ for $1 \le r \le n$ ($r \ne n$ for B_n, $r \ne n-1, n$ for D_n), while $h_{r,a}$ is the q-character of the rth "symmetric" power of the first fundamental representation of $U_q(\hat{\mathfrak{g}})$ for any $r \ge 1$. The tableaux description is given as follows:

$$e_{r,a} = \sum_{i_1,\dots,i_r \in I;(\mathbf{V})} \prod_{l=1}^{r} z_{i_l,a+2(1-l)} = \sum_{i_1,\dots,i_r \in I;(\mathbf{V})} \begin{array}{|c|l} \hline i_1 & a \\ \hline i_2 & a-2 \\ \hline \vdots & \\ \hline i_r & a-2r+2 \\ \hline \end{array} \;,$$

$$h_{r,a+2r-2} = \sum_{i_1,\dots,i_r \in I;(\mathbf{H})} \prod_{l=1}^{r} z_{i_l,a+2(l-r)} = \sum_{i_1,\dots,i_r \in I;(\mathbf{H})} \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_r \\ \hline \multicolumn{1}{c}{a} & \multicolumn{1}{c}{a+2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a+2r-2} \end{array} \;.$$

The rules ($\mathbf{V}$) and ($\mathbf{H}$) for $i_1,\dots,i_r \in I$, called the *vertical* and *horizontal* rules, are given as follows [5, 11, 13]: for any $k = 1,\dots,r$,

$$(3.6)\qquad (\mathbf{V}) \quad \begin{array}{lr} i_k < i_{k+1}, & (A_n) \\ i_k \prec i_{k+1} \text{ or } i_k = i_{k+1} = 0, & (B_n) \\ i_k \prec i_{k+1} \text{ and } [(i_k, i'_k) = (c, \overline{c}) \Rightarrow k' - k \le n - c], & (C_n) \\ i_k \prec i_{k+1} \text{ or } (i_k, i_{k+1}) = (n, \overline{n}) \text{ or } (i_k, i_{k+1}) = (\overline{n}, n). & (D_n) \end{array}$$

$$(3.7)\qquad (\mathbf{H}) \quad \begin{array}{lr} i_k \le i_{k+1}, & (A_n) \\ i_k \preceq i_{k+1} \text{ and } i_k = i_{k+1} \ne 0, & (B_n) \\ [i_k \preceq i_{k+1} \text{ or } (i_k, i_{k+1}) = (\overline{n}, n)] & \\ \qquad \text{and } (i_k, i_{k+1}, i_{k+2}) \ne (\overline{n}, \overline{n}, n), (\overline{n}, n, n), & (C_n) \\ i_k \preceq i_{k+1} \text{ and } n \text{ and } \overline{n} \text{ do not appear simultaneously.} & (D_n) \end{array}$$

Due to the relation (3.5), it holds that [14]

$$(3.8)\qquad \det(h_{\lambda_i-\mu_j-i+j,a+2(\lambda_i-i)\delta})_{1\le i,j\le l} = \det(e_{\lambda'_i-\mu'_j-i+j,a-2(\mu'_j-j+1)\delta})_{1\le i,j\le l'}$$

for any partitions (λ,μ), where l and l' are any non-negative integers such that $l \ge d(\lambda), d(\mu)$ and $l' \ge d(\lambda'), d(\mu')$. For any skew diagrams λ/μ, let $\chi_{\lambda/\mu,a}$ denote the determinant on the left or right hand side of (3.8). We call it the *Jacobi-Trudi determinant* of $U_q(\hat{\mathfrak{g}})$ associated to λ/μ and $a \in \mathbb{C}$. Note that $\chi_{(r),a} = h_{r,a}$ and $\chi_{(1^r),a} = e_{r,a}$.

We remark that the determinant (3.8) appears in [5] for A_n and [11] for B_n in the context of transfer matrices.

We conjecture that

Conjecture 3.1. (1) *If $\mathfrak{g}$ is of type A_n or B_n and λ/μ is a skew diagram of $d(\lambda) \le n$, then $\chi_{\lambda/\mu,a} = \chi_q(V(\lambda/\mu,a))$.*

(2) *If $\mathfrak{g}$ is of type C_n or D_n and λ/μ is a skew diagram of $d(\lambda) \le n$ when $\mathfrak{g}$ is of type C_n (resp. $d(\lambda) \le n-1$ when $\mathfrak{g}$ is of type D_n), then $\chi_{\lambda/\mu,a}$ is the q-character of certain (not necessarily irreducible) representation V of $U_q(\hat{\mathfrak{g}})$ which has $V(\lambda/\mu,a)$ as a subquotient; furthermore, if $\mu = \phi$, then $\chi_{\lambda,a} = \chi_q(V(\lambda,a))$.*

Remark 3.2. An analogue of Conjecture 3.1 is true for the representations of Yangian $Y(\mathfrak{sl}_n)$, which can be proved [2] using the results in [3, 4].

Remark 3.3. The representation V in Conjecture 3.1 for C_n and D_n do not coincide with the irreducible representation $V(\lambda/\mu, a)$, in general. The case $(\lambda, \mu) = ((3,1),(2))$ is a counter-example for C_2 and for D_4, which can be shown from the singularities of the R-matrices (see [1] for example).

In the following sections, we give the tableaux description by applying the path method of Gessel-Viennot [10] (see also [16]). First, we introduce the method for the A_n case in Section 4, and then we refer to the modification of this method to apply it for C_n in Section 5.

4. Tableaux description of type A_n

The method was originally introduced to derive the well-known semistandard tableaux description of the Schur function from the (original) Jacobi-Trudi determinant. For A_n, this method works out without any modification (except for inserting the spectral parameters).

During this section, I is of type A_n in (2.3).

4.1. **Paths description.** In this subsection, we give a paths description of of $\chi_{\lambda/\mu,a}$ (3.8) in terms of the method of Gessel-Viennot.

A *path* p in the lattice $\mathbb{Z}\times\mathbb{Z}$ is a sequence of steps $(s_1, s_2, \dots)$ such that each step s_i is of unit length with the northward (N) or eastward (E) direction. For example, see Figure 1.

Let p be any path such that the initial point is at height 0 and the final point is at height n, and set $E(p) := \{s \in p \mid s \text{ is an eastward step}\}$. The maps

$$L_a^1 : E(p) \to I, \qquad L_a^2 : E(p) \to \{a + 2k \mid k \in \mathbb{Z}\},$$

called the *h-labeling of type A_n* associated to $a \in \mathbb{C}$, are defined as in Figure 1. For example, $z_a^p = z_{2,a}z_{2,a+2}z_{3,a+4}z_{3,a+6}$ for p in Figure 1. We define

$$z_a^p := \prod_{s\in E(p)} z_{L_a^1(s),L_a^2(s)}. \tag{4.1}$$

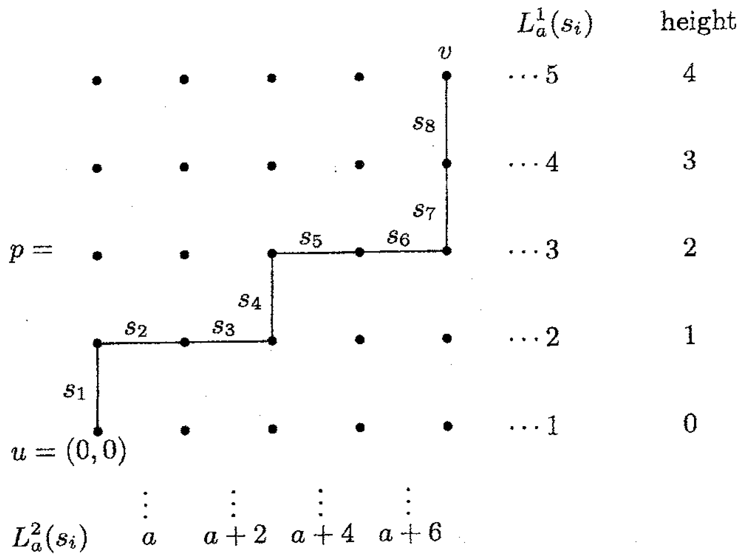

FIGURE 1. An example of a path p and its h-labeling.

Then we have

$$h_{r,a+2r-2}(z) = \sum_{(0,0)\xrightarrow{p}(r,n)} z_a^p. \tag{4.2}$$

For any skew diagrams λ/μ, let $l = d(\lambda)$, and let $\mathbf{u}_\mu = (u_1, \dots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \dots, v_l)$ be l-tuples of fixed initial and final points defined as $u_i = (\mu_i + 1 - i, 0)$ and $v_i = (\lambda_i + 1 - i, n)$. Let $\mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ be the set of l-tuples of paths $\mathbf{p} = (p_1, \dots, p_l)$ such that $u_i \xrightarrow{p_i} v_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_l$. We define the *weight* $z_a^{\mathbf{p}}$ and the *signature* $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ and $a \in \mathbb{C}$ by

$$z_a^{\mathbf{p}} = \prod_{i=1}^{l} z_a^{p_i}, \qquad (-1)^{\mathbf{p}} = \operatorname{sgn} \pi.$$

Then, the determinant (3.8) can be written as

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}}, \tag{4.3}$$

by (4.2). Applying the method of [10], we have

Proposition 4.1. *For any skew diagrams λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in P(A_n; \mu, \lambda)} z_a^{\mathbf{p}}, \tag{4.4}$$

where $P(A_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \dots, p_l) \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any intersecting pairs of paths (p_i, p_j).

The idea of its proof is to consider a weight-preserving, sign-changing involution ι on all $\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which possess an intersecting pair of paths. It immediately follows that the contributions of all such $\mathbf{p}$ to the right hand side of (4.3) are canceled with each other.

4.2. **Tableaux description.** We define the weight z_a^T for any tableau T of shape λ/μ as

$$z_a^T := \prod_{(i,j) \in \lambda/\mu} z_{T(i,j), a+2(j-i)},$$

where $T(i,j)$ is the entry of T at (i,j), namely, the entry at the ith row and the jth column.

A tableau T which satisfies the rules (**V**) and (**H**) of A_n in (3.6) and (3.7) is called an *A_n-tableau*. Namely, an A_n-tableau is nothing but a semistandard tableau. We write the set of all the A_n-tableaux of shape λ/μ by $\mathrm{Tab}(A_n, \lambda/\mu)$.

For any $\mathbf{p} = (p_1, \dots, p_l) \in P(A_n; \mu, \lambda)$, we associate a tableau $T(\mathbf{p})$ of shape λ/μ such that the ith row of $T(\mathbf{p})$ is given by $\{L_a^1(s) \mid s \in E(p_i)\}$ listed in the increasing order. See Figure 2 for an example. Clearly, $T(\mathbf{p})$ satisfies the horizontal rule because of the h-labeling rule of $\mathbf{p}$, and $T(\mathbf{p})$ satisfies the vertical rule since $\mathbf{p} \in P(A_n; \mu, \lambda)$ do not have any intersecting pairs of paths. Therefore, we obtain a map

$$T : P(A_n; \mu, \lambda) \ni \mathbf{p} \mapsto T(\mathbf{p}) \in \mathrm{Tab}(A_n, \lambda/\mu) \tag{4.5}$$

for any skew diagrams λ/μ. In fact,

Proposition 4.2. *The map T (4.5) is a weight-preserving bijection.*

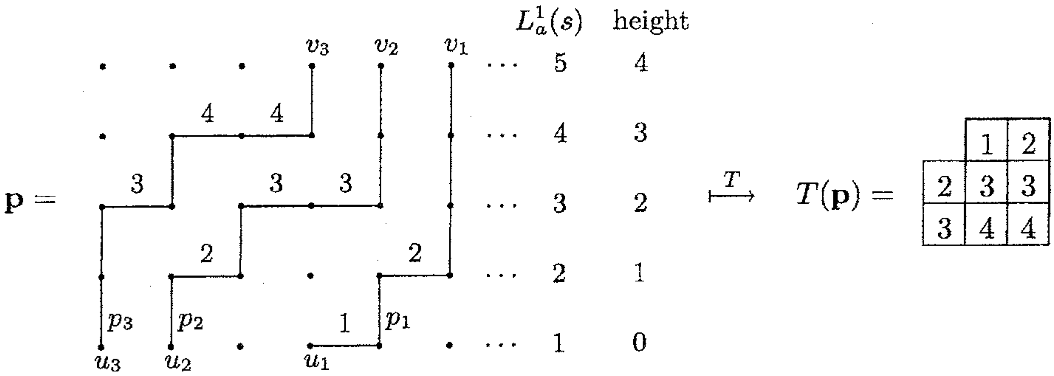

FIGURE 2. An example of $\mathbf{p}$ and the tableau $T(\mathbf{p})$ for $(\lambda, \mu) = ((3^3), (1))$.

By Proposition 4.1 and 4.2, we reproduce the result of [5].

Theorem 4.3 ([5]). *If λ/μ is a skew diagram, then*

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(A_n, \lambda/\mu)} z_a^T.$$

5. Tableaux description of type C_n

In this section, we consider the case that $\mathfrak{g}$ is of type C_n.

5.1. **Paths description.** In view of the definition of the generating function of $H_a(z, X)$ in (3.2), we define an h-path and its h-labeling as follows:

Definition 5.1. Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. An *$h$-path of type C_n* is a path $u \xrightarrow{p} v$ such that the initial point u is at height $-n$ and the final point v is at height n, and the number of the eastward steps at height 0 is even. Let $P(C_n)$ denote the set of all the h-paths of type C_n.

For any $p \in P(C_n)$, the *h-labeling (L_a^1, L_a^2) of type C_n* associated to $a \in \mathbb{C}$ is defined as in Figure 3.

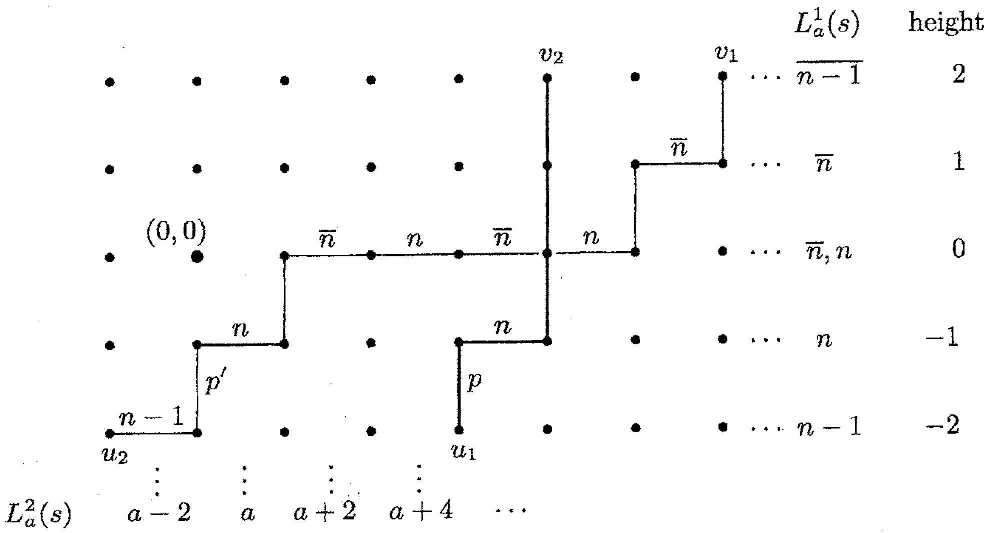

FIGURE 3. An example of h-paths of type C_n and their h-labelings.

By (3.2), we have

$$h_{r,a+2r-2}(z) = \sum_{(0,-n)\xrightarrow{p}(r,n)} z_a^p, \tag{5.1}$$

where z_a^p is defined as in (4.1) by the h-labeling of type C_n.

For any skew diagram λ/μ, let $l = d(\lambda)$, and let $\mathbf{u}_\mu = (u_1, \ldots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \ldots, v_l)$ be l-tuples of initial and final points defined as $u_i = (\mu_i + 1 - i, -n)$ and $v_i = (\lambda_i + 1 - i, n)$. As in the case of type A_n, we set $\mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ be the set of l-tuples of paths $\mathbf{p} = (p_1, \ldots, p_l)$ such that $u_i \xrightarrow{p_i} v_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_l$. and define $z_a^{\mathbf{p}}$ and $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ and $a \in \mathbb{C}$ by the h-labeling of type C_n. Then, the determinant (3.8) can be written as

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

by (5.1).

The first difference from the A_n case is that, the involution ι is not defined on all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which possess an intersecting pair of paths (p_i, p_j), because of the definition of the h-paths of type C_n (Definition 5.1). To define the involution for the C_n case, we give the definition of the specially (resp. ordinarily) intersecting pair of paths (see [15]). Applying the method of [10], all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ with an ordinarily intersecting pair (p_i, p_j) are canceled, and we have

Proposition 5.2. *For any skew diagrams λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in P(C_n; \mu, \lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}}, \tag{5.2}$$

where $P(C_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which does not have any ordinarily intersecting pair of paths (p_i, p_j).

The second difference from A_n is that, the sum (5.2) is alternating. However, we conjecture that the sum (5.2) turns out to be a positive sum for any skew diagram λ/μ. To obtain a positive sum, we use the relations

$$z_{i,a} z_{\bar{i},a-2n+2i-4} = z_{i-1,a} z_{\overline{i-1},a-2n+2i-4}, \quad i = 1, \ldots, n; \ a \in \mathbb{C}. \tag{5.3}$$

The third difference is that there exist some $\mathbf{p} \in P(C_n; \mu, \lambda)$ that have a "transposed" pair of paths (p_i, p_j). (Roughly speaking, if the initial points $\mathbf{u} = (u_1, u_2)$ and final points $\mathbf{v} = (v_1, v_2)$ are in order and $u_1 \xrightarrow{p_1} v_2$, $u_2 \xrightarrow{p_2} v_1$, then (p_1, p_2) is called a transposed pair of paths. For example, (p, p') in Figure 3 is transposed.) For such $\mathbf{p}$, we cannot define a tableau of shape λ/μ as same as $T(\mathbf{p})$ in (4.5), and therefore, the map T on $P(C_n; \mu, \lambda)$ to a certain set of tableaux of shape λ/μ cannot be defined as same as (4.5) of A_n.

5.2. **Tableaux description.** To formulate the tableaux description, we define a tableau $T(\mathbf{p})$ for any $\mathbf{p} = (p_1, \ldots p_l) \in \mathfrak{P}(C_n; \mu, \lambda)$ such that any pair of paths (p_i, p_j) is neither ordinarily intersecting nor transposed. Then we have a weight-preserving bijection

$$T : \tilde{P}(C_n; \mu, \lambda) \to \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu) := \{T(\mathbf{p}) | \mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)\} \tag{5.4}$$

as in (4.5), where $\tilde{P}(C_n;\mu,\lambda)$ is the set of all $\mathbf{p} \in \mathfrak{P}(C_n;\mathbf{u}_\mu,\mathbf{v}_\lambda)$ which do not have any transposed pair of paths. The set $\widetilde{\mathrm{Tab}}(C_n,\lambda/\mu)$ can also be described by the set of all tableaux T of shape λ/μ which satisfy the following "horizontal" and "vertical" rules as in the A_n case, which reduce to the rules (**H**) and (**V**) for C_n in (3.7) and (3.6), when λ/μ is a one-row diagram for (**H**) and one-column diagram for (**V**).

(**H**) Each $(i,j) \in \lambda/\mu$ satisfies both of the following conditions:
- $T(i,j) \preceq T(i,j+1)$ or $(T(i,j),T(i,j+1)) = (\overline{n},n)$.
- $(T(i,j-1),T(i,j),T(i,j+1)) \neq (\overline{n},\overline{n},n),(\overline{n},n,n)$.

(**V**) Each $(i,j) \in \lambda/\mu$ satisfies at least one of the following conditions:
- $T(i,j) \prec T(i+1,j)$.
- $T(i,j) = T(i+1,j) = n$, $(i+1,j-1) \in \lambda/\mu$ and $T(i+1,j-1) = \overline{n}$.
- $T(i,j) = T(i+1,j) = \overline{n}$, $(i,j+1) \in \lambda/\mu$ and $T(i,j+1) = n$.

We expect that the alternating sum (5.2) can be translated into a positive sum by a certain set of tableaux $\mathrm{Tab}(C_n,\lambda/\mu) \subset \widetilde{\mathrm{Tab}}(C_n,\lambda/\mu)$ as

$$\chi_{\lambda/\mu,a} = \sum_{T\in\mathrm{Tab}(C_n,\lambda/\mu)} z_a^T. \tag{5.5}$$

Thus, the tableaux in $\mathrm{Tab}(C_n,\lambda/\mu)$ are described by a horizontal rule and the vertical rule with an "extra" rule which selects them out of $\widetilde{\mathrm{Tab}}(C_n,\lambda/\mu)$.

The idea in [15] to obtain from (5.2) is as follows: We introduce some weight-preserving, sign-inverting injections which "resolves" a transposed pair of paths, which is well-defined if $d(\lambda) \le n$. The relation in (5.3) plays a crucial role for the weight-preserving property of these injections. Then, we can show that the contributions for (5.2) from all $\mathbf{p}$ with a transposed pair of paths almost cancel with each other.

In the following subsection, we give the tableaux description for λ/μ of $d(\lambda) = 2$. In this case, the set $\tilde{P}(C_n;\mu,\lambda)$ is exactly the set of all $\mathbf{p} = (p_1,p_2) \in P(C_n;\mu,\lambda)$ such that (p_1,p_2) is not transposed. Moreover, we do not have to consider any cancellations between all the transposed pairs $\mathbf{p} = (p_1,p_2)$. The contributions for all the transposed pairs $\mathbf{p} = (p_1,p_2)$ are all negative, which turn into the extra rule.

5.3. **Skew diagrams of two rows.** Let $\mathrm{Tab}(C_n,\lambda/\mu)$ be the set of all the HV-tableaux T with the following extra condition:

(**E-2R**) If T contains a subtableau (excluding a and b)

$$\begin{array}{cccccc} & \multicolumn{4}{c}{\overbrace{\hphantom{n\ n\ \cdots\ n}}^{k}} & \\ & \boxed{n} & \boxed{n} & \cdots & \boxed{n} & a \\ b & \boxed{\overline{n}} & \boxed{\overline{n}} & \cdots & \boxed{\overline{n}} & \end{array} \tag{5.6}$$

where k is an odd number, then at least one of the following conditions holds:

(1) Let (i_1,j_1) be the position of the top-right corner of the subtableau (5.6). Then $(i_1,j_1+1) \in \lambda/\mu$ and $a := T(i_1,j_1+1) = n$.

(2) Let (i_2,j_2) be the position of the bottom-left corner of the subtableau (5.6). Then $(i_2,j_2-1) \in \lambda/\mu$ and $b := T(i_2,j_2-1) = \overline{n}$.

Then

Theorem 5.3. *For any skew diagrams λ/μ with $d(\lambda) = 2$ and $n \ge 2$, the equality (5.5) holds.*

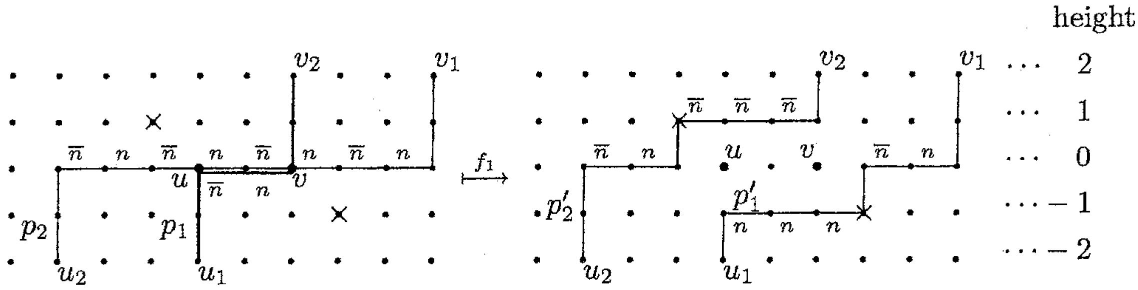

FIGURE 4. An example of $\mathbf{p} = (p_1, p_2) \in P_1(C_n; \mu, \lambda)$ and the map f_1

Proof. Let $P_1(C_n; \mu, \lambda)$ (resp. $P_0(C_n; \mu, \lambda)$) be the set of all $\mathbf{p} = (p_1, p_2) \in P(C_n; \mu, \lambda)$ such that (p_1, p_2) is transposed (resp. (p_1, p_2) is not transposed). In this case, we have $P_0(C_n; \mu, \lambda) = \tilde{P}(C_n; \mu, \lambda)$.

Define a weight-preserving, sign-inverting injection

$$f_1 : P_1(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda)$$

as follows (see Figure 4 for example): Let u (resp. v) be the leftmost (resp. rightmost) intersecting point of p_1 and p_2 at height 0. We assume that $u - (0,1)$ and $v + (0,1)$ is on p_1 while $u - (1,0)$ and $v + (1,0)$ is on p_2. Set $f_1(p_1, p_2) = (p_1', p_2')$ by

$$p_1' : u_1 \xrightarrow{p_1} u + (0,-1) \longrightarrow v + (1,-1) \longrightarrow v + (1,0) \xrightarrow{p_2} v_1,$$
$$p_2' : u_2 \xrightarrow{p_2} u + (-1,0) \longrightarrow u + (-1,1) \longrightarrow v + (0,1) \xrightarrow{p_1} v_2.$$

Roughly speaking, f_1 resolves the transposed pair (p_1, p_2). From (5.2) and (5.4), we have

$$\chi_{\lambda/\mu, a} = \sum_{T \in \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)} z_a^T - \sum_{\mathbf{p} \in \mathrm{Im}\, f_1} z_a^{T(\mathbf{p})}.$$

The set $\{T(\mathbf{p}) \mid \mathbf{p} \in \mathrm{Im}\, f_1\}$ consists of all the tableaux in $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ prohibited by the extra rule (**E-2R**). $\square$

5.4. **Conjecture on the implicit form of the extra rule.** Let $P_1(C_n; \mu, \lambda)$ be the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in P(C_n; \mu, \lambda)$ ($l = d(\lambda)$) such that one pair of paths (p_i, p_j) is transposed. Let $P_1^{i,i+1}(C_n; \mu, \lambda)$ be the set of $\mathbf{p} \in P_1(C_n; \mu, \lambda)$ such that the pair (p_i, p_{i+1}) is transposed. Then we have $P_1(C_n; \mu, \lambda) = \sum_{i=1}^{l-1} P_1^{i,i+1}(C_n; \mu, \lambda)$. We conjecture that one can always resolve the transposed pair (p_i, p_{i+1}) (without producing any ordinarily intersections) and define a weight-preserving, sign-inverting injection $f_1^{i,i+1} : P_1^{i,i+1}(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda)$. Furthermore, we conjecture that

Conjecture 5.4. *For any skew diagram λ/μ of $d(\lambda/\mu) \le n$,*

$$\mathrm{Tab}(C_n, \lambda/\mu) = \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu) \setminus \{T(\mathbf{p}) \mid \mathbf{p} \in \textstyle\bigcup_{i=1}^{d(\lambda)-1} \mathrm{Im}\, f_1^{i,i+1}\}.$$

In other words, for a tableau $T \in \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$, the extra rule (**E**), which gives the condition for T to be in $\mathrm{Tab}(C_n, \lambda/\mu)$, is implicitly stated in terms of paths as follows:

(**E**) If $\mathbf{p} \in P(C_n; \mu, \lambda)$ corresponds to T, then $\mathbf{p}$ is not obtained from some $\mathbf{p}' \in P_1(C_n; \mu, \lambda)$ as the resolution of the transposed pair.

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