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Infinite games and set existence axioms

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Abstract

The aim of this paper is to investigate the logical strength of the determinacy of infinite games whose complexity lies in the finite difference hierarchy over Σ_2^0. This is a continuation of Tanaka's papers [12] and [13], and will be succeeded by our further studies on Δ_3^0 games.

1 Introduction

For $A \subset \omega^\omega$, we associate a two-person *game* G_A (or simply denote A) described as follows: player I and player II alternately choose natural numbers (starting with I) to form an infinite sequence $f \in \omega^\omega$ and I (resp. II) wins iff $f \in A$ (resp. $f \notin A$). We say that A is *determinate* if one of the players has a *winning strategy*, namely, a map $\sigma : \omega^{<\omega} \to \omega$ such that the player is guaranteed to win each play f in which he makes a move $f(n) = \sigma((f(0), \ldots, f(n-1)))$ at each stage n of his turn.

Much work has been done regarding determinacy in set theory. Gale and Stewart [4] proved that open sets are determined. Their result were soon extended by Wolfe [14] to the F_σ-sets, then by Davis [2] to $F_{\sigma\delta}$-sets. The next major advance was made by Martin [7] who showed that all Borel sets are determinate. It has also shown that one must assume some strong large cardinal axioms to prove the determinacy of games beyond the class of Borel sets. See [6] for more information.

The determinacy of games with various logical complexities can be formally stated within second order arithmetic. Then, we are interested in the following question motivated by Reverse Mathematics (cf. Simpson [10]): what set existence axioms are needed to prove the determinacy of some classes of formulas in second order arithmetic. While Friedman [3] proved that Σ_5^0-determinacy is not provable in the full system of second order arithmetic, Steel [11] (cf. Tanaka [12] and Simpson [10]) proved that $\mathbf{ATR}_0$ is necessary and sufficient for Σ^0_1-determinacy. Later, Tanaka [13] proved that the axiom of Σ^1_1-monotone inductive definitions is equivalant to Σ^0_2-determinacy over $\mathbf{ATR}_0$.

Meanwhile, Burgess [1] has obtained a charecterization of R-sets by applying the game quantifier to the difference hierarchy over G_δ-sets. Hinman [4] defined an effective counterpart of the finite level of R-hierarchy and showed that this hierarchy can be defined by finite iteration of Kolmogorov's operator $\mathcal{R}$. These provide us with useful indication to extend Tanaka's characterization of Σ^0_2-determinacy, though it does not seem to be easy to transform them directly into our setting.

The paper is structured as follows. Section 2 contains some preliminaries. Section 3 represents the main part in this paper, we introduce the inductive definition of a combination of finately many Σ^1_1-operators, then show that it is equivalent to the determinacy of a Boolean combination of Σ^0_2-sets.

The proofs here are only sketched. A full treatment will appear elsewhere.

2 Preliminaries

In this section, we introduce some subsystems of second order arithmetic.

Definition 2.1 The *language of second order arithmetic* consists of the following symbols:

- *number variables* $x, y, z \ldots$ intended to range over ω,
- *set variables* $X, Y, Z, \ldots$ intended to range over the subsets of ω,
- *constant symbols* $0, 1$,
- *function symbols* $+, \cdot$,
- *relation symbols* $=, <, \in$.

Number terms are build from number variables and constant symbols $0, 1$ by using $+, \cdot$. *Atomic formulas* are of the form $t_1 = t_2, t_1 < t_2, t_1 \in X$, where t_1, t_2 are number terms. *Formulas* are built up from atomic formulas by means of propositional connectives, number quantifiers $\forall x, \exists x$, and set quantifiers $\forall X, \exists X$. The formulas can be classified as follows:

- φ is *bounded* (Π^0_0) if it is built up from atomic formulas by using propositional connectives and bounded number quantifiers $(\forall x < t), (\exists x < t)$,

- φ is Π^1_0 if it does not contain any set quantifier. Π^1_0 formulas are called *arithmetical* formulas,
- φ is Σ^i_n if $\varphi \equiv \neg\psi$ where ψ is a Π^i_n-formula ($i \in \{0,1\}, n \in \omega$),
- φ is Π^0_{n+1} if $\varphi \equiv \forall x_1 \cdots \forall x_k \psi$ where ψ is a Σ^0_n-formula ($n \in \omega$),
- φ is Π^1_{n+1} if $\varphi \equiv \forall X_1 \cdots \forall X_k \psi$ where ψ is a Σ^1_n-formula ($n \in \omega$).

The Π^i_n-*comprehension axiom*, denoted (Π^i_n-CA), is defined to be $\exists X \forall x(x \in X \leftrightarrow \varphi(x))$, where φ is Π^i_n.

Definition 2.2 The system $\mathbf{ACA}_0$ consists of the discretely ordered semi-ring axioms for $(\mathbf{N}, +, \cdot, 0, 1, <)$ together with (Π^1_0-*CA*) and the following *induction axiom*:

$$\forall X((0 \in X \wedge \forall x(x \in X \to (x+1) \in X)) \to \forall x(x \in X)).$$

$\mathbf{ACA}_0$ is a conservative extension of Peano-arithmetic. It is also a nice base theory when we deal with classical analysis.

Finally, we define the subsystem $\mathbf{ATR}_0$. We start by defining the axiom ($\Pi^i_n - TR$).

Definition 2.3 The axiom ($\Pi^i_n - TR$) is defined as follows: for any set $X \subset \omega$ and for any well-ordering $\prec$, there exists a set $H \subset \omega$ such that

1. if b is the $\prec$-least element, then $(H)_b = X$,
2. if b is the immediate successor of a w.r.t. $\prec$, then $\forall n(n \in (H)_b \leftrightarrow \psi(n, (H)_a))$,
3. if b is a limit, then $\forall a \forall n((n, a) \in (H)_b \leftrightarrow (a \prec b \wedge n \in (H)_a))$,

where ψ is a Π^i_n-formula, $(H)_a = \{n : (n, a) \in H\}$ and, $(n, a) = ((n + a)(n + a + 1)/2) + n$.

The sybsystem $\mathbf{ATR}_0$ consists of $\mathbf{ACA}_0 + (\Pi^1_0 - TR)$. Obviously, $\mathbf{ATR}_0$ is stronger than $\mathbf{ACA}_0$.

3 $(\Sigma^0_2)_k$-determinacy and $[\Sigma^1_1]^k$-ID

An operator $\Gamma : P(\omega) \to P(\omega)$ is said to be *monotone* iff $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y$. Γ is a Σ^1_1-operator iff its graph $\{(x, X) : x \in \Gamma(X)\}$ is Σ^1_1. We will use $mon\Sigma^1_1$ to denote the class of Σ^1_1 monotone operators. A relation R is a *pre-ordering* iff it is reflexive, connected and transitive. R is a *pre-wellordering* iff it is pre-ordering and well-founded. The *field* of R is the set $\{x : \exists y(x, y) \in R \vee (y, x) \in R\}$.

We characterize the determinacy of a Boolean combination of Σ^0_2-formulas in term of inductive definition of a finite combination of Σ^1_1-operators. First, let us look how to iterate a pair of operators Γ_1, Γ_2. We assume Γ_1 has a distinct parameter X, and so it is also denoted by Γ_1^X. We start with the empty set, and iterate applying $\Gamma_1^{\emptyset}$ until we get a fixed point F_0, i.e, $\Gamma_1^{\emptyset}(F_0) \subset F_0$. Then re-start iterating $\Gamma_1^{\Gamma_2(F_0)}$, get a fixed point F_1. Again, iterate $\Gamma_1^{\Gamma_2(F_0)\cup\Gamma_2(F_1)}$ and so on. The procedure stops when we get a fixed point F such that if F' is the least fixed point of Γ_1^F then $\Gamma_2(F') \subset F$. This procedure can be viewed as an inductive definition by a single operator $[\Gamma_1^X, \Gamma_2]$.

Generally, the iteration of $\Gamma_1, \cdots, \Gamma_{k-1}, \Gamma_k$ can be described as follows. We iterate $[\Gamma_1, \cdots, \Gamma_{k-1}]^{\emptyset}$ until we get a fixed point, say F_0. Then restart iterating $[\Gamma_1, \cdots, \Gamma_{k-1}]^{\Gamma_k(F_0)}$, get another fixed point F_1, again iterate $[\Gamma_1, \cdots, \Gamma_{k-1}]^{\Gamma_k(F_0)\cup\Gamma_k(F_1)}$ and so on. Eventually, we stop when we get a fixed point F of $[\Gamma_1, \cdots, \Gamma_{k-1}, \Gamma_k]$.

Now, let $S_1, \cdots, S_k$ be collections of operators. By $[S_1, S_2, \cdots, S_k]$-ID, we denote the axiom which guarantees the existence of a fixed point of $[\Gamma_1, \cdots, \Gamma_k]$ for any $\Gamma_1 \in S_1, \cdots, \Gamma_k \in S_k$. We here give a formal definition for $k = 2$ as follows.

Definition 3.1 $[S_1, S_2]$-ID asserts that for any $\Gamma_1 \in S_1, \Gamma_2 \in S_2$, there exist pre-wellorderings W, V' on their fields F, F' respectively such that

$$\Gamma_2(F') \subset F \wedge \Gamma_1^F(F') \subset F',$$

and also a pre-wellordering V^m on its field F^m for each $m \in F$ such that

- $\forall y \in F^m, \;\; V^m_y = \Gamma_1^{W_{<m}}(V^m_{<y}) \cup V^m_{<y}$
- $W_m = \Gamma_2(F^m) \cup W_{<m}$
- $\Gamma_1^{W_{<m}}(F^m) \subset F^m$.

For convenience, we introduce some abbreviations. By $[S_1^{k_1}, S_2^{k_2}, \cdots, S_i^{k_i}]$-ID, we mean

$$[\overbrace{S_1, \ldots, S_1}^{k_1 \text{ times}}, \overbrace{S_2, \ldots, S_2}^{k_2 \text{ times}}, \cdots, \overbrace{S_i, \ldots, S_i}^{k_i \text{ times}}]\text{-ID}.$$

We also write $[S]^k$-ID for $[S^k]$-ID.

We here remark that our defition of combinations of inductive operators is different from one studied in Richter and Aczel [8].

Next, we define the class $(\Sigma^0_2)_k$ of formulas as follows. For $k = 1$, $(\Sigma^0_2)_1 = \Sigma^0_2$. For $k > 1, \psi \in (\Sigma^0_2)_k$ iff it can be written as $\psi_1 \wedge \psi_2$, where $\neg\psi_1 \in (\Sigma^0_2)_{k-1}$ and $\psi_2 \in \Sigma^0_2$. The goal of this section is to prove the equivalence between $(\Sigma^0_2)_k$-determinacy and $[\Sigma^1_1]^k$-ID. We start by the following theorem.

Theorem 3.1 $\mathbf{ATR}_0 \vdash [mon\Sigma^1_1, (\Sigma^0_1)^{k-1}]\text{-ID} \to (\Sigma^0_2)_k\text{-Det}$

Proof. We only show the case $k = 2$ since the case $k > 2$ can be treated by simple meta-induction. Let $A(f)$ be a $(\Sigma^0_2)_2$-formula. There are a Σ^0_2-formula φ_0 and a Π^0_2-formula φ_1 such that for all $f \in \omega^\omega$, $A(f) \equiv \varphi_0(f) \wedge \varphi_1(f)$. Since φ_0 is a Σ^0_2, there is a Π^0_0 relation R such that for all $f \in \omega^\omega$, $\varphi_0(f) \equiv \exists x \forall y R(x, f[y])$, where $f[y]$ is a code for $\langle f(0), f(1), \cdots, f(y-1)\rangle$.

By $[mon\Sigma^1_1, \Sigma^0_1]$-ID, we define a transfinite sequence $\langle W_\alpha, \alpha \in Y\rangle$ of sure winning positions for player I as follows: for any ordinal $\alpha \in Y$,

$$u \in W_\alpha \leftrightarrow \underbrace{\exists x}_{(1)} \underbrace{(\text{I has a winning strategy for } A_{u,\alpha,x})}_{(2)},$$

where $A_{u,\alpha,x} = \{f \in \omega^\omega : \forall y(R(x, (u*f)[y]) \vee (u*f)[y] \in W_{<\alpha}) \wedge \varphi^u_1(f)\}$. Here, $u*f$ denotes the concatenation of u and f, $\varphi^u_1(f) \leftrightarrow \varphi_1(u*f)$ and $W_{<\alpha} = \bigcup\{W_\beta : \beta < \alpha\}$. We put $W_\infty = \bigcup_{\alpha\in Y} W_\alpha$.

Now, let us look at the above definition of W_α. Clearly, part(1) of the right hand side of the definition is Σ^0_1-operator. Part(2) can be viewed as the complement of a fixed point of Σ^1_1-monotone operator, since $A_{u,\alpha,x}$ is a Π^0_2 game (see [4]). Thus, W_α is defined by a combination of Σ^0_1-operator and Σ^1_1-monotone operator. We talked about a set of ordinals Y, but there is no harm of this at all, because we can simply identify the set Y with the pre-wellordering constructed by $[mon\Sigma^1_1, \Sigma^0_1]$-ID. Moreover, it is not difficult to show that for any $u \in \omega^{<\omega}$, we have

1. $u \in W_\infty \to$ player I has a winning strategy for A^u,
2. $u \notin W_\infty \to$ player II has a winning strategy for A^u.

Here, A^u is defined by $f \in A^u \leftrightarrow u * f \in A$.

From this claim, one of the players has a winning strategy for A^u. Thus $A = A^\emptyset$ is determinate. This completes the proof of the main theorem. □

Next, we prove the converse.

Theorem 3.2 $\mathbf{ACA}_0 \vdash (\Sigma^0_2)_k\text{-Det} \to [\Sigma^1_1]^k\text{-ID}$.

Proof. As in the previous theorem, we only sketch the prove for the case $k = 2$. Let Γ_1, Γ_2 be Σ^1_1-operators.

First, we construct a $(\Sigma^0_2)_2$-game G as follows:

- Player I starts by playing y^* asking whether y^* is in the field of the structure inductively defined by $[\Gamma_1, \Gamma_2]$, then II has to answer either *yes* (in this case, he will be called **Pro**) or *no* (in this case, I will be called **Pro**).

- Pro is requested to construct a pre-ordering W, and $\langle V^m, m \in field(W)\rangle$ such that every V^m is pre-wellorderings, with $y \in field(W)$ or $y \in field(V^{m,y})$ for some m.
- We give Con (the opponent of Pro) a chance to win if he points out a mistake in the $(\leq y)$-segment.
- We show that I has no winning strategy and then by the $(\Sigma^0_2)_2$-determinacy, we deduce that player II has a winning strategy.
- Using II's winning strategy, we build $\tilde{W}, \langle\tilde{V}^m, m \in field(\tilde{W})\rangle$ such that for any y accepted by II, $\tilde{W}_{<y} = W_{<y}$ and $\tilde{V}^m = V^m$ for any $m \in \tilde{W}_{<y}$.
- Using the fact that I can never win, we show that $\tilde{W}, \langle\tilde{V}^m, m \in field(\tilde{W})\rangle$ satisfy the condition of Definition 3.1.□

Finally, we have the following corollary:

Corollary 3.1 *For any $k > 0$, we have*
$\mathbf{ATR}_0 \vdash (\Sigma^0_2)_k\text{-Det} \leftrightarrow [mon\Sigma^1_1, (\Sigma^0_1)^{k-1}]\text{-ID} \leftrightarrow [\Sigma^1_1]^k\text{-ID}$.

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