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Pricing Lookback Options with Knock-out Boundaries *

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Abstract.

This paper describes a new kind of exotic options, lookback options with knock-out boundaries. These options are knock-out options whose pay-offs depend on the extrema of a given securitie's price over a certain period of time. Closed form expressions for the price of seven kinds of lookback options with knock-out boundaries are obtained in this article. The numerical studies has also been presented.

Key words: exotic options, lookback options, knock-out boundaries
JEL classification:G13

1 Introduction

Many kinds of exotic options, such as Asian options, knock-out options and lookback options, have been studied in mathematical finance. The pricing problems of a new kind of options, lookback options with knock-out boundaries, have been introduced in this article. Lookback options with knock-out boundaries are contingent claims which have both features of lookback options and knock-out options at the same time.

Lookback options are contingent claims which were studied by Conze and Viswanathan (1991) and Goldman, Sosin and Gatto (1979). The pay-off of lookback options depends on the maximum and/or minimum of a given securitie's price over a certain period of time. The advantageous point of lookback options is that holders of options can lock in the most favorable profit during the life of options. Knock-out options are contingent claims which

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become worthless at the occasion that the price of underlying asset touches the certain boundaries. The pricing problems of knock-out options have already been considered in early 1970s by Merton (1973). Pricing problems of double knock-out options have been considered in Kunitomo and Ikeda (1994) and Ikeda (2000), for example. An advantageous point of knock-out options is that they are cheaper than ordinary options. There is an advantageous point for lookback options with knock-out boundaries. Althogh lookback options are usually very expensive, it is possible to make the price of lookback options much cheaper by equipping the knock-out features. The analytic formulas for the price of float strike double knock-out lookback options are obtained in this article. The pricing formulas for other kinds of lookback options with knock-out boundaries can be found in Muroi (2004).

2 Lookback Options with knock-out boundaries

The pricing problems for lookback options with double knock-out boundaries are discussed in this section. This is considered in the Black-Scholes economy with the probability space, $(\Omega, \mathcal{F}, P)$. There are two kinds of securities in this market, the risk securities and the risk-free securities. The risk-free security earns interest continuously compounded at the constant rate, $r(\geq 0)$, with a dollar invested at time 0 accumulating to $B(t)$ by time t. The risk-neutral probability measure, Q, has to be equiped to calculate the rational value of contingent calims. On the risk-neutral probability measure, Q, the price process of risk assets is assumed to follow the SDE,

$$\begin{aligned} dS_t &= S_t(rdt + \sigma d\tilde{W}_t) \\ S_0 &= s\,. \end{aligned} \tag{2.1}$$

In order to define the price of lookback options with double knock-out boundaries, following variables are introduced:

$$L = \inf_{0\leq r\leq t} S_r,\ L_T = \inf_{t\leq r\leq T} S_r,\ L(T) = \min\{L_T, L\}$$
$$M = \sup_{0\leq r\leq t} S_r,\ M_T = \sup_{t\leq r\leq T} S_r,\ M(T) = \max\{M_T, M\}\,.$$

Float strike double knock-out lookback options are defined.

Definition 2.1 *Float strike double knock-out lookback options with the maturity date, T, are options which have a cashflow at the maturity date, T, if the price of underlying assets touch neither the lower boundary, l, nor the upper boundary, m, during the life of options. If the lower or upper boundary is breached by the price process of underlying assets, options expire worthless. The cashflow for call options at the maturity date equals $S_T - L(T)$ and the cashflow for put options at the maturity date is given by $M(T) - S_T$.*

In this section, the pricing problems of options with knock-out boundaries are considered under the conditions,

$$S_t = x,\ l < L,\ M < m\ . \tag{2.2}$$

The price of float strike double knock-out lookback call options at time t is denoted by $C_{FL}(t)$. It is possible to derive the option premiums by using the expectation operator, $E[\cdot]$, which is a conditional expectaions with the measure, Q, conditioned by (2.2). The price of options is given by

$$\begin{aligned} C_{FL}(t) &= E[e^{-r\tau}(S_T - L(T))1_{\{l<L_T,M_T<m\}}] \\ &= e^{-r\tau}\{E[S_T 1_{\{l<L_T,M_T<m\}}] - LQ[L < L_T, M_T < m] \\ &\quad -E[L_T 1_{\{l<L_T\leq L,M_T<m\}}]\}\ , \end{aligned} \tag{2.3}$$

where $\tau = T-t$. The probability that the price process of underlying assets reach neither the lower level, p, nor the upper level, q $(p < s < q)$, which is denote by $F(p,q)$. The closed form formula of this probability is given by

$$\begin{aligned} &F(p,q) = P[p < L_T, M_T < q] \\ &= \sum_{n=-\infty}^{\infty} (\frac{q^n}{p^n})^{\frac{2r}{\sigma^2}-1}\{\Phi(\frac{\log(\frac{xq^{2n}}{p^{2n+1}}) + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - \Phi(\frac{\log(\frac{xq^{2n-1}}{p^{2n}}) + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\} \\ &- \sum_{n=-\infty}^{\infty} (\frac{p^{n+1}}{xq^n})^{\frac{2r}{\sigma^2}-1}\{\Phi(\frac{\log(\frac{p^{2n+1}}{xq^{2n}}) + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - \Phi(\frac{\log(\frac{p^{2n+2}}{xq^{2n+1}}) + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\} \end{aligned} \tag{2.4}$$

where $\Phi(\cdot)$ is a distribution function for standard normal random variables. The first term in (2.3) is represented by D:

$$\begin{aligned} D &= E[e^{-r\tau}S_T 1_{\{l<L_T,M_T<m\}}] \\ &= x\sum_{n=-\infty}^{\infty}\{(\frac{m^n}{l^n})^{\frac{2r}{\sigma^2}+1}(\Phi(d_{1n}) - \Phi(d_{2n})) - (\frac{l^{n+1}}{xm^n})^{\frac{2r}{\sigma^2}+1}(\Phi(d_{3n}) - \Phi(d_{4n}))\}\ , \end{aligned} \tag{2.5}$$

where d_{1n}, d_{2n}, d_{3n} and d_{4n} are given by

$$\begin{aligned} d_{1n} &= \frac{\log(\frac{xm^{2n}}{l^{2n+1}}) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},\ d_{2n} = \frac{\log(\frac{xm^{2n-1}}{l^{2n}}) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \\ d_{3n} &= \frac{\log(\frac{l^{2n+1}}{xm^{2n}}) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},\ d_{4n} = \frac{\log(\frac{l^{2n+2}}{xm^{2n+1}}) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\ . \end{aligned}$$

The second and third terms in (2.3) are derived as

$$-LQ[L < L_T, M_T < m] - E[L_T 1_{\{l<L_T<L,M_T<m\}}] = -lF(l,m) - \int_l^L F(y,m)dy\ . \tag{2.6}$$

The first term in (2.6) was already calculated in (2.4) and a remained task is to obatain the explicit formula for the second term in (2.6). In order to derive the explicit representation of this term, the function, $G(\cdot)$, is introduced as

$$G(z)=\int_l^z F(y,m)dy\ .$$

The function, $G(\cdot)$, is given by

$$G(z)=\sum_{n=-\infty}^{\infty}\{G_n^1(z)-G_n^2(z)\}-\sum_{n=-\infty}^{\infty}\{G_n^3(z)-G_n^4(z)\}\ . \tag{2.7}$$

In order to derive the explicit representation formula for lookback options with knock-out boundaries, the following assumption has to be imposed.

Assumption 2.1 *For any integer, k, the relation, $\frac{2r}{\sigma^2}=1+\frac{1}{k}$, is not satisfied.*

Even if Assumtion 2.1 is not satisfied, it is possible to obtained the formula for $G(\cdot)$ and this is discussed later in Appendix. Under Assumption 2.1, the explicit representations for $G_n^1(\cdot)$, $G_n^2(z)$, $G_n^2(z)$ and $G_n^2(z)$ are given by

$$\begin{aligned}
G_n^1(z) &= \frac{m}{(2n+1)\alpha_n^1}\{(\frac{x}{m})e^{(r-\frac{\sigma^2}{2})\tau}\}^{\alpha_n^1}[e^{-\sigma\sqrt{\tau}\alpha_n^1 f_n^1}\Phi(f_n^1)-e^{-\sigma\sqrt{\tau}\alpha_n^1 g_n^1}\Phi(g_n^1)-\\
&\quad -e^{\sigma^2\tau(\alpha_n^1)^2/2}\{\Phi(f_n^1+\sigma\sqrt{\tau}\alpha_n^1)-\Phi(g_n^1+\sigma\sqrt{\tau}\alpha_n^1)\}]\\
G_n^2(z) &= \frac{m}{2n\alpha_n^2}\{(\frac{x}{m})e^{(r-\frac{\sigma^2}{2})\tau}\}^{\alpha_n^2}[e^{-\sigma\sqrt{\tau}\alpha_n^2 f_n^2}\Phi(f_n^2)-e^{-\sigma\sqrt{\tau}\alpha_n^2 g_n^2}\Phi(g_n^2)-\\
&\quad -e^{\sigma^2\tau(\alpha_n^2)^2/2}(\Phi(f_n^2+\sigma\sqrt{\tau}\alpha_n^2)-\Phi(g_n^2+\sigma\sqrt{\tau}\alpha_n^2))]\quad (n\neq 0)\\
G_0^2(z) &= (z-l)\Phi(\frac{\log(\frac{x}{m})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}})\\
G_n^3(z) &= \frac{m}{(2n+1)\alpha_n^3}(\frac{m}{x})^{\frac{2r}{\sigma^2}-1}\{(\frac{x}{m})e^{-(r-\frac{\sigma^2}{2})\tau}\}^{\alpha_n^3}[e^{\sigma\sqrt{\tau}\alpha_n^3 f_n^3}\Phi(f_n^3)-e^{\sigma\sqrt{\tau}\alpha_n^3 g_n^3}\Phi(g_n^3)-\\
&\quad -e^{\sigma^2\tau(\alpha_n^3)^2/2}(\Phi(f_n^3-\sigma\sqrt{\tau}\alpha_n^3)-\Phi(g_n^3-\sigma\sqrt{\tau}\alpha_n^3))]\\
G_n^4(z) &= \frac{m}{(2n+2)\alpha_n^4}(\frac{m}{x})^{\frac{2r}{\sigma^2}-1}\{(\frac{x}{m})e^{-(r-\frac{\sigma^2}{2})\tau}\}^{\alpha_n^4}[e^{\sigma\sqrt{\tau}\alpha_n^4 f_n^4}\Phi(f_n^4)-e^{\sigma\sqrt{\tau}\alpha_n^4 g_n^4}\Phi(g_n^4)-\\
&\quad -e^{\sigma^2\tau(\alpha_n^4)^2/2}(\Phi(f_n^4-\sigma\sqrt{\tau}\alpha_n^4)-\Phi(g_n^4-\sigma\sqrt{\tau}\alpha_n^4))]\quad (n\neq -1)\\
G_{-1}^4(z) &= (z-l)(\frac{m}{x})^{\frac{2r}{\sigma^2}-1}\Phi(\frac{\log(\frac{m}{x})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}})
\end{aligned}$$

where

$$\begin{aligned}
f_n^1 &= \frac{\log(\frac{xm^{2n}}{z^{2n+1}})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\ g_n^1=\frac{\log(\frac{xm^{2n}}{l^{2n+1}})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\\
\alpha_n^1 &= \frac{1-n(\frac{2r}{\sigma^2}-1)}{2n+1},\quad f_n^2=\frac{\log(\frac{xm^{2n-1}}{z^{2n}})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\\
g_n^2 &= \frac{\log(\frac{xm^{2n-1}}{l^{2n}})+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\ \alpha_n^2=\frac{1-n(\frac{2r}{\sigma^2}-1)}{2n},
\end{aligned}$$

$$
\begin{aligned}
f_n^3 &= \frac{\log(\frac{z^{2n+1}}{xm^{2n}}) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \; g_n^3 = \frac{\log(\frac{l^{2n+1}}{xm^{2n}}) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\
\alpha_n^3 &= \frac{1+(n+1)(\frac{2r}{\sigma^2}-1)}{2n+1}, \; f_n^4 = \frac{\log(\frac{z^{2n+2}}{xm^{2n+1}}) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\
g_n^4 &= \frac{\log(\frac{l^{2n+2}}{xm^{2n+1}}) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \; \alpha_n^4 = \frac{1+(n+1)(\frac{2r}{\sigma^2}-1)}{2n+2}.
\end{aligned}
$$

These calculations lead to the explicit representation of $G(\cdot)$ and it is given by

$$
G(z) = \sum_{n=-\infty}^{\infty} \{G_n^1(z) - G_n^2(z)\} - \sum_{n=-\infty}^{\infty} \{G_n^3(z) - G_n^4(z)\}. \tag{2.8}
$$

The following theorem is obtained.

Theorem 2.1 *If the price of underlying assets touch neither the lower boundary, l, nor the upper boundary, m, during the time interval, $[0,t]$, the closed form formula for the time t price of float strike double knock-out lookback call options with the maturity date, T, is given by*

$$
C_{FL}(t) = D - e^{-r\tau}(lF(l,m) + G(L)).
$$

The closed form analytic formulas of D is given by (2.5), $F(\cdot,\cdot)$ is given by (2.4) and $G(\cdot)$is given by (2.7).

It has not been derived the pricing formulas for lookback options with knock-out boundaries in case that Assumption 2.1 is not satisfied. The following assumption is imposed.

Assumption 2.2 *For some integer, k, the relation, $\frac{2r}{\sigma^2} = 1 + \frac{1}{k}$, is satisfied.*

Under assumption 2.2, the terms, which needs corrections in $G(\cdot)$, are $G_k^1(\cdot)$,$G_k^2(\cdot)$,$G_{-k-1}^3(\cdot)$ and $G_{-k-1}^4(\cdot)$. They are given by

$$
\begin{aligned}
G_k^1(z) &= -\frac{m\sigma\sqrt{\tau}}{2k+1}\{f_k^1\Phi(f_k^1) - g_k^1\Phi(g_k^1) + \phi(f_k^1) - \phi(g_k^1)\} \\
G_k^2(z) &= -\frac{m\sigma\sqrt{\tau}}{2k}\{f_k^2\Phi(f_k^2) - g_k^2\Phi(g_k^2) + \phi(f_k^2) - \phi(g_k^2)\} \\
G_{-k-1}^3(z) &= -\frac{m\sigma\sqrt{\tau}}{2k+1}(\frac{m}{x})^{\frac{1}{k}}\{f_{-k-1}^3\Phi(f_{-k-1}^3) - g_{-k-1}^3\Phi(g_{-k-1}^3) + \phi(f_{-k-1}^3) - \phi(g_{-k-1}^3)\} \\
G_{-k-1}^4(z) &= -\frac{m\sigma\sqrt{\tau}}{2k}(\frac{m}{x})^{\frac{1}{k}}\{f_{-k-1}^4\Phi(f_{-k-1}^4) - g_{-k-1}^4\Phi(g_{-k-1}^4) + \phi(f_{-k-1}^4) - \phi(g_{-k-1}^4)\}.
\end{aligned}
$$

where $\phi(\cdot)$ is a density function for the Normal random variables. It is also possible to obtain the pricing formulas for other kind of lookback options with knock-out boundaries and it is discussed in Muroi (2004). The numerical results are also shown in that paper.

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