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# On the Computation of Cores of Games with Punishment-Dominance Relations

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**Abstract**

In this paper, we introduce the main result of Masuzawa (2005).

## 1 Introduction

Games with punishment-dominance relations are strategic form games introduced by Masuzawa (2003). The voluntary contribution game for production of a public good, the $n$-person prisoners' dilemma game, Cournot's oligopoly model of quantity competition, are all in this class. Masuzawa (2003) has shown that any game in this class yields a balanced and ordinally convex $\alpha$-coalitional game, and that the $\alpha$-core is, therefore, nonempty.

The aim of this paper is to introduce the result of Masuzawa (2005), which has shown an efficient algorithm to find a payoff vector in the $\alpha$-core and the associated $\alpha$-core strategy for games with punishment-dominance relations.

First, I mention a reason why the efficient computation of a solution concept is needed. There are various kinds of purposes of solution concept in economics:

1. To prescribe agreements for conflict resolution or for the normative purposes such as the equality.

2. To evaluate economic situations.

3. To describe the behavior of economic agents.

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It is obvious that the efficient computation of solution is in need if one uses a solution for prescription or evaluation. Even for describing the behavior of economic agent, it is also important. The reason for this involves the players' ability to choose a strategy described by the solution. If the solution is too complex for players to find it, it is unconvincing that this solution describes the players' behavior.

The computational difficulties arise especially in $n$-person games because the number of combinations of strategies increases exponentially as the population of players increases. This problem arises not only in cooperative theory but also in non-cooperative theory. However, it is more critical for cooperative theory because, in $n$-person non-cooperative theory, it is easy to check whether a given strategy profile is a Nash equilibrium or not.

Usually a solution concept is defined for the general class of game. However, for the general class of games, it is difficult, or sometimes impossible, to obtain a method to find a non-trivial solution of a game in the class. In other words, if one has little amount of information on a game in advance, it is very difficult to find a solution of the game. Therefore, it is important to specify the information such that, if it is known in advance, it is possible to compute the solution efficiently. Indeed, some game theorists have focused on the computation of solutions of special classes of games.

On the one hand, some negative results are obtained : some solution concepts on some classes of games have been proved to be in the class of difficult problems. However, on the other hand, efficient algorithms are offered for other classes of games. They terminate within polynomial time with respect to the number of players or the length of "data" that characterize instances. (See, Bilbao (2000, chapter 4) for a survey of computational complexity of solution concepts on classes of TU games.)

In Masuzawa (2005), it has been shown that the $\alpha$-core of our class of games also overcomes such computational difficulties.

## 2 The $\alpha$-core

We introduce notations and definitions. We refer to the set of real numbers by $\Re$. A *strategic form game* is a list $G := (N, (X^i)_{i\in N}, (u^i)_{i\in N})$ such that $N := \{1, 2, \dots n\}$ is a nonempty finite set of players, and, for all $i \in N$, $X^i$ is a nonempty set of strategies of $i \in N$, $u^i : \prod_{i\in N} X^i \to \Re$ is a payoff function of $i \in N$. A subset of $N$ is called a *coalition*. For all $S \subset N$ ($S \neq \emptyset$), we refer to the Cartesian product of $X^i$ in $S$ by $X^S$. Typical elements of $X^S$ are denoted by $x^S, y^S, z^S, \dots$. Sometimes, we refer to the restrictions of

$x^S, y^S, z^S, \ldots$ to $X^T$ by $x^T, y^T, z^T, \ldots$ respectively, where $T \subset S$. Typical elements of $\Re^S$ are denoted by $a^S, b^S, \ldots$, and called *payoff vectors* of $S$, the restriction into $\Re^T$ of which are sometimes denoted by $a^T, b^T, \ldots$ respectively for all $T \subset S$. We write that, for all $a^S, b^S \in \Re^S$, $a^S \geq b^S$ if and only if, for all $i \in S$, $a^i \geq b^i$, and $a^S \gg b^S$ if and only if, for all $i \in S$, $a^i > b^i$. By $u^S$, we refer to a vector valued function defined by $u^S(x^N) := (u^i(x^N))_{i \in S}$ for all $S \subset N$.

The notion of the $\alpha$-core can be stated in terms of the coalitional game. The *$\alpha$-coalitional game associated with* $G$, $V_\alpha : 2^N \setminus \{\emptyset\} \twoheadrightarrow \Re^N$, is defined by

$$V_\alpha(S) := \begin{cases} \bigcup_{x^S \in X^S.} \bigcap_{z^{N \setminus S} \in X^{N \setminus S}} \{b^N \in \Re^N : b^S \leq u^S(x^S, z^{N \setminus S})\} & \text{if } S \neq N, \\ \bigcup_{x^S \in X^S.} \{b^N \in \Re^N : b^S \leq u^S(x^S)\} & \text{otherwise.} \end{cases}$$

We say that $S \subset N$ *improves upon* $a^N \in \Re^N$ *via* $b^N \in \Re^N$ if and only if $b^N \in V_\alpha(S)$, and $b^S \gg a^S$. A payoff vector $a^N \in \Re^N$ is in the *$\alpha$-core of* $G$ if (i) $a^N \in V_\alpha(N)$ and (ii) there exists no $S \neq \emptyset$ which improves upon $a^N$.

## 3 Games with Punishment-Dominance Relations

A game with punishment-dominance relation is a strategic form game $(X^i, u^i)_{i \in N}$ that satisfies the following axiom:

**PD**:
For all $i \in N$ and all $x^i, y^i \in X^i$, one of the followings holds:

$$\begin{array}{ll} (1) \forall j \in S \forall z^S \in X^S & u^j(y^i, x^S) \geq u^j(y^i, x^S), \\ (2) \forall j \in S \forall z^S \in X^S & u^j(x^i, x^S) \geq u^j(x^i, x^S), \end{array}$$

where $S := N \setminus \{i\}$.

We say that $x^i$ is punishment-dominant over $y^i$ and write $P(x^i, y^i)$ if (1) holds.

The case when a game is finite is our present concern:

**F**(finiteness of strategy space)
For all $i \in N$, $X^i$ is finite.

While economic situations are represented usually and conventionally by games with infinite strategy sets, they can also be represented as finite games by taking the minimum unit of a scale into the consideration. For example, in case of an oligopoly game of Cournot's with bounded capacity, which satisfies **PD**, the minimum unit of a scale of the product makes the game finite.

We assume that strategic form game $G$ is given by list $(X^i)_{i \in N}$, an algorithm "oracle" which (1), for all $i \in N$ and all $x^N \in X^N$, returns the value of $u^i(x^N)$. We count not only an elementary operation but also a call to the oracle for the value of $u^i(x^N)$ as one step for the evaluation of the time complexity of an algorithm. For example, a payoff comparison is counted as $O(1)$. Even if the cost of the payoff comparison depends on $n$, the analysis of the time complexity may be useful unless the cost is not so rapidly increasing in $|N|$.

We also assume other oracles. One of them, for all $i \in N$ and all $x^i \in X^i$, returns the predecessor of $x^i$ with respect to $\hat{P}^i$, and the other returns the successor with respect to $\hat{P}^i$, where $\hat{P}^i$ is a liner order on $X^i$ such that $\hat{P}^i(y^i, z^i)$ only if $P(y^i, z^i)$. In some economic situation, this condition is plausible. For example, in a public good provision game discuss in Masuzawa (2003), for a contribution level of the private good for public good provision, the next punishment-dominant strategy is the next higher level of the private good.

Let $\mathcal{PF}$ be the class of strategic form game that satisfy **PD** and **F**. Then, we can state the main result of Masuzawa (2005).

**Theorem 1**
(1) For all $G = (X^i, u^i) \in \mathcal{PF}$ and all $a^N \in \Re^N$, the algorithm defined in Masuzawa (2005) finds a vector $b^N$ in the $\alpha$-core, and a strategy profile $x^N \in X^N$ such that $u^N(x^N) \geq b^N$.
(2) The algorithm also finds coalition $S \subset N$ such that

$$S \neq \emptyset,\ b^N \in V_\alpha(S) \text{ and } b^S \gg a^S \text{ if } a^S \text{ is not in the } \alpha\text{-core},$$

$$S = \emptyset \text{ and } a^N = b^N \text{ if } a^N \text{ is in the } \alpha\text{-core}.$$

(3) The algorithm terminates within $O(|N|^3 \cdot \max\{|X^i| : i \in N\})$ times elementary operations and oracle calls.

Note that the result of (1) and (3) is not trivial while the $\alpha$-core of any strategic game in this class includes all marginal worth vector. The reasons for this are twofold. First, we will focus on the core of an NTU-game, the structure of which is more complicated than that of a TU-coalitional game.

Second, we assume that the initial data is given not by coalitional games but by strategic form games and we will find not only payoffs but also the associated strategies.

Furthermore, the marginal worth vector does not have coalition $S$ which satisfies (2). Note that the result of (2) is related to v-NM stable set.

# 4 Concluding Remarks

The algorithm obtained is useful if the utility functions can be described by simple functions, can be inputted into the memory of a calculator, and can be computed efficiently. Indeed, Masuzawa (2005) provided a asymmetric numerical example of 20-person game and found an element in the $\alpha$-core.

On the other hand, in real situations, the evaluation of the utility may costs much time because the utility function is not necessarily described in simple form. Thus, the time complexity $O(|N|^3 \cdot \max\{|X^i| : i \in N\})$ may be not small enough to compute the $\alpha$-core of a game in real situation.

# Reference

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