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A Note on Twisted Dual Games*

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Abstract. Kikuta (2005) proposed a duality of solutions in cooperative TU-games. In this note we try to extend this duality and see that, in a sense, the von Nuemann-Morgenstern solution satisfies the extended property.

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1 Introduction and Preliminaries

Many papers examined relations between a cooperative game with transferable utility and its dual, or relations between solutions for a game and its dual. Shapley (1962) outlines theoretical structures of simple games, and states beautiful properties of simple games and compound simple games with respect to their dual games. A solution for games is sometimes called to satisfy the duality if the solutions for a game and its dual coincide. The Shapley value (Shapley, 1953) and the prenucleolus (Schmeidler, 1969) are single-valued solutions. It is well-known that the Shapley value satisfies the duality. On the other hand, it is easy to find a game where the prenucleolus of a game is different from the prenucleolus of the dual game of it. Is there another duality which is satisfied by the prenucleolus ?

Kikuta (2002, 2005) proposed another kind of duality of solutions which is satisfied by the prenucleolus. This duality is, however, satisfied by other solutions as well as the Shapley value. So it is still a problem to find a duality-like property which distinguishes the prenucleolus from the Shapley value. On the other hand, this duality is mathematically an extension of the self-duality of rules in bankruptcy problems to a property of solutions for games. Nevertheless, this duality is called a twisted duality since there may be other extensions of the self-duality.

It is known that the von Neumann-Morgenstern solution does not satisfy twisted duality. In this note, in Section 3 we try to extend twisted duality and see that, in a sense, the von Neumann-Morgenstern solution satisfies the extended property. In Sections 2 and 4 we state a part of results in Kikuta (2002, 2005) .

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A cooperative game with transferable utility (abbreviated as a *game*) is an ordered pair (N, v), where N is a finite set of *players* and v, called the *characteristic function*, is a real-valued function on the power set of N, satisfying $v(\emptyset) = 0$. For simplicity of notation, we frequently express a game (N, v) as v. A *coalition* is a subset of N. For a finite set Z, $|Z|$ denotes the cardinality of Z. For a coalition S, R^S is the $|S|$-dimensional product space $R^{|S|}$ with coordinates indexed by players in S. The ith component of $x \in R^S$ is denoted by x_i. For $S \subseteq T \subseteq N$ and $x \in R^T$, $x|_S$ means the restriction of x to S. For $x \in R^N$ and $Y \subseteq R^N$, we let $x + Y \equiv \{x + y : y \in Y\}$ and $-Y \equiv \{-y : y \in Y\}$. We call $x \in R^N$ a *(payoff) vector*. For $S \subseteq N$ and $x \in R^N$, we define $x(S) = \sum_{i \in S} x_i$ (if $S \neq \emptyset$) and $= 0$ (if $S = \emptyset$). A game v is called *additive* if there exists $x \in R^N$ such that $v(S) = x(S)$ for all $S \subseteq N$. We identify sometimes a vector $x \in R^N$ with an additive game (N, v) in which $v(S) = x(S)$ for all $S \subseteq N$. A *pre-imputation* for a game v is a vector $x \in R^N$ that satisfies

$$x(N) = v(N). \tag{1.1}$$

$\mathcal{X}(v) \equiv \mathcal{X}(N, v)$ is the set of all pre-imputations for a game v. For games v, w and $k \in R$, define games $v + w$ and kv by $(v + w)(S) = v(S) + w(S)$ and $(kv)(S) = kv(S)$ for all $S \subseteq N$ respectively. By these two operations, the set of all games with the player set N can be identified as the real linear space with dimension $2^{|N|} - 1$. For a game v, define $a(v) \in R^N$ by[1] $a(v)_i = v(i)$ for all $i \in N$. The *dual* game v^* of a game v is defined by: For all $S \subseteq N$,

$$v^*(S) = v(N) - v(N \backslash S). \tag{1.2}$$

The *zero-normalization* of a game v, denoted by v^0, is defined by $v^0(S) = v(S) - \sum_{i \in S} v(i)$ for all $S \subseteq N$.

2 Twisted Duality

In this section we state that some solutions satisfy twisted duality.

Definition 1. Games (N, v) and (N, w) are called *twisted* (to each other) if the game $(N, v^* + w)$ is additive.

By Definition 1, it is easy to see that games (N, v) and (N, w) are twisted to each other if and only if $(v^* + w)^0(S) = 0$ for all $S \subseteq N$. So it is trivially clear that for any game (N, v), there exists a game (N, w) such that games (N, v) and (N, w) are twisted. The twisted games (N, v) and (N, w) are well-defined since $(N, v^* + w)$ is additive if and only if the game $(N, v + w^*)$ is additive (Remark 3.3 of Kikuta (2005)). Furthermore, $v^*(i) + w(i) = v(i) + w^*(i)$ for all $i \in N$ if $(N, v^* + w)$ is additive .

Let ϕ be a correspondence which associates with each game (N, v) a subset $\phi(v) \equiv \phi(N, v)$ of R^N. ϕ is called a *solution*. For a game v, when $\phi(v)$ consists of a unique vector, we identify $\phi(v)$ as the unique vector.

Definition 2. A solution ϕ satisfies the *twisted duality* if for any twisted games (N, v) and (N, w),

$$\phi(N, v) = a(v^* + w) - \phi(N, w).$$

[1] For simplicity we write $v(\{i\})$, $v(\{i, j\})$ etc. as $v(i), v(ij)$ etc.

For a game v and a coalition S, we let

$$M_i(v) \equiv M_i(N,v) \equiv max\{v(S \cup \{i\}) - v(S) : S \subseteq N \setminus \{i\}\} \tag{2.1}$$

and

$$m_i(v) \equiv m_i(N,v) \equiv min\{v(S \cup \{i\}) - v(S) : S \subseteq N \setminus \{i\}\}. \tag{2.2}$$

$M_i(v)$ and $m_i(v)$ are the largest and the smallest amounts that the player i can contribute to a coalition. For a game (N,v), the *reasonable set of outcomes*, denoted by $\mathcal{R}(N,v)$, is defined by adding a lower bound to the *reasonable set* defined by Milnor (1952). In other words, $\mathcal{R}(N,v)$ is the set of all pre-imputations that give each player an amount between the largest and the smallest amounts he can contribute to a coalition (See Gerard-Valet and Zamir (1987)).

$$\mathcal{R}(N,v) \equiv \{x \in \mathcal{X}(N,v) : m(v) \leq x \leq M(v)\}. \tag{2.3}$$

For a real number ϵ, the *(strong)* ϵ*-core* of a game (N,v), denoted $\mathcal{C}_\epsilon(N,v)$, is the set of all pre-imputations that give rise to excesses not greater than ϵ, for all coalitions other than $\emptyset$ and N:

$$\mathcal{C}_\epsilon(N,v) = \{x \in \mathcal{X}(N,v) : e_v(S,x) \leq \epsilon \text{ for all } S \neq \emptyset, N\}.$$

The strong 0-core of a game (N,v) is called the *core* of the game and written as $\mathcal{C}(N,v)$.

It is known that the Shapley value, the prenucleolus, the prekernel, the reasonable set of outcomes and the strong ϵ-core satisfy twisted duality (Kikuta, 2005)[2].

In the next section we try to extend twisted duality, in order to see what kind of relation the von Neumann-Morgenstern solution satisfies.

3 Partially Twisted Duality

In this section we extend twisted duality and discuss whether von Neumann Morgenstern solution satisfies the extended property or not.

For $x, y \in \mathcal{X}(N,v)$ and for a coalition $S \subset N$, we say x *dominates* y via S, if

$$x_i > y_i, \forall i \in S; \tag{3.1}$$

$$x(S) \leq v(S). \tag{3.2}$$

For $x, y \in \mathcal{X}(N,v)$, we say x *dominates* y, written as $x \succ_v y$, if there exists $S \subset N$ such that x dominates y via S. For a game (N,v) and for $K \subset \mathcal{X}(N,v)$, we define $Dom_v K$ by

$$Dom_v K \equiv \{x \in \mathcal{X}(N,v) : \exists y \in K \quad s.t. \quad y \succ_v x\}.$$

For a game (N,v) and for $K \subset \mathcal{X}(N,v)$,we say K is a *stable* set (or a von Nuemann-Morgenstern solution) whenever

$$K \cap Dom_v K = \emptyset \text{ and } K \cup Dom_v K = \mathcal{X}(N,v). \tag{3.3}$$

We say a game (N,v) has a *stable* core if the core is nonempty and it is a stable set. If the core is stable, it is the unique stable set. The next example says that $\mathcal{C}(N,v)$ is not empty

[2] We give definitions of the Shapley value and the prenucleolus in Section 4. We omit the definition of the prekernel. See Peleg and Sudhölter (2003).

and it is not stable, while $C(N,w)$ is not empty and it is stable, which means that the von Neumann-Morgenstern solution does not satisfy twisted duality .

Example 1. (Example 8.7 of Kikuta (2005)) Let $n=3, v(1)=v(2)=v(3)=0, v(12)=20, v(13)=10, v(23)=50$, and $v(123)=60$. By definition we can take (N,w) so that $w(1)=w(2)=w(3)=0, w(12)=0, w(13)=-10, w(23)=30$, and $w(123)=40$. The core of (N,w) is stable, while the core of (N,v) is not, as is seen by drawing the figures of the cores.

Definition 3. A pair of two solutions ϕ^1 and ϕ^2 satisfy *partially* twisted duality if for any twisted games (N,v) and (N,w),

$$\phi^1(N,v)=a(v^*+w)-\phi^2(N,w).$$

It is easy to see that a solution ϕ satisfies twisted duality if and only if two solutions ϕ^1 and ϕ^2 satisfy partially twisted duality when $\phi^1=\phi^2=\phi$. For a game (N,v), let

$$\underline{\mathcal{I}}(N,v)\equiv\{x\in\mathcal{X}(v): x_i\geq v(i), \forall i\in N\},$$

and

$$\overline{\mathcal{I}}(N,v)\equiv\{x\in\mathcal{X}(v): x_i\leq v^*(i), \forall i\in N\}.$$

It is well-known that the set $\underline{\mathcal{I}}(N,v)$ is called the imputation set.

Proposition 1. Two solutions $\underline{\mathcal{I}}$ and $\overline{\mathcal{I}}$ satisfy partially twisted duality.
Proof: For $x\in\overline{\mathcal{I}}(N,w)$, let $y=a(v)+a(w^*)-x$. Then

$$y(N)=\sum_{i\in N}v(i)+\sum_{i\in N}w^*(i)-w(N)=v(N),$$

and for every $i\in N$

$$y_i=v(i)+w^*(i)-x_i\geq v(i)+w^*(i)-w^*(i)=v(i).$$

Hence $y\in\underline{\mathcal{I}}(N,v)$. The converse is in the same way. ∎

Example 2. Let

$$v(i)=0, v(12)=4, v(13)=4, v(23)=5, v(123)=6,$$

and

$$w(i)=0, w(12)=-3, w(13)=-3, w(23)=-2, w(123)=-1.$$

Games v and w are twisted dual. By definition,

$$M_1(v)=4, M_2(v)=M_3(v)=5, m_i(v)=0, i=1,2,3.$$

$$M_1(w)=1, M_2(w)=M_3(w)=2, m_i(v)=-3, i=1,2,3.$$

It is easy to see that $\underline{\mathcal{I}}(N,v)$ is the convex hull of points $(6,0,0),(0,6,0)$ and $(0,0,6)$, while $\overline{\mathcal{I}}(N,w)$ is the convex hull of points $(-5,2,2),(1,-4,2)$ and $(1,2,-4)$. It is easy to see

$$\underline{\mathcal{I}}(N,v)=a(v^*)-\overline{\mathcal{I}}(N,w).$$

On the other hand, $\underline{\mathcal{I}}(N,w)=\emptyset$ and $\overline{\mathcal{I}}(N,v)=\emptyset$.

Definition 4. For $x, y \in \mathcal{X}(N,v)$ and for a coalition $S \subset N$, we say x d-dominates y via S, if

$$x_i < y_i, \forall i \in S; \tag{3.4}$$

$$x(N \setminus S) \leq v(N \setminus S). \tag{3.5}$$

For $x, y \in \mathcal{X}(N,v)$, we say x d-dominates[3] y, written as $x \succ_v^d y$, if there exists $S \subset N$ such that x d-dominates y via S. We let

$$Dom_v^d K \equiv \{x \in \mathcal{X}(N,v) : \exists y \in K \quad s.t. \quad y \succ_v^d x\}.$$

Definition 5. For a game (N,v) and for $K \subset \mathcal{X}(N,v)$,we say K is a d-stable set whenever

$$K \cap Dom_v^d K = \emptyset \text{ and } K \cup Dom_v^d K = \mathcal{X}(N,v).$$

In general a game has many stable sets and d-stable sets. The next proposition gives a correspondence between stable sets and d-stable sets for twisted dual games. We omit the proof of it since it is proved straightforward.

Proposition 2. Assume K is a stable set for a game (N,v). Let games v and w be twisted dual. Then $a(v) + a(w^*) - K$ is a d-stable set for (N,w).

For a game v, let $\Phi(v)$ and $\Psi(v)$ be the set of all stable sets and the set of all d-stable sets respectively. Let games v and w be twisted dual. From Proposition 2, we have a relation between $\Phi(v)$ and $\Psi(w)$, which is similar to partially twisted duality :

$$\Psi(w) = \{a(v) + a(w^*) - K : K \in \Phi(v)\}.$$

4 Axiomatization

In this section we discuss axiomatic characterizations of the Shapley value and the prenucleolus. given in Kikuta (2002)[4].

4.1 Shapley value

Let ϕ be a single-valued solution and (N,v) be a game. For $T \subseteq N$, Hart and Mas-Colell (1989) defined the *reduced game* (T, v_T^ϕ) by :

$$v_T^\phi(S) = v(S \cup T^c) - \sum_{i \in T^c} \phi_i(S \cup T^c, v), \text{ for all } S \subseteq T,$$

where $T^c = N \setminus T$. A solution ϕ is called *consistent* if, for every game (N,v) and every $T \subseteq N$, we have

$$\phi_j(T, v_T^\phi) = \phi_j(N,v), \text{ for all } j \in T.$$

A solution ϕ is called *standard for two-person games* if for all games $(\{i,j\},v), i \neq j$, it satisfies $|\phi(\{i,j\},v)| = 1$ and $\phi_k(\{i,j\},v) = v(k) + \frac{1}{2}\{v(ij) - v(i) - v(j)\}$, for $k = i,j$. The Shapley value (Shapley, 1953) is a single-valued solution ϕ defined by: For $i \in N$,

$$\phi_i(v) \equiv \phi(v)_i = \sum_{S: i \in S \subseteq N} \frac{(|N| - |S|)!(|S|-1)!}{|N|!} \{v(S) - v(S \setminus \{i\})\}. \tag{4.1}$$

[3] We omit an interpretation of d-domination.

[4] Kikuta (2005) characterized both solutions by using axioms which are different from those here.

Hart and Mas-Colell (1989) showed that assuming a solution ϕ is single-valued, ϕ is the Shapley value if and only if (i) ϕ is consistent, and (ii) ϕ is standard for two-person games. Now, we have

Proposition 3. Let ϕ be a single-valued solution. Then ϕ is the Shapley value if and only if (i) ϕ is consistent, and (ii) ϕ satisfies twisted duality.

Proof: The Shapley value satisfies the consistency by Proposition 4.5 of (Hart and Mas-Colell, 1989). It satisfies twisted duality. Conversely, if a solution ϕ satisfies twisted duality, it is standard for two-person games by Proposition 4 of (Kikuta, 2002). So by Theorem 5 of (Hart and Mas-Colell, 1989), ϕ is the Shapley value. ∎

We proceed by providing some properties of a solution ϕ.

Weakly positive : If $v(S) - v(S \setminus \{i\}) \geq 0$ for all S such that $i \in S$, then $\phi_i(v) \geq 0$.

Feasibility : $\sum_{i \in N} \phi_i(v) \leq v(N)$.

Superadditivity :For any games v, w, $\phi_i(v + w) \geq \phi_i(v) + \phi_i(w)$, for all $i \in N$.

Let $\pi : N \to N$ be a permutation of N. For all $S \subseteq N$, $\pi v(S) \equiv v(\pi(S))$, where $\pi(S) \equiv \{\pi(i) | i \in S\}$.

Symmetry : For all $i \in N$ and for all permutations π of N, $\phi_{\pi(i)}(\pi v) = \phi_i(v)$.

Efficiency : $\sum_{i \in N} \phi_i(v) = v(N)$.

If $v(S) - v(S \setminus \{i\}) = 0$ for all S such that $i \in S$, $i \in N$ is called a *dummy* player.

Dummy : If i is a dummy player, $\phi_i(v) = 0$.

Additivity :For any games v, w, $\phi(v + w) = \phi(v) + \phi(w)$.

Duality :For any game v, $\phi(v) = \phi(v^*)$.

Proposition 4. A solution ϕ satisfies single-valuedness, feasibility, weak positivity, superadditivity, symmetry and twisted duality, if and only if it is the Shapley value.

Proof: The Shapley value satisfies all properties in Proposition 4. Conversely, twisted duality and superadditivity imply additivity. Twisted duality and weak positivity imply *Dummy*. Twisted duality and feasibility imply efficiency. Hence by (Shapley, 1953), ϕ must be the Shapley value. ∎

4.2 Prenucleolus

Given a game (N, v), for all $S \subseteq N$ and $x \in \mathcal{X}(v)$, define $e_v(S, x) \equiv v(S) - x(S)$. $e_v(S, x)$ is called the excess of S at x. It represents the gain (or loss, if it is negative) to S if its members depart from an agreement that yields x in order to form their own coalition (Maschler, 1992). For $x \in \mathcal{X}(v)$ let $\theta(x)$ be the 2^n-vector whose components are the numbers $e_v(S, x), S \subseteq N$, arranged in nonincreasing order, i.e., $\theta(x)_i \geq \theta(x)_j$ whenever $1 \leq i \leq j \leq 2^n$. We say that $\theta(x)$ is lexicographically smaller than $\theta(y)$, denoted $\theta(x) \prec_L \theta(y)$, if and only if there is an index k such that $\theta(x)_i = \theta(y)_i$ for all $i < k$, and $\theta(x)_k < \theta(y)_k$. We write $\theta(x) \preceq_L \theta(y)$ for not $\theta(y) \prec_L \theta(x)$. The *prenucleolus* of v is the set $\mathcal{PN}(v)$ of pre-imputations that minimize θ in the lexicographical ordering, i.e.,

$$\mathcal{PN}(v) \equiv \{x \in \mathcal{X}(v) : \theta(x) \preceq_L \theta(y) \text{ for all } y \in \mathcal{X}(v)\}.$$

Let ϕ be a solution and (N, v) be a game. For $T \subseteq N$ and $x \in R^N$, define the *reduced game* (T, v^x) by:

$$v^x(S) = max\{v(S \cup Q) - x(Q) : Q \subseteq N \setminus T\}, \text{ for all } S \subset T,$$
$$v^x(T) = v(N) - x(T), \text{ and } v^x(\emptyset) = 0.$$

ϕ is said to have the reduced game property if, for a game (N, v), $\emptyset \neq T \subset N$,and $x \in \phi(N, v)$, we have $x|T \in \phi(T, v^x)$. Here we provide three properties of a solution function ϕ.

Covariance under strategic equivalence :if for a game v, $k > 0$ and $x \in R^N$, $\phi(kv+x) = k\phi(v)+x$.

The players $i, j \in N, i \neq j$, are *substitutes* in a game v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

Equal treatment property: For any substitutes $i, j \in N, i \neq j$ in a game v, $\phi_i(v) = \phi_j(v)$.

Positive homogeneity: For any $k > 0$ and any game v. $\phi(kv) = k\phi(v)$.

Proposition 5. Assume a solution ϕ is single-valued. It satisfies positive homogeneity, twisted duality and reduced game property if and only if it is the prenucleolus.

Proof: It is well-known that the prenucleolus is single-valued and it satisfies reduced game property. It satisfies twisted duality. It is covariant under strategic equivalence. Hence it is positively homogeneous. Conversely, assume a solution ϕ is single-valued, positively homogeneous, satisfies twisted duality and reduced game property. Then twisted duality, combined with positive homogeneity, implies that ϕ is covariant under strategic invariance. Assume i and j are substitutes. Let $T = \{i, j\}$. Let $x = \phi(N, v)$. We have $v^x_T(\{i\}) = v^x_T(\{j\})$ for all $Q \subseteq T^c$ since i and j are substitutes. Hence $\phi_i(T, v^x_T) = \phi_j(T, v^x_T)$, which implies $x_i = x_j = v^x_T(\{i, j\})/2$ by reduced game property. Hence ϕ has equal treatment property. Then ϕ must be the prenucleolus, by Theorem 3.2 of (Orshan, 1993) (The prenucleolus is the unique solution concept ϕ which satisfies single-valuedness, covariance under strategic equivalence, equal treatment property and reduced game property.). ∎

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