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Nonintersecting Paths, Noncolliding Diffusion Processes and Representation Theory

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Abstract The system of one-dimensional symmetric simple random walks, in which none of walkers have met others in a given time period, is called the **vicious walker model**. It was introduced by Michael Fisher and applications of the model to various wetting and melting phenomena were described in his Boltzmann medal lecture. In the present report, we explain interesting connections among representation theory, probability theory, and random matrix theory using this simple diffusion particle system. Each vicious walk of N walkers is represented by an N-tuple of **nonintersecting lattice paths** on the spatio-temporal plane. There is established a simple bijection between nonintersecting lattice paths and **semistandard Young tableaux**. Based on this bijection and some knowledge of symmetric polynomials called the **Schur functions**, we can give a determinantal expression to the partition function of vicious walks, which is regarded as a special case of the **Karlin-McGregor formula** in the probability theory (or the **Lindström-Gessel-Viennot formula** in the enumerative combinatorics). Due to a basic property of Schur function, we can take the diffusion scaling limit of the vicious walks and define a **noncolliding system of Brownian particles**. This diffusion process solves the stochastic differential equations with the drift terms acting as the repulsive two-body forces proportional to the inverse of distances between particles, and thus it is identified with **Dyson's Brownian motion model**. In other words, the obtained noncolliding system of Brownian particles is equivalent in distribution with the **eigenvalue process of a Hermitian matrix-valued process**.

1 Vicious Walks, Young Tableaux and Schur Functions

Let $\left(\{\mathbf{S}(t)\}_{t=0,1,2,\ldots}, \mathrm{P}^{\mathbf{x}}\right)$ be the N-dimensional Markov chain starting from $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, such that the coordinates $S_i(t), i = 1, 2, \ldots, N$, are independent simple random walks on $\mathbb{Z}$. We always take the starting point $\mathbf{x}$ from the set

$$\mathbb{Z}^N_< = \left\{\mathbf{x} = (x_1, x_2, \ldots, x_N) \in (2\mathbb{Z})^N \ : x_1 < x_2 < \ldots < x_N\right\}.$$

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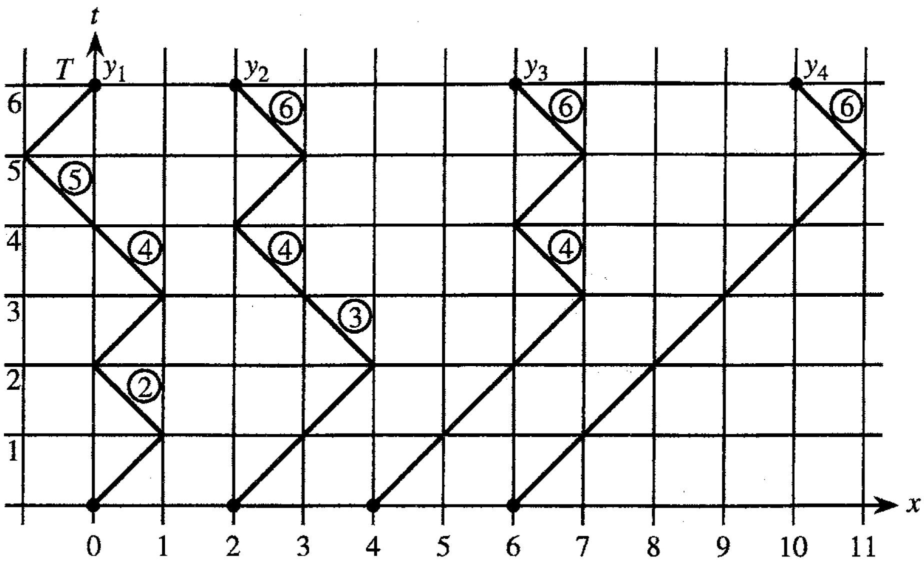


Figure 1: An example of vicious walk in the case $N = 4, T = 6$.

We consider the condition that any of walkers does not meet other walkers up to time $T > 0$, *i.e.*

$$S_1(t) < S_2(t) < \cdots < S_N(t), \quad t = 1, 2, ..., T. \tag{1.1}$$

We denote by $\mathsf{Q}_T^{\mathbf{x}}$ the conditional probability of $\mathsf{P}^{\mathbf{x}}$ under the event $\Lambda_T = \Big\{S_1(t) < S_2(t) < \cdots < S_N(t),\ t = 0, 1, ..., T\Big\}$. M. Fisher called the process $\Big(\{\mathbf{S}(t)\}_{t=0,1,2,...,T}, \mathsf{Q}_T^{\mathbf{x}}\Big)$ the **vicious walker model** in his Boltzmann medal lecture [4].

We will assume the initial positions as

$$S_i(0) = 2(i-1), \qquad i = 1, 2, \cdots, N \tag{1.2}$$

in Sections 1 in this report.

Each realization of vicious walk is represented by an N-tuple of **nonintersecting lattice paths** on the 1+1 spatio-temporal plane, $\mathbb{Z} \times \{0, 1, \cdots, T\}$. An example is given by Figure 1 in the case that four walkers ($N = 4$) perform a noncolliding walk up to time $T = 6$.

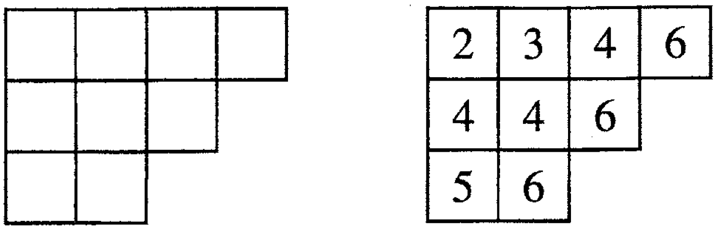
Bijection between such nonintersecting lattice paths and **semistandard Young tableaux (SSYT)**, $\mathcal{T} = (\mathcal{T}(i, j))$, is established by the following procedure [8, 17].

(1) For $1 \leq j \leq N$, let

$$L_j \equiv \text{the number of leftward steps among } T \text{ steps of the } j\text{-th walker.}$$

Draw a collection of boxes with N columns, in which the number of boxes in the j-th column is L_j. (We number columns from the left to the right.) Since $\mathbf{L} \equiv (L_1, L_2, L_3, L_4) = (3, 3, 2, 1)$ in the walk shown in Figure 1, we draw the collection of boxes as shown in Figure 2 (a) for this example.

(2) For each walker, we label each leftward step by the integer $\in \{1, 2, \cdots, T\}$, which is the time when that leftward step was done. See Figure 1, in which labels of leftward



(a) (b)

Figure 2: (a) Young diagram and (b) Young tableau $\mathcal{T}$ corresponding to the vicious walk in Figure 1.

steps are indicated by integers in small circles associated with the line segments showing leftward steps. Then for the j-th column of the collection of boxes, fill the boxes by the labels of leftward steps of the j-th walker, from the top to the bottom, $1 \leq j \leq N$. For the walk given in Figure 1, we have the boxes with integers shown in Figure 2 (b). Let

$$\mathcal{T}(i,j) = \text{the integer in the box located in the } i\text{-th row and } j\text{-th column.}$$

For example, $\mathcal{T}(1,3) = 4$ and $\mathcal{T}(3,1) = 5$ in this case.

Remark 1. The above procedure with the nonintersecting condition (1.1) guarantees the inequalities

$$L_1 \geq L_2 \geq \cdots \geq L_N, \tag{1.3}$$

and

$$\begin{aligned} &\mathcal{T}(i,j) < \mathcal{T}(i+1,j), && \text{strictly increasing in each column,} \\ &\mathcal{T}(i,j) \leq \mathcal{T}(i,j+1), && \text{weakly increasing in each row.} \end{aligned} \tag{1.4}$$

Assume that the number of rows in the collection of boxes is ℓ. Let

$$\lambda_i \equiv \text{the number of boxes in the } i\text{-th row, } i = 1, 2, \cdots, \ell.$$

Then the inequalities (1.3) imply

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell. \tag{1.5}$$

The collections of boxes with such conditions concerning the numbers of boxes in rows (1.5) (and in columns (1.3)) are called **Young diagram (YD)**. The number of rows ℓ in the YD is called the **length** of YD. In the present situation, $\ell \leq T$, in general. (In our example in Figure 2 (b), $\ell = 3$ for $T = 6$.) YD's with integers with the conditions (1.4) are called **semistandard Young tableaux (SSYT)**.

Remark 2. YD with λ_i boxes in the i-th row, $1 \leq i \leq \ell$, is said to be the YD of **shape** $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_\ell)$. The YD with the shape $\mathbf{L} = (L_1, \cdots, L_N)$ is regarded as the **conjugate** of the YD with the shape $\boldsymbol{\lambda}$ and denoted by

$$\mathbf{L} = \widetilde{\boldsymbol{\lambda}}.$$

As shown in Figure 3, they are mirror images with respect to the diagonal line.

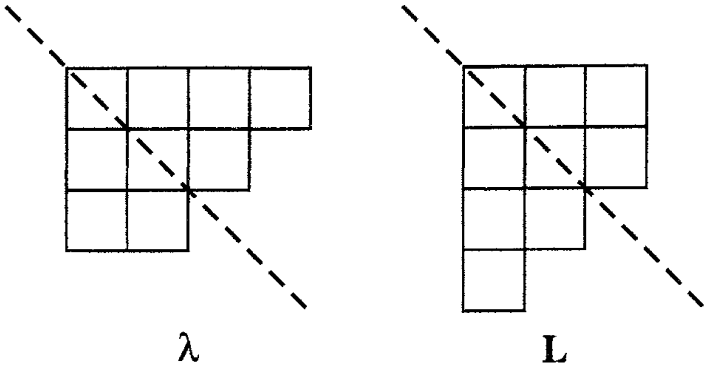


Figure 3: A pair of conjugate YD's.

A sequence of integers with the condition (1.5), that is,

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_T) \quad \text{with} \quad \lambda_i \in \mathbb{N} \equiv \Big\{x \in \mathbb{Z}, x \geq 0\Big\},\ 1 \leq i \leq T, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_T,$$

is regarded as a **partition** of an integer $n = \sum_{i=1}^{T} \lambda_i$. We introduce a set of T variables $\mathbf{z} = \{z_1, z_2, \cdots, z_T\} \in \mathbb{C}^T$ and define a monomial

$$\begin{aligned} \mathbf{z}^{\mathcal{T}} &= \prod_{(i,j)} z_{\mathcal{T}(i,j)} \\ &= \prod_{k=1}^{T} z_k^{\#\text{ of times that the integer } k \text{ occurs in } \mathcal{T}}. \end{aligned}$$

For example, the monomial corresponding to the SSYT $\mathcal{T}$ shown in Figure 2 (b) is

$$\begin{aligned} \mathbf{z}^{\mathcal{T}} &= z_2 \times z_3 \times z_4 \times z_6 \\ &\times z_4 \times z_4 \times z_6 \\ &\times z_5 \times z_6 \\ &= z_2 z_3 z_4^3 z_5 z_6^3. \end{aligned}$$

Notice that for one YD with a given shape $\boldsymbol{\lambda}$, there are different ways of filling boxes with integers to make SSYT's satisfying the conditions (1.4). For each YD with shape $\boldsymbol{\lambda}$, we define a polynomial of $\mathbf{z} = \{z_1, \cdots, z_T\}$ by summing $\mathbf{z}^{\mathcal{T}}$ over all SSYT defined on the YD:

$$s_{\lambda}(z_1, z_2, \cdots, z_T) = \sum_{\mathcal{T}:\text{all SSYT with the same shape } \boldsymbol{\lambda}} \mathbf{z}^{\mathcal{T}}.$$

This polynomial is called the **Schur function** indexed by (the partition/YD with shape) $\boldsymbol{\lambda}$ on $\mathbf{z} = (z_1, \cdots, z_T)$. We can prove the two formulae (**Jacobi-Trudi formulae**). The first one is the following.

Lemma 1

$$s_\lambda(z_1, \cdots, z_T) = \frac{\det_{1\le i,j\le T}\left[z_i^{\lambda_j+T-j}\right]}{\det_{1\le i,j\le T}\left[z_i^{T-j}\right]},$$

where the denominator is the **Vandermonde determinant** *evaluated as the* **product of differences,**

$$\det_{1\le i,j\le T}\left[z_i^{T-j}\right] = \prod_{1\le i<j\le T}(z_i - z_j). \tag{1.6}$$

This formula clarifies that the Schur functions are **symmetric polynomials** in $\mathbf{z} = (z_1, \cdots, z_T)$.

For the second formula, we define the polynomials $e_j(z_1, \cdots, z_T)$'s as the coefficients in the expansion

$$\prod_{i=1}^{T}(1 + z_i\xi) = \sum_{j=0}^{T} e_j(z_1, \cdots, z_T)\xi^j. \tag{1.7}$$

Then

$$\begin{aligned} e_j(z_1, \cdots, z_T) &= \text{sum of all monomials in the form } z_{i_1}z_{i_2}\cdots z_{i_j} \\ &\quad \text{for all strictly increasing sequences } 1 \le i_1 < i_2 < \cdots < i_j \le T. \end{aligned}$$

$e_j(z_1, \cdots, z_T)$'s are also symmetric polynomials in $z_1, \cdots, z_T$ and called the j**-th elementary symmetric polynomials**.

Lemma 2 *Assume that the conjugate of* $\boldsymbol{\lambda}$ *is given by* $\widetilde{\boldsymbol{\lambda}} = (\widetilde{\lambda}_1, \cdots, \widetilde{\lambda}_N)$ *with length* N. *Then*

$$s_\lambda(z_1, \cdots, z_T) = \det_{1\le i,j\le N}\left[e_{\widetilde{\lambda}_j+(i-j)}(z_1, \cdots, z_T)\right].$$

More details for YD, SSYT and symmetric polynomials, see *e.g.* Fulton (1997) [5].

Now we go back to the vicious walker model. We notice a simple relation between the partition $\mathbf{L} = (L_1, \cdots, L_N)$ and the final positions of the N vicious walkers at time T,

$$\mathbf{y} = (y_1, \cdots, y_N) \equiv (S_1(T), \cdots, S_N(T)),$$

given by

$$y_i = T - 2L_i + 2(i-1), \quad 1 \le i \le N,$$

on the initial condition (1.2). Then we have established the following relation between the vicious walks and YD/SSYT/Schur functions.

Set the initial positions as $\mathbf{x}_0 = (0, 2, \cdots, 2(N-1))$. For $\mathbf{y}$ such that $\mathbf{y} \in \mathbb{Z}_<^N$, if $T \in 2\mathbb{N}$, $\mathbf{y}+1 \equiv (y_1+1, \cdots, y_N+1) \in \mathbb{Z}_<^N$, if $T+1 \in 2\mathbb{N}$, let $\mathbf{L} = (L_1, \cdots, L_N)$ with $L_i = (T + x_{0i} - y_i)/2$, $1 \leq i \leq N$, and $\boldsymbol{\lambda} = \widetilde{\mathbf{L}}$. Then

- positions $\mathbf{y}$ of vicious walkers at the finial time T $\iff$ YD with the shape $\boldsymbol{\lambda}$
- a realization of vicious walk from $\mathbf{S}(0) = \mathbf{x}_0$ to $\mathbf{S}(T) = \mathbf{y}$ $\iff$ an SSYT $\mathcal{T}$ with the shape $\boldsymbol{\lambda}$
- a set of all vicious walks from $\mathbf{S}(0) = \mathbf{x}_0$ to $\mathbf{S}(T) = \mathbf{y}$ $\iff$ a Schur function $s_{\lambda}(z_1, \cdots, z_T)$.

For $\mathbf{y} = (y_1, y_2, \cdots, y_N)$ such that $\mathbf{y} \in \mathbb{Z}_<^N$, if $T \in 2\mathbb{N}$, $\mathbf{y}+1 \in \mathbb{Z}_<^N$, if $T+1 \in 2\mathbb{N}$, and $\mathbf{x}_0 = (0, 2, \cdots, 2(N-1))$, define the number

$$
\begin{aligned}
M_{N,T}(\mathbf{y}) \;=\;& \text{total number of distinct realizations of vicious walk of } N \text{ walkers} \\
& \text{from the positions } \mathbf{S}(0) = \mathbf{x}_0 \text{ to } \mathbf{S}(T) = \mathbf{y}.
\end{aligned}
$$

The above relations prove the following identity.

Assume that $L_i = \{T + 2(i-1) - y_i\}/2, \mathbf{L} = (L_1, \cdots, L_N)$, and $\boldsymbol{\lambda} = \widetilde{\mathbf{L}}$. Then

$$
M_{N,T}(\mathbf{y}) = s_{\lambda}(z_1, \cdots, z_T)\Big|_{z_1=z_2=\cdots=z_T=1}.
$$

Lemma 1 gives the following estimate for $M_{N,T}(\mathbf{y})$, via appropriate q-**factorization** and the formula for the Vandermonde determinant (1.6).

Proposition 3

$$
\begin{aligned}
M_{N,T}(\mathbf{y}) &= \lim_{q \to 1} s_{\lambda}(1, q, q^2, \cdots, q^{T-1}) \\
&= \lim_{q \to 1} \frac{\det_{1 \leq i,j \leq T} \left[q^{(i-1)(\lambda_j + T - j)} \right]}{\det_{1 \leq i,j \leq T} \left[q^{(i-1)(T-j)} \right]} \\
&= \lim_{q \to 1} q^{\sum_{k=1}^{T} (k-1)\lambda_k} \prod_{1 \leq i < j \leq T} \frac{q^{\lambda_i - \lambda_j + j - i} - 1}{q^{j-i} - 1} \\
&= \prod_{1 \leq i < j \leq T} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\end{aligned}
$$

On the other hand, by definition (1.7), it is easy to see that

$$e_j(z_1, \cdots, z_T)\Big|_{z_1=\cdots=z_T=z} = \binom{T}{j} z^T \equiv \frac{T!}{j!(T-j)!} z^T.$$

Then Lemma 2 gives the following.

Proposition 4

$$\begin{aligned} M_{N,T}(\mathbf{y}) &= \det_{1 \le i,j \le N} \left[\binom{T}{\widetilde{\lambda}_j + i - j} \right] \\ &= \det_{1 \le i,j \le N} \left[\binom{T}{L_j + i - j} \right] \\ &= \det_{1 \le i,j \le N} \left[\binom{T}{\{T + 2(i-1) - y_j\}/2} \right]. \end{aligned}$$

2 Determinantal Formula for Nonintersecting Paths

Since we have assumed the initial positions as (1.2), we can see that the (i,j)-element of the matrix in the determinant in Proposition 4 is

$$\begin{aligned} \binom{T}{\{T + 2(i-1) - y_j\}/2} &= \#\Big\{ \textbf{lattice path} \text{ from } 2(i-1) \text{ at time } 0 \text{ to } y_j \text{ at time } T \Big\} \\ &= \sum_{\text{all lattice paths: } (2(i-1),0) \rightsquigarrow (y_j,T)} w(\text{path}) \Big|_{w(\text{path})=1}. \end{aligned}$$

If we define an appropriate weight function $w(\text{path})$ on single lattice paths, the summation of $w(\text{path})$ will give the **Green function** of single lattice paths,

$$G\Big((x,0),(y,T)\Big) = \sum_{\text{all lattice paths: } (x,0) \rightsquigarrow (y,T)} w(\text{path}).$$

Proposition 4 can be regarded as a special case of the **Karlin-McGregor formula** in the probability theory [10, 11], and the **Lindström-Gessel-Viennot formula** in the enumerative combinatorics (see [19, 17] and references therein). In order to explain this fact, here we introduce some definitions and notations for describing **lattice paths**.

Let $V = \{\text{vertex}\}, E = \{\text{directed edge}\}$, $D = (V, E) =$ an **acyclic** directed graph, where acyclic means that any cycles of directed edges are forbidden. For $u, v \in V$,

$$\begin{aligned} \text{a } \textbf{lattice path } u \to v &= \text{a sequence of directed edges from } u \text{ to } v, \\ \mathcal{P}(u,v) &= \text{the set of all lattice paths from } u \text{ to } v. \end{aligned}$$

A weight function $w : E \to \mathbb{Z}[[x_e : e \in E]]$ is introduced, where $\mathbb{Z}[[x_e : e \in E]]$ denotes a ring of formal power series of $\{x_e : e \in E\}$ and the **weight on a lattice path** P is defined by

$w(P) \equiv \prod_{e \in P} w(e)$. Then the **Green function of lattice paths** from u to v is defined by

$$G(u,v) = \sum_{P:P \in \mathcal{P}(u,v)} w(P).$$

Let $I = \{u_1, u_2, \cdots, u_N\}$, $J = \{v_1, v_2, \cdots, v_N\}$ with $u_i, v_i \in V, i = 1, 2, \cdots, N$. The sets I, J are ordered; $u_1 < u_2 < \cdots < u_N, v_1 < v_2 < \cdots < v_N$. Then we consider a set of **N-tuples of lattice paths**

$$\mathcal{P}(I,J) = \Big\{ \mathbf{P} = (P_1, \cdots, P_N) : P_i \in \mathcal{P}(u_i, v_i), i = 1, 2, \cdots, N \Big\}.$$

The weight for each N-tuple of lattice paths is given by

$$w(\mathbf{P}) = \prod_{i=1}^{N} w(P_i) = \prod_{i=1}^{N} \prod_{e \in P_i} w(e).$$

We say 'lattice paths P and Q **intersect**', if P and Q share at least one common vertex. Then, for ordered sets of vertices I and J, we say 'I is **D-compatible** with J' in the case that, whenever $u_i < u_j$ in I and $v_i < v_j$ in J, every lattice path $P \in \mathcal{P}(u_i, v_j)$ intersects every lattice path $Q \in \mathcal{P}(u_j, v_i)$. A set of **$N$-tuples of nonintersecting lattice paths** is denoted by

$$\mathcal{P}_0(I,J) = \Big\{ \mathbf{P} \in \mathcal{P}(I,J) : \text{any lattice paths in } \mathbf{P} \text{ do not intersect with others} \Big\},$$

and the **Green function of N-tuples of nonintersecting lattice paths** is defined by

$$G_{\text{nonint}}(I,J) = \sum_{\mathbf{P} \in \mathcal{P}_0(I,J)} w(\mathbf{P}).$$

Theorem 5 *Let $I = (u_1, \cdots, u_N)$ and $J = (v_1, \cdots, v_N)$ be two ordered sets of vertices in an acyclic graph D. If I is D-compatible with J, then the Green function of the nonintersecting N-tuples of lattice paths is given by*

$$G_{\text{nonint}}(I,J) = \det_{1 \le i,j \le N} \Big[G(u_i, v_j) \Big],$$

where $G(u,v)$ denotes the Green function of single lattice paths from u to v on D.

The proof of Theorem 5 is given in Appendix A following Stembridge (1990) [19]. For our vicious walker model, consider the directed graph $D = (V, E)$, where

$$V = \Big\{ (x,t) \in \mathbb{Z}^2 : x + t = \text{even}, \ t = 0, 1, 2, \cdots, T \Big\},$$

and all edges connecting the nearest-neighbor pairs of vertices in V are oriented to the positive direction of t axis. Set the weight function

$$w(e) = \begin{cases} 1 & \text{for } e = \langle (x,t-1) \to (x-1,t) \rangle \\ 1 & \text{for } e = \langle (x,t-1) \to (x+1,t) \rangle \\ 0 & \text{otherwise.} \end{cases}$$

Set $T \in \mathbb{N}$ and $u_i = (x_i, 0), v_i = (y_i, T) \in V, i = 1, 2, \cdots, N$. Then the Green function of single lattice paths from u_i to v_i is

$$G(u_i, v_i) = \Big|\mathcal{P}(u_i \to v_i)\Big| = \binom{T}{(T + x_i - y_i)/2}.$$

Theorem 5 then gives the Green function of the N-tuples of nonintersecting lattice paths from $I = \Big\{(x_i, 0)\Big\}_{i=1}^N$ to $J = \Big\{(y_i, T)\Big\}_{i=1}^N$ as $\det_{1 \le i,j \le N} \left[\binom{T}{(T + x_i - y_j)/2}\right]$.

For $\mathbf{x} \in \mathbb{Z}^N_<$, and $\mathbf{y}$ such that $\mathbf{y} \in \mathbb{Z}^N_<$, if $T \in 2\mathbb{N}$, $\mathbf{y} + 1 \in \mathbb{Z}^N_<$, if $T + 1 \in 2\mathbb{N}$, define

$M_N(T, \mathbf{y}|\mathbf{x})$ = total number of distinct realizations of vicious walk of N walkers from the positions $\mathbf{x} = (x_1, \cdots, x_N)$ to $\mathbf{y} = (y_1, \cdots, y_N)$ during time T.

Proposition 4 is now generalized as follows.

Proposition 6 $$M_N(T, \mathbf{y}|\mathbf{x}) = \det_{1 \le i,j \le N} \left[\binom{T}{(T + x_i - y_j)/2}\right].$$

3 Diffusion Scaling Limit

Recall that $(\{\mathbf{S}(t)\}_{t=0,1,2,\ldots,T}, \mathrm{Q}^{\mathbf{x}}_T)$ denotes the vicious walk with the noncolliding condition up to time $T > 0$ starting from the positions $\mathbf{x} \in \mathbb{Z}^N_<$. For $L \ge 1$, we consider probability measures $\mu^{\mathbf{x}}_{L,T}$ on the space of continuous paths $C([0,T] \to \mathbb{R}^N)$ defined by

$$\mu^{\mathbf{x}}_{L,T}(\cdot) = \mathrm{Q}^{\mathbf{x}}_{L^2T}\left(\frac{1}{L}\mathbf{S}(L^2t) \in \cdot\right),$$

where $\mathbf{S}(t), t \ge 0$, is now considered to be the interpolation of the N-dimensional random walk $\mathbf{S}(t), t = 0, 1, 2, \ldots$. We study the limit of the probability measure $\mu^{\mathbf{x}}_{L,T}$, $L \to \infty$.

We put

$$\mathbb{R}^N_< = \Big\{\mathbf{x} \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\Big\},$$

which can be called the **Weyl chamber of type** A_{N-1} (see, for example, [6]). By virtue of the Karlin-McGregor formula [10, 11], the transition density function $f_N(t, \mathbf{y}|\mathbf{x})$ of the **absorbing Brownian motion** in $\mathbb{R}^N_<$ and the probability $\mathcal{N}_N(t, \mathbf{x})$ that the Brownian motion starting from $\mathbf{x} \in \mathbb{R}^N_<$ does not hit the boundary of $\mathbb{R}^N_<$ up to time $t > 0$ are given by

$$f_N(t, \mathbf{y}|\mathbf{x}) = \det_{1 \le i,j \le N}\left[\frac{1}{\sqrt{2\pi t}}\, e^{-(x_j - y_i)^2/2t}\right], \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N_<,$$

and $\mathcal{N}_N(t,\mathbf{x}) = \int_{\mathbb{R}^N_<} d\mathbf{y} f_N(t,\mathbf{y}|\mathbf{x})$, respectively. We put $h_N(\mathbf{x}) = \prod_{1\le i<j\le N} (x_j - x_i)$, and let $\mathbf{0} = (0,0,\cdots,0) \in \mathbb{R}^N$, which describes the state that all N particles are at the origin 0.

Theorem 7 (i) *For any fixed* $\mathbf{x} \in \mathbb{Z}^N_<$ *and* $T > 0$, *as* $L \to \infty$, $\mu^{\mathbf{x}}_{L,T}(\cdot)$ *converges weakly to the law of the* **temporally inhomogeneous diffusion process** $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t)), t \in [0,T]$, *with transition probability density* $g_{N,T}(s,\mathbf{x};t,\mathbf{y})$;

$$g_{N,T}(0,\mathbf{0};t,\mathbf{y}) = c_N T^{N(N-1)/4} t^{-N^2/2} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h_N(\mathbf{y})\mathcal{N}_N(T-t,\mathbf{y}),$$

$$g_{N,T}(s,\mathbf{x};t,\mathbf{y}) = \frac{f_N(t-s,\mathbf{y}|\mathbf{x})\mathcal{N}_N(T-t,\mathbf{y})}{\mathcal{N}_N(T-s,\mathbf{x})},$$

for $0 < s < t \le T$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^N_<$, *where* $c_N = 2^{-N/2}/\prod_{i=1}^N \Gamma(i/2)$ *with the gamma function* Γ.
(ii) *The diffusion process* $\mathbf{X}(t)$ *solves the following equation:*

$$dX_i(t) = dB_i(t) + b_i^T(t,\mathbf{X}(t))dt, \quad t \in [0,T], \quad i = 1,2,\ldots,N,$$

where $B_i(t)$, $i = 1,2,\ldots,N$, *are independent one-dimensional Brownian motions and*

$$b_i^T(t,\mathbf{x}) = \frac{\partial}{\partial x_i} \ln \mathcal{N}_N(T-t,\mathbf{x}), \quad i = 1,2,\ldots,N.$$

Figure 4 illustrates the process $\mathbf{X}(t), t \in [0,T]$, when all N particles start from the origin; $\mathbf{X}(0) = \mathbf{0}$.

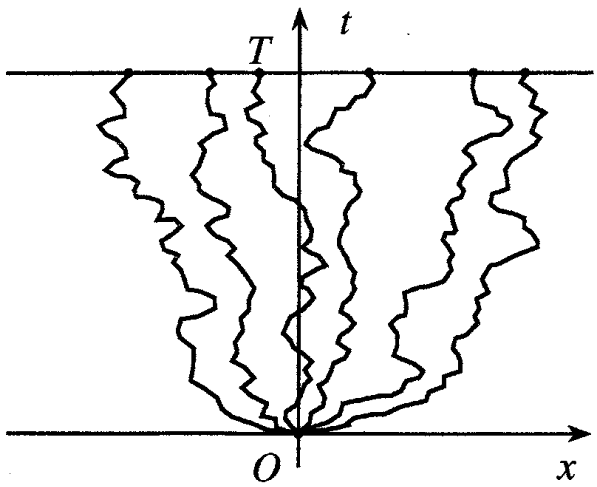


Figure 4: The process $\mathbf{X}(t), t \in [0,T]$, starting from $\mathbf{0}$.

Although $\mu^{\mathbf{x}}_{L,T}(\cdot)$ is the probability measure defined on $C([0,T] \to \mathbb{R}^N)$, it can be regarded as that on $C([0,\infty) \to \mathbb{R}^N)$ concentrated on the set $\{w \in C([0,\infty) \to \mathbb{R}^N) : w(t) = w(T), t \geq T\}$. Next we consider the case that $T = T(L)$ goes to infinity as $L \to \infty$.

Corollary 8 (i) *Let $T(L)$ be an increasing function of L with $T(L) \to \infty$ as $L \to \infty$. For any fixed $\mathbf{x} \in \mathbb{Z}^N_<$, as $L \to \infty$, $\mu^{\mathbf{x}}_{L,T(L)}(\cdot)$ converges weakly to the law of the* **temporally homogeneous diffusion process** $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \dots, Y_N(t)), t \in [0,\infty)$, *with transition probability density $p_N(s,\mathbf{x};t,\mathbf{y})$;*

$$p_N(0,\mathbf{0};t,\mathbf{y}) = c'_N t^{-N^2/2} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h_N(\mathbf{y})^2,$$

$$p_N(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{h_N(\mathbf{x})} f_N(t-s,\mathbf{y}|\mathbf{x}) h_N(\mathbf{y}), \tag{3.1}$$

for $0 < s < t < \infty$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^N_<$, where $c'_N = (2\pi)^{-N/2}/\prod_{i=1}^N \Gamma(i)$.
(ii) *The diffusion process $\mathbf{Y}(t)$ solves the equations of* **Dyson's Brownian motion model** *with the parameter $\beta = 2$,*

$$dY_i(t) = dB_i(t) + \sum_{1 \leq j \leq N, j \neq i} \frac{1}{Y_i(t) - Y_j(t)} dt, \quad t \in [0,\infty), \quad i = 1,2,\dots,N.$$

Here we only give the proof of a key lemma used to prove Theorem 7, in order to demonstrate that the Schur function plays an important role. See Katori and Tanemura [12] for the complete proofs of Theorem 7 and Corollary 8.

For $L > 0$ we introduce the following functions:

$$\phi_L(x) = 2\left[\frac{Lx}{2}\right], \; x \in \mathbb{R}, \text{ and } \phi_L(\mathbf{x}) = \Big(\phi_L(x_1), \phi_L(x_2), \dots, \phi_L(x_N)\Big), \; \mathbf{x} \in \mathbb{R}^N,$$

where $[a]$ denotes the largest integer not greater than a. Let $V_N(T,\mathbf{y}|\mathbf{x}) = 2^{-NT} M_N(T,\mathbf{y}|\mathbf{x})$, where $M_N(T,\mathbf{y}|\mathbf{x})$ is given by Proposition 6.

Lemma 9 *For $t > 0$, $\mathbf{x} \in \mathbb{Z}^N_<$ and $\mathbf{y} \in \mathbb{R}^N_<$.*

$$\left(\frac{L}{2}\right)^N V_N\Big(\phi_{L^2}(t), \phi_L(\mathbf{y})\Big|\mathbf{x}\Big) = c'_N t^{-N^2/2} h_N\left(\frac{\mathbf{x}}{L}\right) \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h_N(\mathbf{y}) \left(1 + \mathcal{O}\left(\frac{|\mathbf{y}|}{L}\right)\right),$$

as $L \to \infty$.

Proof. It will be enough to consider the case that $\mathbf{x} = 2\mathbf{u} = (2u_1, \cdots, 2u_N) \in \mathbb{Z}^N_<, \mathbf{y} = 2\mathbf{v} =$

$(2v_1, \cdots, 2v_N) \in \mathbb{Z}^N_<$, and $\phi_{L^2}(t) = 2\ell, \ell \in \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{1, 2, 3, \cdots\}$. Then

$$M_N\Big(\phi_{L^2}(t), \phi_L(\mathbf{y})\Big|\mathbf{x}\Big) = M_N(2\ell, 2\mathbf{v}|2\mathbf{u}) = \det_{1\le i,j\le N}\left[\binom{2\ell}{\ell + u_j - v_i}\right],$$

and

$$\begin{aligned}\binom{2\ell}{\ell + u_j - v_i} &= \frac{(2\ell)!}{(\ell + u_j - v_i)!(\ell - u_j + v_i)!} \\ &= \frac{(2\ell)!}{(\ell - v_i)!(\ell + v_i)!} A_{ij}(\ell, \mathbf{v}, \mathbf{u}),\end{aligned}$$

with

$$A_{ij}(\ell, \mathbf{v}, \mathbf{u}) = \frac{(\ell + v_i - u_j + 1)_{u_j}}{(\ell - v_i + 1)_{u_j}},$$

where we have used the Pochhammer symbol; $(a)_0 \equiv 1$, $(a)_i = a(a+1)\cdots(a+i-1)$, $i \ge 1$. Then

$$M_N\Big(\phi_{L^2}(t), \phi_L(\mathbf{y})\Big|\mathbf{x}\Big) = \prod_{i=1}^N \frac{(2\ell)!}{(\ell - v_i)!(\ell + v_i)!} \det_{1\le i,j\le N}\Big[A_{ij}(\ell, \mathbf{v}, \mathbf{u})\Big]. \tag{3.2}$$

The leading term of $\det_{1\le i,j\le N}\Big[A_{ij}(\ell, \mathbf{v}, \mathbf{u})\Big]$ in $L \to \infty$ is

$$\begin{aligned}D_1(\mathbf{v}, \mathbf{u}) &= \det_{1\le i,j\le N}\left[\left(\frac{\ell + v_i}{\ell - v_i}\right)^{u_j}\right] \\ &= (-1)^{N(N-1)/2} \det_{1\le i,j\le N}\left[\left(\frac{\ell + v_i}{\ell - v_i}\right)^{u_{N-j+1}}\right].\end{aligned}$$

Let $\boldsymbol{\xi}(\mathbf{u}) = (\xi_1(\mathbf{u}), \ldots, \xi_N(\mathbf{u}))$ be a partition specified by the starting point $2\mathbf{u}$ defined by

$$\xi_j(\mathbf{u}) = u_{N-j+1} - (N - j), \ j = 1, 2, \ldots, N.$$

We have

$$\begin{aligned}D_1(\mathbf{v}, \mathbf{u}) &= (-1)^{N(N-1)/2} \det_{1\le i,j\le N}\left[\left(\frac{\ell + v_i}{\ell - v_i}\right)^{N-j}\right] s_{\xi(\mathbf{u})}\left(\frac{\ell + v_1}{\ell - v_1}, \ldots, \frac{\ell + v_N}{\ell - v_N}\right) \\ &= (-1)^{N(N-1)/2} \prod_{1\le i<j\le N}\left(\frac{\ell + v_i}{\ell - v_i} - \frac{\ell + v_j}{\ell - v_j}\right) s_{\xi(\mathbf{u})}\left(\frac{\ell + v_1}{\ell - v_1}, \ldots, \frac{\ell + v_N}{\ell - v_N}\right) \\ &= \prod_{1\le i<j\le N} \frac{2\ell(v_j - v_i)}{(\ell - v_i)(\ell - v_j)} s_{\xi(\mathbf{u})}\left(\frac{\ell + v_1}{\ell - v_1}, \ldots, \frac{\ell + v_N}{\ell - v_N}\right),\end{aligned}$$

where we have used Lemma 1 for the Schur function associated to the partition $\boldsymbol{\xi}(\mathbf{u})$. Proposition 3 gives

$$s_{\xi(\mathbf{u})}(1, 1, \ldots, 1) = \prod_{1\le i<j\le N} \frac{\xi_i(\mathbf{u}) - \xi_j(\mathbf{u}) + j - i}{j - i}.$$

Therefore the leading term of $D_1(\mathbf{v},\mathbf{u})$ in $L\to\infty$ is

$$\begin{aligned}D_2(\mathbf{v},\mathbf{u}) &= \prod_{1\le i<j\le N}\frac{2(v_j-v_i)}{\ell}\times s_{\xi(\mathbf{u})}(1,1,\dots,1)\\ &= \ell^{-N(N-1)/2}2^{N(N-1)/2}h_N(\mathbf{v})h_N(\mathbf{u})\prod_{1\le i<j\le N}\frac{1}{j-i}\\ &= h_N\left(\frac{\mathbf{v}}{\ell}\right)h_N(2\mathbf{u})\prod_{i=1}^{N}\frac{1}{\Gamma(i)}. \end{aligned}\tag{3.3}$$

On the other hand, by Stirling's formula we see that

$$\prod_{i=1}^{N}\frac{(2\ell)!}{(\ell-v_i)!(\ell+v_i)!}=(\ell\pi)^{-N/2}2^{2N\ell}\prod_{i=1}^{N}\left(1-\frac{v_i^2}{\ell^2}\right)^{-\ell-1/2}\left(\frac{1-v_i/\ell}{1+v_i/\ell}\right)^{v_i}\left(1+\mathcal{O}\left(\frac{1}{\ell}\right)\right). \tag{3.4}$$

From (3.2), (3.3) and (3.4)

$$\begin{aligned}&V_N\Big(\phi_{L^2}(t),\phi_L(\mathbf{y})\Big|\mathbf{x}\Big)=2^{-2N\ell}M_N\Big(\phi_{L^2}(t),\phi_L(\mathbf{y})\Big|\mathbf{x}\Big)\\ &=c_N'\left(\frac{2}{\ell}\right)^{N/2}h_N\left(\frac{\mathbf{v}}{\ell}\right)h_N\left(2\mathbf{u}\right)\exp\left\{-\frac{|\mathbf{v}|^2}{\ell}\right\}\left(1+\mathcal{O}\left(\frac{|\mathbf{v}|}{\ell}\right)\right)\\ &=c_N'\left(\frac{2}{L}\right)^{N}t^{-N^2/2}h_N\left(\frac{\mathbf{x}}{L}\right)\exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\}h_N(\mathbf{y})\left(1+\mathcal{O}\left(\frac{|\mathbf{y}|}{L}\right)\right).\end{aligned}$$

Then we obtain Lemma 9. ∎

Corollary 8 is obtained from Theorem 7 by the following evaluation of asymptotic [12]. Let $t>0$ and $\mathbf{x}\in\mathbb{R}^N_<$, then

$$\mathcal{N}_N(t,\mathbf{x})=\frac{1}{\overline{c}_N}h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right)\left(1+\mathcal{O}\left(\frac{|\mathbf{x}|}{\sqrt{t}}\right)\right),\quad \text{in the limit}\quad \frac{|\mathbf{x}|}{\sqrt{t}}\to 0,$$

where $\overline{c}_N=\pi^{N/2}\prod_{i=1}^N\Big\{\Gamma(i)/\Gamma(i/2)\Big\}$.

4 Eigenvalue Process of Hermitian Matrix-valued Processes

We consider complex-valued processes $\xi_{ij}(t)\in\mathbb{C}, 1\le i,j\le N, t\in[0,\infty)$, with the condition $\xi_{ji}(t)^*=\xi_{ij}(t)$, and introduce **Hermitian matrix-valued processes** $\Xi(t)=\Big(\xi_{ij}(t)\Big)_{1\le i,j\le N}$. We denote by $U(t)=\Big(u_{ij}(t)\Big)_{1\le i,j\le N}$ the family of unitary matrices which diagonalize $\Xi(t)$ so that

$$U(t)^\dagger\Xi(t)U(t)=\Lambda(t)=\mathrm{diag}\Big\{\lambda_1(t),\lambda_2(t),\cdots,\lambda_N(t)\Big\}, \tag{4.1}$$

where $\Big\{\lambda_i(t)\Big\}_{i=1}^N$ are eigenvalues of $\Xi(t)$ and we assume their increasing order

$$\lambda_1(t)\le\lambda_2(t)\le\cdots\le\lambda_N(t).$$

Define $\Gamma_{ij}(t), 1 \le i,j \le N$, by

$$\Gamma_{ij}(t)dt = \Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{ij} \Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{ji}, \tag{4.2}$$

where $d\Xi(t) = \Big(d\xi_{ij}(t)\Big)_{1\le i,j\le N}$. The indicator function $\mathbf{1}_{\{\omega\}}$ gives $\mathbf{1}_{\{\omega\}} = 1$ if the condition ω is satisfied, and $\mathbf{1}_{\{\omega\}} = 0$ otherwise.

Theorem 10 *Assume that $\xi_{ij}(t), 1 \le i < j \le N$, are continuous semimartingales. The process of eigenvalues $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t))$ satisfies the* **stochastic differential equations**

$$d\lambda_i(t) = dM_i(t) + dJ_i(t), \quad t \in [0,\infty), i = 1, 2, \cdots, N, \tag{4.3}$$

where $M_i(t)$ is the martingale with quadratic variation

$$\langle M_i \rangle_t = \int_0^t \Gamma_{ii}(s) ds \tag{4.4}$$

and $J_i(t)$ is the process with finite variation given by

$$dJ_i(t) = \sum_{j=1}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} \mathbf{1}_{\{\lambda_i(t) \ne \lambda_j(t)\}} \Gamma_{ij}(t) dt + d\Upsilon_i(t) \tag{4.5}$$

where $d\Upsilon_i(t)$ is the finite-variation part of $\Big(U(t)^\dagger d\Xi(t) U(t)\Big)_{ii}$.

This theorem is obtained by simple generalization of Theorem 1 in Bru [1]. A key point to derive the theorem is applying the **Itô rule** for differentiating the product of matrix-valued semimartingales: If X and Y are $N \times N$ matrices with semimartingale elements, then

$$d(X^\dagger Y) = (dX)^\dagger Y + X^\dagger (dY) + (dX)^\dagger (dY).$$

We give the proof in Appendix B. (See Remark 3 below.)

Let $B_{ij}(t)$, $\widetilde{B}_{ij}(t)$, $1 \le i,j \le N$, be independent one-dimensional Brownian motions. For $1 \le i,j \le N$ we set

$$s_{ij}(t) = \begin{cases} \dfrac{1}{\sqrt{2}} B_{ij}(t), & \text{if } i < j, \\ B_{ii}(t), & \text{if } i = j, \\ \dfrac{1}{\sqrt{2}} B_{ji}(t), & \text{if } i > j, \end{cases} \quad \text{and} \quad a_{ij}(t) = \begin{cases} \dfrac{1}{\sqrt{2}} \widetilde{B}_{ij}(t), & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -\dfrac{1}{\sqrt{2}} \widetilde{B}_{ji}(t), & \text{if } i > j. \end{cases}$$

A Hermitian matrix-valued process is defined by

$$\Xi(t) = \Big(\xi_{ij}(t)\Big)_{1\le i,j\le N} = \Big(s_{ij}(t) + \sqrt{-1} a_{ij}(t)\Big)_{1\le i,j\le N}, \quad t \in [0,\infty). \tag{4.6}$$

By definition $d\xi_{ij}(t)d\xi_{k\ell}(t) = \delta_{i\ell}\delta_{jk}dt$, $1 \le i,j,k,\ell \le N$, and thus $\Gamma_{ij}(t) = 1$. Theorem 10 thus implies that the eigenvalue process $\boldsymbol{\lambda}(t)$ of this matrix-valued process (4.6) solves the equations of **Dyson's Brownian motion model** with the parameter $\beta = 2$ [3]

$$d\lambda_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{1 \le j \le N, j \ne i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, \quad t \in [0,\infty),\, i = 1,2,\cdots,N, \tag{4.7}$$

where $B_i(t), i = 1,2,\cdots,N$ are independent one-dimensional Brownian motions.

Remark 3. In general, Eqs.(4.3) with (4.4) and (4.5) for the eigenvalue process $\boldsymbol{\lambda}(t) = (\lambda_1(t),\cdots,\lambda_N(t))$ depend on unitary matrix $U(t)$ through $\Gamma_{ij}(t)$ defined by (4.2). The equations written in the form,

$$d\lambda_i(t) = \sum_j \alpha_{ij}(t,\boldsymbol{\lambda}(t))dB_j(t) + \beta_i(t,\boldsymbol{\lambda}(t))dt, \quad t \in [0,\infty),\, i = 1,2,\cdots,N, \tag{4.8}$$

where the coefficients $\alpha_{ij}(t,\boldsymbol{\lambda})$ and $\beta_i(t,\boldsymbol{\lambda})$ are functions not only of $\boldsymbol{\lambda}$ but also of other variables, are generally called **stochastic differential equations (SDE's)** in [9] (see Definition 1.1 with Eqs.(1.1), (1.1') in Chapter IV 'Stochastic Differential Equations' on page 159.) In the special case, in which these coefficients are only depending on $\boldsymbol{\lambda}(t)$, equations are given in the form

$$d\lambda_i(t) = \sum_j \sigma_{ij}(\boldsymbol{\lambda}(t))dB_j(t) + b_i(\boldsymbol{\lambda}(t))dt, \quad t \in [0,\infty),\, i = 1,2,\cdots,N, \tag{4.9}$$

and they are said to be of the **Markovian type** (see page 172 with Eq. (2.11) in [9]). The condition that the SDE's of eigenvalue process are reduced to be of the Markovian type may be that the matrix-valued process $\Xi(t)$ is **unitary invariant in distribution**. By virtue of properties of Brownian motions, the Hermitian matrix-valued process $\Xi(t)$ defined by (4.6) is unitary invariant in distribution, and thus the obtained SDE's of Dyson's Brownian motion model are of the Markovian type.

5 Concluding Remarks

Corollary 8(ii) and Eq.(4.7) with $\beta = 2$ implies that the temporally homogeneous process $\mathbf{Y}(t)$ obtained as a diffusion scaling limit of vicious walks and the eigenvalue process $\boldsymbol{\lambda}(t)$ of the Hermitian matrix-valued process (4.6) are equivalent in distribution. The formula (3.1) in Corollary 8 shows that it is the **h-transform in the sense of Doob** [2] of the **absorbing Brownian motion in the Weyl chamber $\mathbb{R}^N_<$**, since $h_N(\mathbf{x})$ is a strictly positive harmonic function in $\mathbb{R}^N_<$ [7]. An interesting relationship between this temporally homogeneous process $\mathbf{Y}(t)$ (Dyson's Brownian motion model with the parameter $\beta = 2$) and the temporally inhomogeneous process $\mathbf{X}(t)$ given by Theorem 7 was reported in [12, 14]. A systematic study on the relations among various matrix-valued processes, standard, chiral and non-standard random matrix theories, and families of noncolliding diffusion processes was reported in [15].

For the noncolliding diffusion processes starting from $\mathbf{0}$, the **multi-time correlation functions** were calculated by Nagao and the present authors using the quaternion determinants of self-dual quaternion matrices (*i.e.* **pfaffians**) and the scaling limits of the infinite particles $N \to \infty$ and the infinite time-interval $T \to \infty$ were investigated [18, 13]. Further study of **infinite systems of noncolliding diffusion particles** will be reported elsewhere [16].

A Proof of Theorem 5

By definition of determinant

$$\det_{1 \le i,j \le N} \left[G(u_i, v_j) \right] = \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) G(u_1, v_{\sigma(1)}) \cdots G(u_N, v_{\sigma(N)}), \tag{A.1}$$

where $\mathfrak{S}_N$ is the set of all permutations of $\{1, 2, \cdots, N\}$. We may interpret (A.1) as a generating function for $(N+1)$-tuples $(\sigma, P_1, \cdots, P_N)$, where $\sigma \in \mathfrak{S}_N, P_i \in \mathcal{P}(u_i, v_{\sigma(i)}), i = 1, 2, \cdots, N$.

Consider an arbitrary configuration $(\sigma, P_1, \cdots, P_N)$ with at least one pair of intersecting lattice paths. We set the order of all vertices in V. Let v denote the last vertex among all vertices that occur as points of intersection among the lattice paths. Among the lattice paths that pass through v, assume that P_i and P_j are the two whose indices i and j are the smallest (see Figure 5).

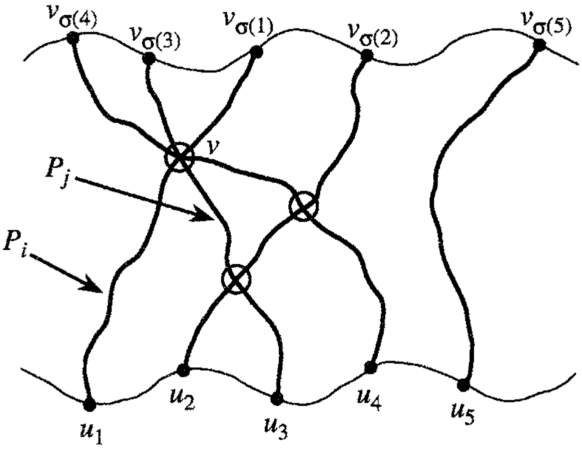


Figure 5: A configuration $(\sigma, P_1, \cdots, P_N)$. The last intersecting vertex is denoted by v and lattice paths P_i and P_j are chosen, both of which pass through v.

Write

$$P_i = P_i(\to v)P_i(v \to), \quad \text{and} \quad P_j = P_j(\to v)P_j(v \to).$$

For a configuration $(\sigma, P_1, \cdots, P_N)$, define as shown in Figure 6

$$\begin{aligned} P_i' &= P_i(\to v)P_j(v \to), \\ P_j' &= P_j(\to v)P_i(v \to), \\ P_k' &= P_k \quad \text{for } k \neq i, j, \\ \sigma' &= \sigma \circ (i, j), \quad \text{where } (i,j) \text{ denotes an exchange of } i \text{ and } j. \end{aligned}$$

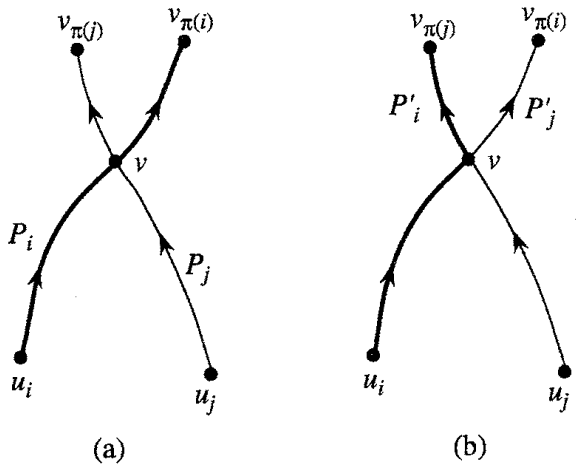


Figure 6: (a) Paths P_i and P_j. (b) Paths P'_i and P'_j.

The operation

$$(\sigma, P_1, \cdots, P_N) \mapsto (\sigma', P'_1, \cdots, P'_N)$$

preserves the set of vertices of intersection, and is an involution. The weight of lattice paths is the same, but the sign is changed. So any such pair $\left\{(\sigma, P_1, \cdots, P_N), (\sigma', P'_1, \cdots, P'_N)\right\}$ appear in (A.1) is canceled out.

The only configurations remain in (A.1) are nonintersecting lattice paths. Since I is assumed to be D-compatible with J, for nonintersecting lattice paths $\sigma = \mathrm{id}$, *i.e.*, $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\mathrm{id}) = 1$. ∎

B Proof of Theorem 10

We consider a matrix-valued process $A(t) = (\alpha_{ij}(t))_{1 \le i,j \le N}$ defined by

$$dA(t) = U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t), \quad t \in [0, \infty)$$

with $A(0) = 0$. Since $U(t)^\dagger U(t) = I_N$ for all t, where I_N denotes the $N \times N$ unit matrix,

$$0 = d(U(t)^\dagger U(t)) = dU(t)^\dagger U(t) + U(t)^\dagger dU(t) + dU(t)^\dagger dU(t).$$

Then

$$\begin{aligned} dA(t)^\dagger &= dU(t)^\dagger U(t) + \frac{1}{2} dU(t)^\dagger dU(t) \\ &= -U(t)^\dagger dU(t) - \frac{1}{2} dU(t)^\dagger dU(t) = -dA(t), \end{aligned}$$

that is, $dA(t)$ is anti-Hermitian. We also see that

$$\begin{aligned} -dA(t)dA(t) &= dA(t)^\dagger dA(t) \\ &= \left(U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t)\right)^\dagger \left(U(t)^\dagger dU(t) + \frac{1}{2} dU(t)^\dagger dU(t)\right) \\ &= dU(t)^\dagger U(t) U(t)^\dagger dU(t) = dU(t)^\dagger dU(t). \end{aligned} \tag{B.1}$$

This implies

$$dU(t) = U(t)\left(dA(t) + \frac{1}{2}dA(t)dA(t)\right). \tag{B.2}$$

By (4.1)

$$\begin{aligned}
d\Lambda(t) &= dU(t)^\dagger\Xi(t)U(t) + U(t)^\dagger d\Xi(t)U(t) + U(t)^\dagger\Xi(t)dU(t) \\
&\quad + dU(t)^\dagger d\Xi(t)U(t) + dU(t)^\dagger\Xi(t)dU(t) + U(t)^\dagger d\Xi(t)dU(t) \\
&= U(t)^\dagger d\Xi(t)U(t) + \left\{\Lambda(t)U(t)^\dagger dU(t) + (\Lambda(t)U(t)^\dagger dU(t))^\dagger\right\} \\
&\quad + \left\{U(t)^\dagger d\Xi(t)dU(t) + (U(t)^\dagger d\Xi(t)dU(t))^\dagger\right\} + dU(t)^\dagger\Xi(t)dU(t).
\end{aligned}$$

Each term in the RHS is rewritten as follows:

$$\begin{aligned}
\Lambda(t)U(t)^\dagger dU(t) &= \Lambda(t)\left(U(t)^\dagger dU(t) + \frac{1}{2}dU(t)^\dagger dU(t)\right) - \frac{1}{2}\Lambda(t)dU(t)^\dagger dU(t) \\
&= \Lambda(t)dA(t) + \frac{1}{2}\Lambda(t)dA(t)dA(t),
\end{aligned}$$

where (B.1) was used, and

$$\begin{aligned}
U(t)^\dagger d\Xi(t)dU(t) &= U(t)^\dagger d\Xi(t)U(t)dA(t), \\
dU(t)^\dagger\Xi(t)dU(t) &= dA(t)^\dagger U(t)^\dagger\Xi(t)U(t)dA(t) = dA(t)^\dagger\Lambda(t)dA(t),
\end{aligned}$$

where (B.2) was used. Then we have the equality

$$\begin{aligned}
d\Lambda(t) &= U(t)^\dagger d\Xi(t)U(t) + \Lambda(t)dA(t) + (\Lambda(t)dA(t))^\dagger \\
&\quad + \frac{1}{2}\Lambda(t)dA(t)dA(t) + \frac{1}{2}(\Lambda(t)dA(t)dA(t))^\dagger \\
&\quad + U(t)^\dagger d\Xi(t)U(t)dA(t) + (U(t)^\dagger d\Xi(t)U(t)dA(t))^\dagger + dA(t)^\dagger\Lambda(t)dA(t).
\end{aligned} \tag{B.3}$$

The diagonal elements of (B.3) give

$$\begin{aligned}
d\lambda_i(t) &= \sum_{k,\ell} u_{ki}(t)^* u_{\ell i}(t) d\xi_{k\ell}(t) \\
&\quad + 2\lambda_i(t)d\gamma_{ii}(t) + d\phi_{ii}(t) + d\phi_{ii}(t)^* + d\psi_{ii}(t), \quad 1 \le i \le N,
\end{aligned} \tag{B.4}$$

and the off-diagonal elements of (B.3) give

$$\begin{aligned}
0 &= \sum_{k,\ell} u_{ki}(t)^* u_{\ell j}(t) d\xi_{k\ell}(t) + \lambda_i(t)d\alpha_{ij}(t) + \lambda_j(t)d\alpha_{ji}(t)^* \\
&\quad + \lambda_i(t)d\gamma_{ij}(t) + \lambda_j(t)d\gamma_{ji}^*(t) + d\phi_{ij}(t) + d\phi_{ji}(t)^* + d\psi_{ij}(t), \quad 1 \le i < j \le N,
\end{aligned} \tag{B.5}$$

where we have used the notations

$$\begin{aligned}
d\gamma_{ij}(t) &\equiv \left(\frac{1}{2}dA(t)dA(t)\right)_{ij} = \frac{1}{2}\sum_k d\alpha_{ik}(t)d\alpha_{kj}(t) = d\gamma_{ji}(t)^*, \\
d\phi_{ij}(t) &\equiv \left(U(t)^\dagger d\Xi(t)U(t)dA(t)\right)_{ij} = \sum_{k,\ell,m} u_{ki}(t)^* d\xi_{k\ell}(t) u_{\ell m}(t) d\alpha_{mj}(t), \\
d\psi_{ij}(t) &\equiv \left(dA(t)^\dagger\Lambda(t)dA(t)\right)_{ij} \\
&= \sum_k d\alpha_{ki}(t)^*\lambda_k(t)d\alpha_{kj}(t) = -\sum_k d\alpha_{ik}(t)\lambda_k(t)d\alpha_{kj}(t).
\end{aligned} \tag{B.6}$$

Since $(\gamma_{ij}(t))_{1\le i,j\le N}$, $(\phi_{ij}(t))_{1\le i,j\le N}$, $(\psi_{ij}(t))_{1\le i,j\le N}$ are functions of finite variations, (B.3) gives

$$
\begin{aligned}
d\langle M_i\rangle_t &= \Big(U(t)^\dagger d\Xi(t)U(t)\Big)^\dagger_{ii}\Big(U(t)^\dagger d\Xi(t)U(t)\Big)_{ii} \\
&= \sum_{k,\ell}\sum_{m,n} u_{ki}(t)^* u_{\ell i}(t)u_{mi}(t)^* u_{ni}(t)d\xi_{k\ell}(t)d\xi_{mn}(t).
\end{aligned}
$$

On the other hand, for $dA(t)$ is anti-Hermitian, (B.5) gives

$$
\begin{aligned}
(\lambda_j(t)-\lambda_i(t))d\alpha_{ij}(t) &= \sum_{k,\ell} u_{ki}(t)^* u_{\ell j}(t)d\xi_{k\ell}(t) \\
&+ (\lambda_i(t)+\lambda_j(t))d\gamma_{ij}(t)+d\phi_{ij}(t)+d\phi_{ji}(t)^*+d\psi_{ij}(t), \qquad \text{(B.7)}
\end{aligned}
$$

and using this equality we can rewrite (B.6) as

$$
d\phi_{ij}(t)=\sum_k(\lambda_k(t)-\lambda_i(t))d\alpha_{ik}(t)d\alpha_{kj}(t).
$$

Then the finite-variation part of (B.4) is written as

$$
\begin{aligned}
&2\lambda_i(t)d\gamma_{ii}(t)+d\phi_{ii}(t)+d\phi_{ii}(t)^*+d\psi_{ii}(t) \\
&= \sum_j\Big\{\lambda_i(t)+2(\lambda_j(t)-\lambda_i(t))-\lambda_j(t)\Big\}d\alpha_{ij}(t)d\alpha_{ji}(t) \\
&= \sum_j(\lambda_j(t)-\lambda_i(t))d\alpha_{ij}(t)d\alpha_{ji}(t) \\
&= \sum_j \frac{1}{\lambda_i(t)-\lambda_j(t)}\mathbf{1}_{\{\lambda_i(t)\neq\lambda_j(t)\}}\sum_{k,\ell,m,n} u_{ki}(t)^* u_{\ell j}(t)u_{mj}(t)^* u_{ni}(t)d\xi_{k\ell}(t)d\xi_{mn}(t),
\end{aligned}
$$

where (B.7) was used in the last equation. This completes the proof. ∎

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