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PERSPECTIVES OF ALGEBRAIC GAME THEORY (AN INFORMAL REPORT)

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1. Games and Mathematics

Up to the present, we have several mathematical theories on games. Below we list some of them in chronological order of birth:

(i) "Probability Theory" originated by P. de Fermat and B. Pascal in the 17th century, a theory of gambling having applications to statistical physics, finance and pure mathematics (e.g. number theory),

(ii) "Game Theory" originated by J. von Neumann and O. Morgenstern in 1944 ("Theory of Games and Economic Behavior"), a theory of equilibrium (such as Nash equilibrium) in a game-like situation among (more than two) players, having applications to economics and evolutional biology,

(iii) "Combinatorial Game Theory" originated by J. H. Conway in 1976 ("Games and Numbers" [5]), a theory of 2-person games with "no chance moves" having applications to the analysis of end-games of Go and chess, code theory and to pure mathematics (surreal numbers).

To this honorable list, we dare to add another one:

(iv) "Algebraic Game Theory" which is a theory of algorithms = probabilistic algorithms = impartial two-person games with representation-theoretic flavour.

Each of (i)-(iii) (and perhaps (iv)) is not merely an application of mathemathics to games, but an application of (the notion of) games to mathematics.

2. A game as an abstract algebraic system

2.1. (P,φ) : a *game* $\Longleftrightarrow$ P : a non-empty set, $\varphi\colon P \longrightarrow 2^P$
$\nexists$ infinite sequence $p_0, p_1, p_2, p_3, \ldots,\quad p_i \in P$
satisfying $p_i \in \varphi(p_{i-1}),\quad i = 1,2,3,\ldots$
(P,φ): a *finitary* game $\Longleftrightarrow$ a game such that $|\varphi(p)| < \infty$ for any $p \in P$
$p\ (\in P)$: an *ending position* $\Longleftrightarrow$ $\varphi(p) = \emptyset$

A game is a non-deterministic discrete dynamical system which comes to an end after a finite number of steps. A game can also be considered as an algebraic system with a non-deterministic unary operation φ.

2.2. A game models:

(i) a one-person game = an algorithm,
(ii) a probabilistic one-person game = a probabilistic algorithm, after assigning a suitable probabilistic measure on each set $\varphi(p)$, $p \in P$,
(iii) a two-person ("impartial" [5]) game, in which a player who reaches an ending position wins.

2.3. Let $(P, \varphi), (Q, \psi)$ be games. A map $f\colon P \longrightarrow Q$ lifts to $\tilde{f}\colon 2^P \longrightarrow 2^Q$.
$f\colon P \longrightarrow Q$ is a *homomorphism* $\Longleftrightarrow$ $\tilde{f}(\varphi(p)) \subset \psi(f(p))$ for any $p \in P$
$f\colon P \longrightarrow Q$ is a *full homomorphism* $\Longleftrightarrow$ $\tilde{f}(\varphi(p)) = \psi(f(p))$ for any $p \in P$
$f\colon P \longrightarrow Q$ is an *isomomorphism* $\Longleftrightarrow$ f is a bijective full homomorphism

2.4. Let (P, φ) be a game, and $\emptyset \neq Q \subset P$. Let $\varphi_Q\colon Q \longrightarrow 2^Q$ be defined by

$$\varphi_Q(q) = \varphi(q) \cap Q, \qquad (q \in Q).$$

Then (Q, φ_Q) is a game, and is called a *subgame* of (P, φ). If $\varphi(q) \subset Q$ for any $q \in Q$, then $\varphi_Q = \varphi|_Q$, and (Q, φ_Q) is called a *full subgame* of (P, φ). Since the intersection of full subgames is again a full subgame if it is non-empty, we have the notion of the full sugame $\langle A \rangle$ of P *generated* by a non-empty subset $A \subset P$.

2.5. Let $\varphi\colon P \longrightarrow 2^P$. For $i \in \mathbf{Z}$, we define $\varphi^i\colon P \longrightarrow 2^P$ by:

$$\varphi^0(p) = \{p\}, \quad \varphi^i(p) = \varphi(\varphi^{i-1}(p) \ (i \geq 1),$$

$$\varphi^{-1}(p) = \{q \in P \mid p \in \varphi(q)\}, \quad \varphi^{-i} = (\varphi^{-1})^i \ (i \geq 2).$$

2.6. Back to the situation in 2.4, we have

$$\langle A \rangle = \bigcup_{i=0}^{\infty} \varphi^i(A),$$

for any non-empty subset A of P.
The set P has the structure of a partially ordered set defined by

$$q \leq p \Longleftrightarrow q \in \langle p \rangle \Longleftrightarrow \langle q \rangle \subset \langle p \rangle.$$

2.7. Let (P, φ) and (Q, ψ) be games. We put

$$P + Q = \{\, (p, q) \mid p \in P,\ q \in Q \,\}.$$

We also put

$$(\varphi + \psi)(p, q) = (\varphi(p), q) \cup (p, \psi(q)),$$

where

$$(\varphi(p), q) = \{\, (p', q) \mid p' \in \varphi(p) \,\}$$

and

$$(p, \psi(q)) = \{\, (p, q') \mid q' \in \psi(q) \,\}.$$

Then $(P + Q, \varphi + \psi)$ is a game, and is called the *sum* $(P, \varphi) + (Q, \psi)$ of the games (P, φ) and (Q, ψ). (This notion is classical; see e.g. [1, p. 23],[5, p. 78].)

2.8. Let (P, φ) be a game. For $p, q, p_0, p_1, \ldots, p_n \in P$,
(q,p) : a *transition* $\Longleftrightarrow q \in \varphi(p)$
(q,p) : a *simple* transition $\Longleftrightarrow$ a transition such that

$$\varphi(p) \cap \varphi^{-k}(q) = \emptyset \text{ for any } k = 1,2,3,...$$

$(p_0, p_1, \ldots, p_n)$: a *(simple) path* $\Longleftrightarrow \varphi(p_0) = \emptyset$, (p_i, p_{i+1}): a (simple) transition

If (P, φ) is finitary, then any path has a (not necessarily unique) "simple refinement".

If (P, φ) is finitary, then its *simplification* (P, φ_{simp}) is a game defined by

$$\varphi_{simp}(p) = \{q \in \varphi(p) \mid (q,p) \text{ simple}\}.$$

2.9. Let (P, φ) be a finitary game.
(P, φ) : a *ranked* game $\Longleftrightarrow \exists r \colon P \longrightarrow \mathbf{N}_0 = \{0,1,2,\ldots\}$ such that, for $p \in P$

$$r(p) = n \text{ if } \exists \text{ a simple path } (p_0, p_1, \ldots, p_n = p) \text{ of length } n$$

For a ranked game, we are interested in knowing the number of simple paths $(p_0, p_1, \ldots, p_n = p)$ for each $p \in P$.

2.10. Let (P, φ) be a game.
(P, φ) : a *triangular* game $\Longleftrightarrow \varphi(p) \cap \varphi^{-1}(q) \neq \emptyset$ if (q,p): non-simple transition

2.11. Let (P, φ) be a finitary triangular game.
Then one can naturally construct a probabilistic version $(P, \varphi_{simp})_{prob}$ of the simplification (P, φ_{simp}) of (P, φ).
For that purpose we need to define a probabilistic measure on each $\varphi_{simp}(p)$ $(p \in P)$ whenever $\varphi_{simp}(p) \neq \emptyset$. This is done as follows using an auxiliary game.
Assume

$$\varphi_{simp}(p) \neq \emptyset \ (\Leftrightarrow \varphi(p) \neq \emptyset).$$

The purpose of this game is to select an element $q \in \varphi_{simp}(p)$.

(1) Select $q_1 \in \varphi(p)$ with uniform probability $\frac{1}{|\varphi(p)|}$.
(2) If (q_1, p) is simple, then we put $q = q_1$. The game is over.
(3) If (q_1, p) is not simple, then by the triangularity condition, we have

$$\varphi(p) \cap \varphi^{-1}(q_1) \neq \emptyset.$$

Hence one can select $q_2 \in \varphi(p) \cap \varphi^{-1}(q_1)$ with uniform probability $\frac{1}{|\varphi(p) \cap \varphi^{-1}(q_1)|}$.
(4) If (q_2, p) is simple, then we put $q = q_2$. The game is over.
(5) If (q_2, p) is not simple, then, since $\varphi(p) \cap \varphi^{-1}(q_2) \neq \emptyset$, one can select $q_3 \in \varphi(p) \cap \varphi^{-1}(q_2)$ with uniform probability.
(6) ...
(7) Finally one selects an element $q_n \in P$ such that (q_n, p) is a simple transition. We put $q = q_n$. The game is over.

Note that the game $(P, \varphi_{simp})_{prob}$ can be considered as a probanilistic machine selecting a simple path in (P, φ). We are interested in knowing the probability in which the machine selects a given simple path.

2.12. A finitary ranked triangular game is particulary interesting. This class of games is closed under the addition defined in 2.7.

2.13. We define an addition $\oplus$ in $\mathbf{N}_0$, which is called the *Nim addition* by game theorists, and the *xor* (exclusive-or) *addition* by computer scientists. We understand that the set $\mathbf{N}_0$ is well-ordered in the usual (and canonical) way. For $a, b \in \mathbf{N}_0$, we define $a \oplus b$ recursively by

$$a \oplus b = \min[\mathbf{N}_0 \setminus \{\, a' \oplus b,\ a \oplus b' \mid a', b' \in \mathbf{N}_0,\ a' < a, b' < b \,\}].$$

Then $(\mathbf{N}_0, \oplus)$ is an additive group.
A more practical (rather than aesthetic) definition of $\oplus$ is as follows. (In fact, this latter definition is also of theoretical importance, because it enables us to extend the addition $\oplus$ to $\mathbf{Z}$.) We shall write the binary expression of an element a of $\mathbf{Z}$ as

$$a = [a_i] = [a_i]_{i \in \mathbf{N}_0} = [\ldots, a_i, \ldots, a_3, a_2, a_1, a_0]. \tag{2.1}$$

For example,

$$\begin{aligned} 11 &= 1 + 2 + 0 \ \ + 2^3 + 0 \ + \cdots = [\cdots 00001011], \\ -1 &= 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = [\cdots 11111111], \\ -2 &= 0 + 2 + 2^2 + 2^3 + 2^4 + \cdots = [\cdots 11111110]. \end{aligned}$$

For $a = [a_i], b = [b_i]$, and $c = [c_i]$ in $\mathbf{Z}$, we write

$$a \oplus b = c$$

if

$$a_i + b_i \equiv c_i \pmod 2, \qquad i \in \mathbf{N}_0.$$

2.14. Let (P, φ) be a finitary game. The *Sprague-Grundy function* $F_P \colon P \longrightarrow \mathbf{N}_0$ is recursively defined by

$$F_P(p) = \min[\mathbf{N}_0 \setminus \{SG(q) \mid q \in \varphi(p)\,\}], \quad p \in P.$$

As is well-known, in the two-person version of the full subgame $\langle p \rangle$ generated by $p \in P$, the first (resp. second) player has a winning strategy if and only if $F_P(p) \neq 0$ (resp. $F_P(p) = 0$). Moreover, in the situation of 2.7, the Sprague-Grundy function F_{P+Q} of $P + Q$ is given by

$$F_{P+Q}(p, q) = F_P(p) \oplus F_Q(q), \quad p \in P, q \in Q.$$

(This is a classical theorem due to R. P. Sprague and P. M. Grundy. See [1],[5].)
A full homomorphism between finitary games preserves Sprague-Grundy functions.

2.15. Let (P, φ) be a game such that P is a finite set. Extend P to $\tilde{P} = \{a\} \amalg P$ by adding a new position a. Extend φ to $\tilde{\varphi} \colon \tilde{P} \longrightarrow 2^{\tilde{P}}$ by putting $\tilde{\varphi}(a) = P$. Then $(\tilde{P}, \tilde{\varphi})$ is a finitary game. Hence we can extend the Sprague-Grundy function F_P of (P, φ) to the one $F_{\tilde{P}}$ of $(\tilde{P}, \tilde{\varphi})$. In that case we denote the value $F_{\tilde{P}}(a)$ by $F_P(P)$ and call it the *opening value* of F_P.

2.16. Let (P, φ) be a finitary game. Assume that there exists a function $E \colon P \longrightarrow \mathbf{N}_0$ satisfying

$$\sum_{q \in \varphi(p)}^{\oplus} (t | E(q)) = t \oplus (t - E(p)), \quad p \in P,$$

and

$$E(q) \neq E(p), \quad q \in \varphi(p).$$

where t is a variable taking values in $\mathbf{Z}$, and

$$(a|b) = a \oplus b \oplus (a \oplus b - 1), \quad a, b \in \mathbf{Z}.$$

If such a function E exists, then it coincides with the Sprague-Grundy function of (P,φ). We call E the *energy function* of (P,φ). In the situation of 2.15, if the energy function E of (P,φ) can be extended to the one $\tilde{E}$ of $(\tilde{P},\tilde{\varphi})$, we denote the value $\tilde{E}(a)$ by $E(P)$, and call it the *opening value* of E.
The significance of the energy function lies in the empirical fact that in many known cases, the energy function, if it exists, can be written down in an explicit formula. The class of games with energy functions is closed under the addition given in 2.7.

3. Examples of Games

3.1. Let Y be a (Young or Ferrers) diagram of a partition, and q a natural number. We call a hook at a node (or a 'box') belonging to Y a *q-hook* if its length is a multiple of q. *Nakayama's q-hook game* is a one-person game in which the player removes q-hooks from a given diagram Y successively as far as possible so as to obtain finally a diagram possessing no q-hook. In the terminology of Section 2, this game can be described as a pair (P,φ_q) such that

$$P = \text{ the set of diagrams } Y \text{ of partitions },$$

and

$$\varphi_q(Y) = \text{ the set of diagrams obtained from Y by removing a } q\text{-hook }.$$

Let $Y \in P$. By a well-known theorem due to T. Nakayama and G. de Robinson (see e.g. [12, pp. 75–87]), the full subgame $\langle Y\rangle_q$ of (P,φ_q) has the following remarkable properties :

(i) The ending position Z is uniquely determined by the opening position Y. (The diagram Z is called the *q-core* of Y.)

(ii) The game $\langle Y\rangle_q$ is isomorphic to a sum $\langle W_1\rangle_1 + \langle W_2\rangle_1 + \cdots + \langle W_{q-1}\rangle_1$ of 1-hook games whose opening positions are determined by a series
$$\mathcal{W} = \{ W_0, W_1, \ldots, W_{q-1} \}$$
of diagrams, some of which may be empty. (The series $\mathcal{W}$ is called the *q-quotient* of Y.)

(iii) The diagram Y is uniquely recovered by its q-core Z and q-quotient $\mathcal{W}$.

It is also known [13],[16] that an analogous one-person game exists for the shifted Young diagrams.

3.2. Nakayama's 1-hook game $\langle Y\rangle_1$, and hence Nakayama's q-hook game $\langle Y\rangle_q$ also, is a finitary ranked triangular game (see 2.1, 2.9 and 2.10). Hence we can consider the probabilistic version of the simplification of $\langle Y\rangle_1$. Then it can be shown that the resulting game is isomorphic (as probabilistic games) to the game invented by C. Greene, A. Nijenhuis and H. S. Wilf [10]. In the terminology of Section 2, the main result of [10] can be restated as follows:
The probabilistic version of the simplification of $\langle Y\rangle_1$ selects a simple path of $\langle Y\rangle_1$ (connecting $\emptyset$ and Y) uniform randomly with the probability

$$\frac{\prod_{X\in\varphi_1(Y)}(|\varphi_1(Y)\cap\varphi_1^{-1}(X)|+1)}{|\varphi_1(Y)|!}.$$

Hence the number of simple paths in $\langle Y\rangle_1$ is equal to

$$\frac{|\varphi_1(Y)|!}{\prod_{X\in\varphi_1(Y)}(|\varphi_1(Y)\cap\varphi_1^{-1}(X)|+1)},$$

which is equivalent to the famous hook formula (see e.g. [12]) for the number of standard tableaux on Y.

3.3. According to [20], the two-person version of Nakayama's 1-hook game $\langle Y\rangle_1$ was invented and analyzed by M. Sato around 1950. His beautiful result with full proof first appeared in 1970 in an informal Japanese journal [21] published by a math-student circle. Sato [19] formulated the game in another way, using physical metaphors such as "fermion" and "energy level". In this second formulation, the game is essentially the same as the one generally known as Welter's game, because C. P. Welter [26] gave a complete analysis of it in 1954. See also [5]. Sato [19][21] formulated his main result in two different ways. In its second formulation, the result is essentially equivalent to that of Welter [26], though their proofs are quite different. In its first formulation, the result was stated in terms of hooks of a Young diagram Y. In the prsent paper, considering the importance of Sato's point of view (i.e. the connection with Young diagrams), we decide to call the two-person game $\langle Y\rangle_1$ the Sato-Welter game. In the terminology of Section 2, Sato's result (in its first formulation) can be restated as follows:
The game $\langle Y\rangle_1$ has the energy function $E(Y)$ given by

$$E(Y) = \sum_{X\in\varphi_1(Y)}^{\oplus} (E(\varphi_1(Y)\cap\varphi_1^{-1}(X)) + 1|0),$$

where $(a|0) = a\oplus(a-1)$ as in 2.16, and $E(\varphi_1(Y)\cap\varphi_1^{-1}(X))$ is the opening value of the energy function of the subgame $\varphi_1(Y)\cap\varphi_1^{-1}(X)$ of $\langle Y\rangle_1$.
The close similarity with the result in 3.2 is truely striking, as Sato himself remarked in [22].

3.4. We can also consider probabilistic version and two-person version of Nakayama's q-hook game. But by virtue of Nakayama-Robinson Theorem stated in 3.1, the analysis of such games can easily be reduced to the case of 1-hook game.

4. Constructing Games

4.1. Let (W,S,T,P,p_*) be a quintet consisting of:
- (i) a group W,
- (ii) a set $S=\{\,s_i \mid i\in I\,\}$ of generators of W,
- (iii) a subset T of W,
- (iv) a set P on which W acts transitively, and
- (v) an element p_* of P.

For $p,q\in P$, let $d(p,q)$ be the *distance* between $p\in P$ and q defined by

$$d(p,q)=0, \qquad d(p,q)=\min\{\,l \mid p=s_{i_l}\cdots s_{i_2}s_{i_1}q\,\},\ p\neq q.$$

We also define a mapping

$$\Phi_T\colon P\longrightarrow 2^P$$

by

$$\Phi_T(p)=\{\,tp \mid t\in T,\ d(tp,p_*)<d(p,p_*)\,\}.$$

Then (P,Φ_T) is a game, which we write $\mathcal{G}(W,S,T,P,p_*)$. Let (W',S',T',P',p'_*) be another such quintet. The following proposition is obvious.

Proposition 1. *We have*
$\mathcal{G}(W,S,T,P,p_*)+\mathcal{G}(W',S',T',P',p'_*)\cong\mathcal{G}(W\times W',S\amalg S',T\amalg T',P\times P',p\times p'_*)$.

Note that the traditional game addition (defined in 2.7) naturally enters into our picture.

4.2. Basic references for this subsection are [3] and [11]. A *Coxeter system* (W, S) is a pair of a group W and a set $S = \{ s_i \mid i \in I \}$ of generators of W subject to the defining relations:

$$(s_i s_j)^{m(i,j)} = 1, \quad i, j \in I, \tag{4.1}$$

where $m(i,i) = 1$, and $2 \leq m(i,j) = m(j,i) \leq \infty$ for $i \neq j$. If $m(i,j) = \infty$ for some (i,j), then we understand the corresponding relation in (4.1) is vacant. We assume, for simplicity, S is finite. An element of S is called a *simple reflection*, and an element of the set

$$T = \{wsw^{-1} | \ s \in S, w \in W\}$$

is called a *reflection*. A subgroup W_J of W generated by a subset J of S is called a *standard parabolic subgroup*. A Coxeter system (W, S) is *reducible*, if there exists a partition

$$S = J' \amalg J'', \quad J', J'' \neq \emptyset, \tag{4.2}$$

such that

$$s's'' = s''s', \quad s' \in J', s'' \in J''. \tag{4.3}$$

Then we cleary have

$$W = W_{J'} \times W_{J''}, \tag{4.4}$$

and

$$T = (T \cap W_{J'}) \amalg (T \cap W_{J''}). \tag{4.5}$$

For any $w \in W$, there exists a sequence

$$(s_{i_1}, s_{i_2}, \ldots, s_{i_l}) \tag{4.6}$$

of elements of S such that

$$w = s_{i_1} s_{i_2} \cdots s_{i_l}. \tag{4.7}$$

If a sequence (4.6) is chosen so that l is as small as possible, then (4.7) is called a *reduced expression* of w, and $l = l(w)$ the *length* of w. (For the identity element e, we define its length by $l(e) = 0$.) Let V be a real vector space with basis $\Pi = \{\alpha_i | \ i \in I\}$. We define a symmetric bilinear form $(\ , \ )$ on V by

$$(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m(i,j)}, \quad i, j \in I.$$

Then W can be identified with the subgroup of $GL(V)$ generated by the elements s_i $(i \in I)$ of $GL(V)$ defined by

$$s_i v = v - 2(\alpha_i, v)\alpha_i, \quad v \in V. \tag{4.8}$$

The bilinear form $(\ , \ )$ is invariant under the action of W. We put

$$\Sigma = W\Pi = \{ w\alpha_i \mid w \in W, i \in I \}.$$

Let

$$V^+ = \{ v \in V \mid v = \sum_i c_i \alpha_i, \ c_i \geq 0 \}.$$

For $\alpha, \beta \in V$, we write

$$\alpha \geq \beta$$

if $\alpha - \beta \in V^+$. Then we have

$$\Sigma = \Sigma^+ \amalg \Sigma^-,$$

where

$$\Sigma^+ = \{\, \alpha \in \Sigma \mid \alpha > 0 \,\} \quad \text{and} \quad \Sigma^- = -\Sigma^+.$$

An element of Σ (resp. Π, resp. Σ^+) is called a *root* (resp. *simple root*, resp. *positive root*) of the Coxeter system (W, S). For $w \in W$, we put

$$\Sigma^+(w) = \Sigma^+ \cap w^{-1}(\Sigma^-).$$

If $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is reduced, then $\Sigma^+(w)$ consists of the following l distinct roots:

$$s_{i_l} s_{i_{l-1}} \cdots s_{i_{k+1}} \alpha_{i_k}, \quad 1 \le k \le l. \tag{4.9}$$

Conversely, an element $w \in W$ is uniquely determined by $\Sigma^+(w)$. For $\alpha \in \Sigma^+$, we define $t_\alpha \in GL(V)$ by

$$t_\alpha(v) = v - 2(\alpha, v)\alpha, \quad v \in V.$$

Then

$$t_\alpha = w s_{\alpha_i} w^{-1} \quad \text{if} \quad \alpha = w\alpha_i.$$

The correspondence $\alpha \mapsto t_\alpha$ gives a bijection from Σ^+ to T. For $w \in W$, we consider

$$N(w) = \{\, t \in T \mid l(wt) < l(w) \,\}.$$

Then we have

$$N(w) = \{\, t_\alpha \mid \alpha \in \Sigma^+(w) \,\}.$$

Hence

$$l(w) = |N(w)|, \quad w \in W.$$

4.3. We keep notation in the previous subsection. A subgroup of W generated by a subset of T is called a *reflection subgroup*. Let W_1 be a reflection subgroup of W. We put

$$S_1 = \{\, t \in T \mid N(t) \cap W_1 = \{t\} \,\}. \tag{4.10}$$

Then, by Dyer [8] (see also Deodhar [7]), (W_1, S_1) is a Coxeter system. We call S_1 the *canonical set of Coxeter generators* of W_1. It is also known [8] that $T_1 = T \cap W_1$ is the set of reflections of (W_1, S_1). Let

$$\Sigma_1^+ = \{\, \alpha \in \Sigma^+ \mid t_\alpha \in T_1 \,\}, \quad \Sigma_1 = \Sigma_1^+ \amalg (-\Sigma_1^+), \quad \Pi_1 = \{\, \alpha \in \Sigma^+ \mid t_\alpha \in S_1 \,\}.$$

These are the set of positive roots, roots and simple roots of (W_1, S_1), respectively.

We put

$$D_{W_1} = \{\, w \in W \mid N(w) \cap W_1 = \emptyset \,\}.$$

Proposition 2 (Dyer [8]). *Let W_1 be a reflection subgroup of W, and $T_1 = T \cap W_1$. We have:*

(i) *Let $w \in W$. The coset wW_1 contains a unique element $y \in D_{W_1}$.*
(ii) *Let $y \in D_{W_1}$. Then y is a unique element of minimal length in the coset yW_1.*

The following two results are also due to M. Dyer and given in Appendix A of [23].

Proposition 3 (Dyer). *Let $J \subset S$. Let W_1 be a reflection subgroup of W, and S_1 the canonical set of Coxeter generators of W_1. If $y \in D_{W_1}$, then $y^{-1}W_Jy \cap W_1$ is generated by $y^{-1}W_Jy \cap S_1$. In particular, it is a standard parabolic subgroup of (W_1, S_1).*

Proposition 4 (Dyer). *Let $J \subset S$ and W_1 a reflection subgroup of W. We have:*

(i) *Every $w \in W$ can be factored uniquely in the form:*

$$w = xyz,$$

where $y \in D_{W_J}^{-1} \cap D_{W_1}$, $x \in W_J \cap D_{W_J \cap yW_1y^{-1}}$ and $z \in W_1$.

(ii) *Let $y \in D_{W_J}^{-1} \cap D_{W_1}$. Then y is a unique element of minimal length in the double coset W_JyW_1.*

4.4. We still keep notation in 4.2 and 4.3. Let $J \subset S$. Let

$$S \setminus J = \{ s_{01}, s_{02}, \dots, s_{0p} \} \quad (p = |S \setminus J|).$$

We extend the Coxeter system (W, S) to (W_*, S_*) by adding a new element $s_* = s_{*J}$ to S (hence, $S_* = S \amalg \{s_*\}$), and adding new relations:

$$(s_* s_{0k})^{m(k)} = 1, \quad s_{0k} \in S \setminus J \tag{4.11}$$

and

$$(s_* s)^2 = 1, \quad s \in J \tag{4.12}$$

to (4.1), where $m(k) \geq 3$. (The exact values of $m(k)$ are irrelevant for us.) The set Π_* of simple roots of (W_*, S_*) is given by

$$\Pi_* = \Pi \amalg \{\alpha_*\}$$

where $\alpha_* = \alpha_{*J}$ is the simple root corresponding to s_*. The set of roots (resp. positive roots) of (W_*, S_*) is denoted by Σ_* (resp. Σ_*^+). For each $1 \leq k \leq p$, let $\alpha_{0k} \in \Pi$ be the simple root corresponding to s_{0k}. Then, by (4.11) and (4.12), we have

$$(\alpha_*, \alpha_{0k}) = -\cos\frac{\pi}{m(k)} \leq -\frac{1}{2}, \quad 1 \leq k \leq p, \tag{4.13}$$

and

$$(\alpha_*, \alpha) = 0, \quad \alpha \in \Pi \setminus \{ \alpha_{01}, \dots, \alpha_{0p} \}. \tag{4.14}$$

A sequence $(s_{i_1}, s_{i_2}, \dots, s_{i_l})$ of elements of S, or an expression $w = s_{i_1}s_{i_2}\cdots s_{i_l} \in W$ is said to be *increasing* (resp. *weakly increasing*), if

$$\alpha_* < s_{i_1}\alpha_* < s_{i_2}s_{i_1}\alpha_* < \dots < s_{i_l}\cdots s_{i_2}s_{i_1}\alpha_*$$
$$(\text{ resp. } \alpha_* \leq s_{i_1}\alpha_* \leq s_{i_2}s_{i_1}\alpha_* \leq \dots \leq s_{i_l}\cdots s_{i_2}s_{i_1}\alpha_*).$$

For an element w of W, w^J denotes the element of minimal length in the coset W_Jw.

Lemma 5. *We have:*

(i) *Any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ of any element $w \in W$ is weakly increasing.*

(ii) *For $w, w' \in W$, $w^{-1}\alpha_* = w'^{-1}\alpha_*$ if and only if $W_Jw = W_Jw'$.*

(iii) *For $w \in W$, we have $w = w^J$ if and only if w is the shortest element of $\{ v \in W \mid v^{-1}\alpha_* = w^{-1}\alpha_* \}$.*

(iv) *If a sequence* $(s_{i_1}, s_{i_2}, \ldots, s_{i_l})$ *of elements of* S *is increasing, then the expression* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *is reduced.*

(v) *Let* $\gamma \in W\alpha_*$. *Then there exists an element* $w \in W$ *with an increasing expression* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *such that* $\gamma = w^{-1}\alpha_*$.

(vi) *Assume that* $w = s_{i_1} s_{i_2} \cdots s_{i_l} \in W$ *is an increasing expression. Let* $\gamma = w^{-1}\alpha_*$. *Then*

$$N(w) = \{\, t \in T \mid t\gamma < \gamma \,\}.$$

Moreover, w *is uniquely determined by* γ.

(vii) *For* $w \in W$, *we have* $w = w^J$ *if and only if a reduced expression* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *is increasing.*

(viii) *For* $w \in W$, *we have* $w = w^J$ *if and only if any reduced expression* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *is increasing.*

4.5. Let (W, S) be a Coxeter system, and T the set of reflections of (W, S). Let $J \subset S$. Let $P_J = P_{J \subset S}$ be the quotient space $W_J \backslash W$. We define a map $\Phi_J = \Phi_{J \subset S} \colon P_J \longrightarrow 2^{P_J}$ by

$$\Phi_J(W_J w) = \{\, W_J wt \mid t \in N(w^J) \,\} = \{\, W_J wt \mid t \in T, l(w^J t) < l(w^J) \}, \ w \in W.$$

Then (P_J, Φ_J) is a game, which is a special case considered in 4.1. Note that essentially the same object has been studied [6][9] from differnt viewpoints. A full subgame of (P_J, Φ_J) is called an (*unrestrained*) *reflection game of type* T. By (4.4), (4.5) and Proposition 1, we have

Proposition 6. *Assume that* (W, S) *is reducible as in* (4.2). *Then we have a natural decomposition*

$$(P_{J \subset S}, \Phi_{J \subset S}) \cong (P_{(J' \cap J) \subset J'}, \Phi_{(J' \cap J) \subset J'}) + (P_{(J'' \cap J) \subset J''}, \Phi_{J'' \cap J \subset J''}).$$

Lemma 5 implies the following "root description" of an unrestrained reflection game:

Proposition 7. *Let* $X_J = W\alpha_*$, *where* $\alpha_* = \alpha_{*J}$ *be as in Section* 2.5. *Define* $\Lambda_J \colon X_J \longrightarrow 2^{X_J}$ *by*

$$\Lambda_J(\gamma) = \{\, t\gamma \mid t \in T,\ t\gamma < \gamma \,\}, \quad \gamma \in X_J.$$

Then the mapping

$$f \colon P_J \longrightarrow X_J$$

defined by

$$f(W_J w) = w^{-1}\alpha_*, \quad w \in W$$

gives an isomorphism

$$(P_J, \Phi_J) \cong (X_J, \Lambda_J)$$

of games.

Let W_1 be a reflection subgroup of W, and T_1 the set of reflections of W_1. We define a map $\Phi_{J,W_1} \colon P_J \longrightarrow 2^{P_J}$ by

$$\Phi_{J,T_1}(W_J w) = \{\, W_J wt \mid t \in N(w^J) \cap T_1 \,\}, \quad w \in W.$$

Then (P_J, Φ_{J,T_1}) is a game. A full subgame of (P_J, Φ_{J,T_1}) is called a (*restrained*) *reflection game of type* T_1. If $T_1 = T$, this reduces to an unrestrained game. By Proposition 7, we have the following root description of a restrained reflection game:

Proposition 8. *Let W_1 be a reflection subgroup of (W, S). Let $X_J = W\alpha_*$ as in Proposition 7. Define $\Lambda_{J,W_1} : X_J \longrightarrow 2^{X_J}$ by*

$$\Lambda_{J,W_1}(\gamma) = \{\, t\gamma \mid t \in T_1,\ t\gamma < \gamma \,\}, \quad \gamma \in X_J.$$

Then the mapping $f\colon P_J \longrightarrow X_J$ defined in Proposition 7 induces an isomorphism

$$(P_J, \Phi_{J,W_1}) \cong (X_J, \Lambda_{J,W_1})$$

of games.

For $w \in W$, $\langle W_J w\rangle_{T_1}$ denotes the full subgame of (P_J, Φ_{J,T_1}) generated by $W_J w$.
The following result is an extension of Nakayama-Robinson Theorem mentioned in 3.1.

Theorem 9. *Let W_1 be a reflection subgroup of W. Let w be an element of W. Let y be the unique element of minimal length in the double coset $W_J w W_1$ (see Lemma 4). We have:*

(i) *The restrained reflection game $W_J\backslash W_J w W_1$ of type T_1 is naturally isomorphic to the unrestrained game $(y^{-1}W_J y \cap W_1)\backslash W_1$ of type T_1.*

(ii) *Let*

$$w = xyz, \quad x \in W_J \cap D_{W_J \cap y W_1 y^{-1}},\ z \in W_1$$

be the factorization of w given in Lemma 4. Then the restrained reflection game $\langle W_J w\rangle_{T_1}$ is naturally isomorphic to the unrestrained game $\langle (y^{-1}W_J y \cap W_1)z\rangle_{T_1}$.

(iii) *The game $\langle W_J w\rangle_{W_1}$ has a unique ending position $W_J y$. Moreover, the coset $W_J w$ is uniquely recovered by the cosets*

$$W_J y \quad \text{and} \quad (y^{-1}W_J y \cap W_1)z.$$

The positions $W_J y$ ($\in W_J\backslash W$) and $(y^{-1}W_J y \cap W_1)z$ ($\in (y^{-1}W_J y \cap W_1)\backslash W_1$) described in Theorem 9 are called the T_1-*core* and the T_1-*quotient* of the position $W_J w$ ($\in W_J\backslash W$), respectively.

4.6. For any $J \subset S$, and any $w \in W$, an unrestrained reflection game $\langle W_J w\rangle_T$ (hence a restrained reflection game $\langle W_J w\rangle_{T_1}$ also) is a finitary ranked game. But it is not nescessarily a triangular game.

4.7. Here we show how Nakayama's game can be considered as a reflection game.

Example 1. Let W be the n-th symmetric group acting on the set $\{1, 2, \ldots, n\}$. For $1 \le i \ne j \le n$, let $(i,j) \in W$ be the transposition of i and j. We put $S = \{s_i = (i, i+1) \mid 1 \le i \le n-1\}$ and $T = \{(i,j) \mid 1 \le i < j \le n\}$. Then (W, S) is a Coxeter system and T is the set of reflections of (W, S). For a positive integer q, let

$$T(q) = \{(i,j) \in T \mid j - i \text{ is a multiple of } q\}.$$

Then $T(q)$ is the set of reflections of the reflection subgroup $\langle T(q)\rangle$ ($\subset W$). Fixing $1 \le k \le n$, we put $J = \{s_i \mid 1 \le i \le n, i \ne k\}$. The reflection gama $W_J\backslash W$ of type $T(q)$ is isomorphic to Nakayama's q-hook game whose positions are contained in the $k \times (n-k+1)$ rectangular diagram. The notion of $T(q)$-core and $T(q)$-quotient coincides with the classical notion of q-core and q-quotient.

5. BASIC REFLECTION GAMES

5.1. In view of examples given in 3.2 and 3.3, it is natural to investigate the class of reflection games which are triangular in the sense of 2.10. In this section, we discuss partial results obtained in this direction. By Theorem 9, we can restrict our attention to unrestrained reflection games.

5.2. Note that the game order $\leq$ of the reflection game (P_J, Φ_J) is nothing but the *Bruhat order* on the quotient space $W_J \backslash W$ (see [6],[9]). Motivated by this observation, we define $\Omega_J \colon P_J \longrightarrow 2^{P_J}$ by

$$\Omega_J(W_J w) = \{ W_J ws \mid s \in N(w^J) \cap S \}, \quad w \in W.$$

Then (P_J, Ω_J) is again a game; the corresponding game order coincides with the *weak order* on $W_J \backslash W$ (see [2]). For $w \in W$, let $\langle W_J w \rangle$ and $\langle W_J w \rangle_\Omega$ be the full subgames of (P_J, Φ_J) and (P_J, Ω_J) generated by $W_J w \in P_J$; respectively. As subsets of P_J, they can also be defined by

$$\langle W_J w \rangle = \{ W_J v \in P_J \mid W_J v \leq W_J w \},$$

and

$$\langle W_J w \rangle_\Omega = \{ W_J v \in P_J \mid W_J v \leq_\Omega W_J w \}.$$

Clearly, we have

$$\langle W_J w \rangle_\Omega \subset \langle W_J w \rangle \quad \text{(as sets)}.$$

This suggests studying a game $\langle W_J w \rangle$ satisfying the following condition:

(A) $\langle W_J v \rangle_\Omega = \langle W_J v \rangle$ (as sets) for any $W_J v \in \langle W_J w \rangle$.

A reflection game $\langle W_J w \rangle$ satisfying (A) is called a *basic* reflection game. The following lemma explains why we are interested in this class of games.

Lemma 10. *A basic reflection game is triangular.*

Proposition 11. *Let $w \in W$. Assume $\langle W_J w \rangle$ satisfies* (A). *Assume, moreover, $w^J = w^I$ for some $J \subset I \subset S$. Then we have a natural isomorphism of games:*

$$\langle W_J w \rangle \cong \langle W_I w \rangle.$$

By Proposition 11, in studying a game $\langle W_J w \rangle$ satisfying (A), we may assume the following condition:

(B) $w^J \neq w^I$ for any $J \subsetneq I \subset S$.

Let $Q_J = Q_{J \subset S}$ be the set of elements $W_J w$ of P_J satisfying both (A) and (B).

Proposition 12. *Let $J \subset S$. The set Q_J is non-empty if and only if the elements of $S \setminus J$ are mutually commutative.*

Proposition 13. *Let $J \subset S$. Assume that $\{ s_{01}, s_{02}, \dots, s_{0n} \} = S \setminus J$ are mutually commutative with $n = |S \setminus J|$. Then, for any $W_J w \in Q_J$, there exist $\{I_1, I_2, \dots, I_n\}$ $(I_i \subset S)$ and $(w_1, w_2, \dots, w_n) \in W_{I_1} \times W_{I_2} \times \cdots \times W_{I_n}$ such that*

(i) $s_{0k} \in I_k, \quad k = 1, 2, \dots, n.$

(ii) $I_h \cap I_k = \emptyset$ *if* $h \neq k$.

(iii) *An element of I_h and an element of I_k commute if $h \neq k$.*

(iv) $W_{J_k} w_k \in Q_{J_k \subset I_k}$ $(J_k = I_k \setminus \{s_{0k}\}), \quad k = 1, 2, \dots, n.$

(v) $W_J w = W_J w_1 w_2 \cdots w_n$.
(vi) $\langle W_J w\rangle \cong \langle W_{J_1} w_1\rangle + \langle W_{J_2} w_2\rangle + \cdots + \langle W_{J_n} w_n\rangle$.

Conversely, if $\{I_1, I_2, \ldots, I_n\}$ $(I_k \subset S)$ *and* $(w_1, w_2, \ldots, w_n) \in W_{I_1} \times W_{I_2} \times \cdots \times W_{I_n}$ *satisfy (i)-(iv) above, then we have*

$$\langle W_J w_1 w_2 \cdots w_n\rangle \cong \langle W_{J_1} w_1\rangle + \langle W_{J_2} w_2\rangle + \cdots + \langle W_{J_n} w_n\rangle,$$

and

$$W_J w_1 w_2 \cdots w_n \in Q_J.$$

A full subgame $\langle W_J w\rangle$ generated by $W_J w \in Q_J$ is called a *basic game of type* J. An element $w \in W$ is called a *basic element of type* J if the game $\langle W_J w\rangle$ is basic and $w = w^J$. By Propositions 12 and 13, the study of a basic game $\langle W_J w\rangle$ is reduced to the case $|S \setminus J| = 1$.

5.3. We keep the notation in 3.2. Any root $\gamma \in \Sigma_*$ can be written uniquely as a linear combination of the set Π_* of simple roots; the coefficient of $\alpha \in \Pi$ is denoted by $c(\alpha, \gamma)$. Let $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ be a fixed reduced decomposition of an element w of W. For $0 \le k \le l$, we put $w(k) = s_{i_1} s_{i_2} \cdots s_{i_k}$. For each $\alpha \in \Pi$, the sequence

$$\boldsymbol{c}(\alpha, w(k)) = \{\, c(\alpha, w(h)^{-1}\alpha_*)\,\}_{0 \le h \le k}$$

is weakly increasing by Lemma 5 (i). We consider the following condition on $w \in W$.

(A′) For any $\alpha \in \Pi$, any $0 \le k \le l = l(w)$ and any reduced expression of w, the coefficients

$$c(\alpha, tw(k)^{-1}\alpha_*), \quad t \in N(w(k)^J)$$

are always contained in the sequence $\boldsymbol{c}(\alpha, w(k))$.

Lemma 14. *We put* $\gamma = w^{-1}\alpha_*$ *for an element* $w \in W$ *satisfying* (A′). *Let* $\delta \in W\alpha_*$. *If* $\delta \le \gamma$, *then there exists a reduced expression* $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ *such that* $\delta = w(k)^{-1}\alpha_*$ *for some* $0 \le k \le l$.

Lemma 15. *As a condition on* $w \in W$ *satisfying* $w = w^J$, *the condition* (A) *is equivalent to the condition* (A′).

Let $w \in W$ be a basic element of type J. By the previous lemma, this is equivalent to say that it satisfies : (A′), (B), and $w = w^J$. Let $\gamma = w^{-1}\alpha_* \in \Sigma_*^+$. A root obtained in this way is called a *basic root of type* J. By Lemma 5, a basic element w of type J is uniquely determined by the corresponding basic root γ.
An element $w \in W$ is called *fully commutative* [24] if for every pair of non-commuting generators $s_i, s_j \in S$, there is no reduced expression for w containing a subword of length $m(i,j)$ of the form $s_i s_j s_i s_j \ldots$ As a consequence of Lemma 15, we have:

Lemma 16. *Let* $J \subset S$. *A basic element* $w \in W$ *of type* J *is fully commutative.*

5.4. According to [4][17][25], around 1989, D. Peteson introduced the notion of *minuscule elements* of Weyl groups of Kac-Moody Lie algebras. Let W be a Weyl group, and λ a dominant integral weight. An element $w \in W$ is called λ-minuscule if, for any reduced decomposition $w = s_{i_1} \ldots s_{i_l}$ of w, we have

$$w_{i_k} \ldots w_{i_1}\lambda = \lambda - \sum_{j=1}^{k} \alpha_{i_j}, \quad 1 \le k \le l.$$

An element $w \in W$ is called minuscule if it is λ-minuscule for some dominant integral weight λ. Minuscule elements are classified by R.A. Proctor [18] (simply-laced case) and J.R. Stembridge [25](non-simply-laced case).

Theorem 17. *For a simply-laced Coxeter group, the classification of basic elements coincides with that of minuscule elements. A basic reflection game associated with a general Coxeter group is isomorphic to a basic reflection game associated with a simply-laced Coxeter group.*

For some (non-simply-laced) Coxeter groups, the set of basic elements contains but not coincides with the set of minuscule elements.

5.5. The following theorem, essentially due to S. Okamura [15], generalizes the result of Greene, Nijenhuis and Wilf mentioned in 3.2.

Theorem 18. *Let $\langle W_J w\rangle_T$ be a basic reflection game. Then the probabilistic version of the simplification of $\langle W_J w\rangle_T$ selects a simple path of $\langle W_J w\rangle_T$ (connecting W_J and $W_J w$) uniform randomly with the probability*

$$\frac{\prod_{X\in\varphi_J(W_J w)}(|\varphi_J(W_J w)\cap\varphi_J^{-1}(X)|+1)}{l(w^J)!}.$$

Hence the number of simple paths in $\langle W_J w\rangle_T$ is equal to

$$\frac{l(w^J)!}{\prod_{X\in\varphi_J(W_J w)}(|\varphi_J(W_J w)\cap\varphi_J^{-1}(X)|+1)}.$$

If w^J is minuscule, the last formula is equivalent to the one (due to D. Peterson, see e.g. [4]) for the number of reduced expressions of w^J.

5.6. The following theorem generalizes the result of Sato and Welter mentioned in 3.3.

Theorem 19. *Let $\langle W_J w\rangle_T$ be a basic reflection game. Then the game $\langle W_J w\rangle_T$ has the energy function $E(W_J w)$ given by*

$$E(W_J w) = \sum_{X\in\varphi_J(W_J w)}^{\oplus} (E(\varphi_J(W_J w)\cap\varphi_J^{-1}(X))+1|0),$$

where $(a|0)=a\oplus(a-1)$ as in 2.16, and $E(\varphi_J(W_J w)\cap\varphi_J^{-1}(X))$ is the opening value of the energy function of the subgame $\varphi_J(W_J w)\cap\varphi_J^{-1}(X)$ of $\langle W_J w\rangle_T$.

5.7. An obvious open problem is the study of a general (not necessarily basic) triangular reflection game. The author hopes to report on this in a near future.

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