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The Consistency in Interest Rate Models/Extending the support theorem and the viability theorem. May 27, 2005

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1 abstract

The purpose of this presentation is to state the consistency problem in HJM(Heath-Jarrow-Morton) interest rate models by extending the support theorem and the viability theorem. The role of the support theorem is to express supports of distributions of paths obtained from SDE's. The role of the viability theorem is to give the conditions for SDE's solutions to stay in a subset.
Musiela reduced the following SPDE (stochastic partial differential equation) for instantaneous forward rates $r(t,x)$ at time $t+x$ observed at time t.

$$\begin{cases} dr(t,x) & = \frac{\partial r}{\partial x}(t,x)\ dt + \left(\sum_j \sigma_j(t,x)\int_0^x \sigma_j(t,u)du\right)\ dt + \sum_j \sigma_j(t,x)dB^j(t) \\ r(0,x) & = f(x) \end{cases} \tag{1}$$

In this equation, σ_j are stochastic in general. The initial forward curve $x \mapsto f(x)$ is chosen so that it is consistent with the market price of risk-free bond and so on. The function f is, for example, chosen from the following Nelson-Siegel family which has four parameters z_1, z_2, z_3, z_4.

$$f(x; z_1, z_2, z_3, z_4) = z_1 + z_2 e^{-z_4 x} + z_3 x e^{-z_4 x}$$

The "solution" of SPDE (1) can be mathematically formulated as a H-valued stochastic process $(r(t,\ \cdot\ ))_{t\geq 0}$, where H is a separable Hilbert space.
On the other hand, forward curves observed from the market belong to the narrower set $M \subset H$ such as Nelson-Siegel family. Therefore it is very important when M is formulated as a finite dimensional manifold.
Since the initial forward curve f belongs to M, the forward curve $r(t,\ \cdot\ )$ observed at the future time t should also belong to M. This problem is motivated from the stability of the daily estimation of the model parameters, and is called the consistency problem.
Let $r^f(t,x)$ be the solution in a sense of SPDE (1). Then the consistency can be formulated such that $P(r^f(t,\ \cdot\ ) \in M, \quad \forall t \geq 0) = 1$ if $f \in M$ holds.
When $M \subset H$ and SPDE (1) are given, it is very important to find a criterion of consistency.

By regarding SPDE (1) as a stochastic (ordinary) differential equation in H, the condition of consistency is, roughly speaking, that the modified drift and the dispersion belong to the "tangent space" at all points in M.
This is the role of the viability theorem. So it is useful to establish the viability theorem for SPDE (1).
Björk and Filipović solved this kind of problem by differential geometry approach. However, Nakayama ([2]) proved the viability theorem by using the support theorem proved in Nakayama ([1]).
In Nakayama ([1]) and ([2]), the support theorem and the viability theorem had been proven for the mild solution of the stochastic differential equation in a Hilbert space of the form:

$$\begin{cases} dX^x(t) = AX^x(t)dt + b(X^x(t))dt + \sigma(X^x(t))dB(t), \\ X^x(0) = x. \end{cases}$$

It is driven by a Hilbert space-valued Wiener process B, with the infinitesimal generator A of a (C_0)-semigroup. This equation contains the SPDE (1).

References

[1] Nakayama, T., *Support Theorem for mild solutions of SDE's in Hilbert spaces*, Journal of mathematical sciences, the University of Tokyo, 2004.

[2] Nakayama, T., *Viability Theorem for SPDE's including HJM framework*, Journal of mathematical sciences, the University of Tokyo, 2004.