Cospectral graphs of the Grassmann graphs

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(joint work with Edwin van Dam)
Let $q$ be a prime power, and $V$ be a $n$-dimensional space over the $GF(q)$ the field with $q$ elements. Let $1 \leq e \leq n - 1$ be an integer.

The **Grassmann Graph** $G_q(n, e)$ has as vertices the $e$-dimensional subspaces and $S \sim T$ iff their intersection is $(e - 1)$-dimensional.

To construct graphs with the same spectrum as $G_q(n, e)$ we first will look at a partial linear space.
Let $n, e$ be positive integers such that $4 \leq 2e \leq n$.

Let $V$ be a $n$-dimensional vector space over $GF(q)$ and

let $H$ be a $2e$-dimensional subspace of $V$.
We first construct the partial linear space

$$\mathcal{L}G_q(n, e, e+1).$$

Its points are the $e$-dimensional subspaces of $V$.

There are two kinds of lines:

Lines of the first kind: $(e+1)$-dimensional subspaces $L$ of $V$ which are not a subspace of $H$. A line $L$ as points the $e$-dimensional subspaces contained in $L$.

b Lines of the second kind: $(e-1)$-dimensional spaces $M$ contained in $H$. A line $M$ has as points the $e$-dimensional spaces contained in $H$ which contain $M$ as a subspace.
Now \( \mathcal{LG}_q(n, e, e + 1) \) has
\[
\binom{n}{e} \text{ points,}
\]
\[
\binom{n}{e + 1} \text{ lines,}
\]
each point is incident with \( \binom{n-e}{1} \) lines
and each line is incident with \( \binom{e+1}{1} \) points.

Through any pair of points there is at most one line.
If \( P \) and \( Q \) are points then they are on a line iff \( P \cap Q \)
is \((e - 1)\)-dimensional.
Define $P_q(n, e+1)$ as the line graph of $\mathcal{L}G_q(n, e, e+1)$, that is its vertices are the lines of $\mathcal{L}G_q(n, e, e+1)$ and two lines are adjacent iff they have exactly one point in common.

**Theorem 1**

(i) $P_q(n, e+1)$ is cospectral with $G_q(n, e+1)$,

(ii) $P_q(n, e + 1)$ is distance-regular iff $n = 2e + 1$.

(iii) $P_q(2e + 1, e + 1)$ is not isomorphic to the Grassmann graph $G_q(2e + 1, e + 1)$. 
(i) Let $N$ be the point-line incidence matrix. Then
$NN^T - [\begin{array}{cc} n-e & 1 \\ 1 & 1 \end{array}]I$ is the adjacency matrix of the point graph. As the point graph is clearly $G_q(n,e)$, we know the spectrum of $NN^T$. Now except for the zero eigenvalue the spectrum of $NN^T$ is the same as the $N^TN$. This implies that $P_q(n,e+1)$ is cospectral with $G_q(n,e+1)$ as $NN^T - [\begin{array}{cc} e+1 & 1 \\ 1 & 1 \end{array}]I$ is the adjacency matrix for $P_q(n,e+1)$. 
(ii) If \( n < 2e + 1 \), then there is \( e + 1 \)-dimensional space \( L \) which intersects \( H \) in a \((e - 1)\)-dimensional space \( M \). Now in \( P_q(n, e+1) \) the distance between \( L \) and \( M \) is 2 and it easy to see that they have \( \binom{2}{1}\binom{e+1}{1} \) common neighbours where in the Grassmann graph \( c_2 = \binom{2}{1}^2 \).

If \( n = 2e + 1 \), then it is possible to check that it is distance-regular. An easy way to see this is true we use a result by Fiol and Garriga which states that if a graph has the same spectrum as a distance-regular graph \( \Gamma \) with diameter \( d \) is distance-regular iff for all vertices \( x \) we have \( k_d(x) = k_d(\Gamma) \). And this is easily checked.

(iii) Let \( n = 2e + 1 \). Let \( K \) be an \((e + 2)\)-dimensional space which intersects \( H \) in \( e + 1 \) dimensions. Now the \((e + 1)\)-dimensional subsapces of \( K \) which are not contained in \( H \) forms a maximal clique of size \( \binom{e+2}{1} - 1 \) in \( P_q(2e + 1, e + 1) \), whereas the Grassmann graph \( G_q(2e + 1, e + 1) \) has maximal cliques of sizes \( \binom{e+2}{1} \) and \( \binom{e+1}{1} \).

This shows the theorem.
(i) By looking at maximal cliques in $P_q(2e+1, e+1)$, it is easy to see that it is not vertex-transitive. The group $P\Gamma L(2e + 1)_{2e}$ is an automorphism group of the graph. It was shown by M. Tagami that this is the full automorphism group.

(ii) For large $q$ and $e$ we were able to show that the local graph of a line of type 1 is not cospectral to the local graph of a line of type 2. We suspect that this is always the case. This implies, for example, that the Terwilliger Algebra depends on the base vertex for $P_q(2e + 1, e + 1)$. 