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Rational Expectations and the Modigliani-Miller Theorem

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1. Introduction

In this paper a general model of dynamic economy is presented and the equilibrium of rational expectations for the economy is defined. In the model we re-examine the Modigliani-Miller theorem which asserts that the value of a firm is independent of the debt-equity ratio. We show in the context of a dynamic general equilibrium model that the M-M theorem holds in a much more general framework. The validity of the theorem depends heavily on the rationality of individuals' expectations.

In the proof of the M-M theorem, it was usually assumed that the gross returns of a firm depend only on the state of the world, since it was based on static equilibrium rather than dynamic analysis. In the dynamic economy, the profits of firms are determined depending on the behaviors of all economic agents, especially their expectations.

In this paper, we show that the M-M results are still valid in a dynamic equilibrium of rational expectations. Also, in a simplified model of the economy where all the consumers are identical, we prove the existence of rational expectations equilibrium. Our model is a generalization of the asset pricing model which was presented by Lucas.

2. A General Model

In this section, we consider a general model of economy, where there are infinitely many consumers and finitely many firms. The set of consumers is denoted by an atomless measure space $(A, \mathscr{A}, \nu)$, where A is the set of all consumers, $\mathscr{A}$ is a σ-field of some subsets of A, and ν is a measure defined on $\mathscr{A}$ such that $\nu(A)=1$. On the other hand, we assume that there are finitely many firms j, and the number of firms is J.

In the economy there are n kinds of commodities and the commodity space is denoted by an n-dimensional Euclidian space R^n. All kinds of commodities can be used as consumption goods as well as capital goods.

The consumption set of each consumer is the non-negative orthant of the commodity space, which is denoted by R^n_+. The family of possible utility functions of consumers is denoted by a set, $\mathscr{U}$. The utility function of each consumer is uncertain, but it is an element of $\mathscr{U}$. Family $\mathscr{U}$ is a set of some real-valued functions define on R^n_+ and is endowed with the topology of uniform convergence. The family of possible production sets of firms is denoted by a set $\mathscr{Y}$. The production set of each firm is also uncertain, but it is an element of $\mathscr{Y}$. Family $\mathscr{Y}$ is a set of some subsets of R^n and is endowed with the topology of closed convergence.

The uncertainty in the economy can be described by a stochastic process. We assume that the space of time is discrete and is denoted by a set $T=\{0, 1, 2, \cdots\}$. Let $(\Omega, \mathscr{F}, P)$ be a probability space, i.e., Ω is the set of all the states of nature, $\mathscr{F}$ is the set of all possible events which is a σ-field of some subsets of Ω, and P is a probability measure which is an additive function from $\mathscr{F}$ to $[0, 1]$. The uncertainty in consumers' utility functions and firms' production sets is described by a stochastic process $\{\mathscr{E}_t \mid t \in T\}$ defined on $(\Omega, \mathscr{F}, P)$. At each period t in time, $\mathscr{E}_t$ is a measurable map denoted by

$$\omega \in \Omega \rightarrow (U, Y) \in \mathscr{U}^A \times \mathscr{Y}^J,$$

where $\mathscr{U}^A$ is the set of all the measurable maps from A to $\mathscr{U}$ and $\mathscr{Y}^J$ is a J-time product, of $\mathscr{Y}$. When a state ω of nature is realized at period t, consumers' utility functions and firms' production sets are denoted by $\mathscr{E}_t(\omega)$, say (U, Y). Then, U is a map from A to $\mathscr{U}$, and the value $U(a)$ is the utility function of consumer $a \in A$. Y is an element of set $\mathscr{Y}^J$, and the j-th coordinate Y_j is the production set of the j-th firm. We assume that consumers' utility functions and firms' production sets at each period can be known at the beginning of the period.

Suppose that a state $\omega \in \Omega$ is realized and the utility that a consumer gets at period t is u_t. Let δ be the discount rate of utility, where $0 < \delta < 1$. The sum of discounted utilities the consumer gets is $\sum_{t=0}^{\infty} \delta^t u_t$. However, the consumer can not know the levels of utilities that he will obtain in the future. Therefore, the consumer will guess future utilities and behave to maximize the sum of expected utilities. On the other hand, firms are able to know their production sets at the beginning of each period and production takes place in one period. Therefore, there is no uncertainty for firms and they simply maximize their profits at each period in time.

The process $\{\mathscr{E}_t \mid t \in T\}$ describes a transition of the uncertainty in the economy. We assume that it is a Markov process. Let $S=\mathscr{U}^A \times \mathscr{Y}^J$ and $\mathscr{B}(S)$ be the set of all Borel subsets of S. We denote, by $\mathscr{M}(S)$, the set of all measures defined on $\mathscr{B}(S)$, which is endowed with the weak topology.

Assumption 2.1: There exists a continuous map from S to $\mathscr{M}(S)$,

$$s \in S \quad \rightarrow \mu_s \in \mathscr{M}(S),$$

which has the following properties: For each $s \in S$, μ_s is a transition probability on S, i.e., for each $t \in T$,

$$\mu_s(B) = \text{Prob.}\{\mathscr{E}_{t+1} \in B \mid \mathscr{E}_t = s\} \qquad \text{for all } B \in \mathscr{B}(S).$$

More precisely, for each $t \in T$ and $s \in S$,

$$\int_A \mu_s(B) d(P \cdot \mathscr{E}_t^{-1})(s) = P(\mathscr{E}_{t+1}^{-1}(B) \cap \mathscr{E}_t^{-1}(A)) \qquad \text{for all } A,\ B \in \mathscr{B}(S).$$

The existence of such a transition probability means that the uncertainty at each period does not depend on time, but only on the state at the previous period in time. Therefore, if $s=(U, Y) \in S$ is realized at period t, then the uncertainty in the economy at the periods after period t depends only on $s=(U, Y)$. The transition of uncertainty is the same at all periods in time, and in this sense the economy is stationary.

Because of the stationarity, when we focus on the economy at one period, the period itself does not matter. Thus, we do not have to show explicitly suffix t of time in the arguments.

3. The Definition of Equilibrium

Let $\mathscr{L}_1^J$ be the space of all integrable functions from A to R^J. We use a function in $\mathscr{L}_1^J$ to describe the shares in firms owned by consumers. Let $\theta=(\theta_1, \theta_2, \cdots, \theta_J) \in \mathscr{L}_1^J$. Then, $\theta_j(a)$ denotes the share of the j-th firm' equity owned by consumer a.

Let $\mathscr{L}_1$ be the space of all integrable functions from A to R. We use a function in $\mathscr{L}_1$ to describe the number of bonds owned by consumers. Let $\beta \in \mathscr{L}_1$. Then, $\beta(a)$ denotes the number of bonds owned by consumer a. We denote the numbers of bonds that firms issue by a vector $D=(D_1, D_2, \cdots, D_J) \in R^J$, where D_j is the number of bonds that firm j issues.

The equilibrium of the economy is defined by a pair $\{\psi, V\}$ that describes consumers' expectations, where ψ *is* a correspondence from $\mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S$ to $R^n \times R^J \times R$ and V is a function from $A \times R_+^n \times R_+^J \times R \times \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S$ to R, where $S = \mathscr{U}^A \times \mathscr{Y}^J$. The correspondence ψ: $\mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S \rightarrow R^n \times R^J \times R$ is called *a price correspondence* and is depicted in the following notation.

$$(k, \theta, \beta, D; s) \in \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S \rightarrow \psi(k, \theta, \beta, D; s) \subset R^n \times R^J \times R.$$

To denote an element of set $\psi(k, \theta, \beta, D; s)$, we use a vector $(p, q, r) \in R_+^n \times R_+^J \times R$.

The function $V: A \times R_+^n \times R_+^J \times R \times \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S \rightarrow R$ is called *a value function*" and is depicted in the following notation.

$$(a, z, e, b; k, \theta, \beta, D; s) \in A \times R_+^n \times R_+^J \times R \times \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S$$
$$\to V_a(z, e, b; k, \theta, \beta, D; s) \in R.$$

Definition 3.1: A pair $\{\psi, V\}$ of a price correspondence and a value function is *an equilibrium of the economy*, if $\{\psi, V\}$ has the following property:

Let $(k, \theta, \beta, D, s) \in \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J \times S$, and $(p, q, r) \in R^n \times R^J \times R$ such that

$$\int_A \beta \, dv = \sum_{j=1}^J D_j, \quad \int_A \theta \, dv = \underline{1}, \quad \text{and} \quad (p, q, r) \in \psi(k, \theta, \beta, D; s),$$

where $s=(U, Y) \in S$ and $\underline{1}=(1, 1, \cdots, 1) \in R^J$. Then, there exist $\hat{c} \in \mathscr{L}_{\infty+}^n$, $(\hat{k}, \hat{\theta}, \hat{\beta}, \hat{D}) \in \mathscr{L}_{\infty+}^n \times \mathscr{L}_{1+}^J \times \mathscr{L}_1 \times R^J$, and $\hat{y}_j \in Y_j$ such that the following conditions are satisfied.

(1) Firms are maximizing their profits, i.e., for each $j=1, 2, \cdots, J$,

$$p \cdot \hat{y}_j \geqq p \cdot y_j \quad \text{for all } y_j \in Y_j.$$

(2) Consumers are maximizing their expected utilities subject to their budget constraints, i.e., for almost all $a \in A$,

$$p \cdot (\hat{c}(a) + \hat{k}(a)) + q \cdot \hat{\theta}(a) + \hat{\beta}(a) \leqq p \cdot k(a) + q \cdot \theta(a) + (1+r)\beta(a) + \sum_{j=1}^J \theta_j(a)(p \cdot \hat{y}_j - rD_j)$$

and

$$V_a(k(a), \theta(a), \beta(a); k, \theta, \beta, D; s) = U_a(\hat{c}(a)) + \delta \int_S V_a(\hat{k}(a), \hat{\theta}(a), \hat{\beta}(a); \hat{k}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot) d\mu_s$$

$$\geqq U_a(x) + \delta \int_S V_a(z, e, b; \hat{k}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot) d\mu_s$$

for all $(x, z, e, b) \in R_+^n \times R_+^n \times R_+^J \times R$ with

$$p \cdot (x+z) + q \cdot e + b \leqq p \cdot k(a) + q \cdot \theta(a) + (1+r)\beta(a) + \sum_{j=1}^J \theta_j(a)(p \cdot \hat{y}_j - rD_j).$$

(3) All the markets are in equilibrium, i.e.,

$$\int_A \hat{c} \, dv + \int_A \hat{k} \, dv = \int_A k \, dv + \sum_{j=1}^J \hat{y}_j,$$

$$\int_A \hat{\beta} \, dv = \sum_{j=1}^J \hat{D}_j,$$

$$\int_A \hat{\theta} \, dv = \underline{1}.$$

If consumers' expectations are rational, the value function V satisfies the following conditions.

Condition 1: For each $a \in A$,

$$V_a(z, e, \lambda b; k, \theta, \lambda\beta, \lambda D; s) = V_a(z, e, b; k, \theta, \beta, D; s)$$

for all $(k, \theta, \beta, D; s) \in \mathscr{L}^n_{\infty+} \times \mathscr{L}^J_{1+} \times \mathscr{L}_1 \times R^J \times S$, $(z, e, b) \in R^n_+ \times R^J_+ \times R$, and $\lambda > 0$.

Condition 2: For each $a \in A$, if $(z, e, b) \in R^n_+ \times R^J_+ \times R$ and $(k, \theta, \beta, D; s) \in \mathscr{L}^n_{\infty+} \times \mathscr{L}^J_{1+} \times \mathscr{L}_1 \times R^J \times S$, then

$$V_a(z, e, b + e\cdot\Delta D; k, \theta, \beta + \theta\cdot\Delta D, D + \Delta D; s) = V_a(z, e, b; k, \theta, \beta, D; s)$$

for all $\Delta D \in R^J$.

Theorem 2.1 (Homogeneity): Let $\{\psi, V\}$ be an equilibrium of the economy. Assume that the value function V satisfies Condition 1. If $(p, q, r) \in \psi(k, \theta, \beta, D; s)$, then

$$(\lambda p, \lambda q, r) \in \psi(k, \theta, \lambda\beta, \lambda D; s) \qquad \text{for all } \lambda > 0.$$

Proof: Omitted. ■

Theorem 2.2 (the M-M Theorem): Let $\{\psi, V\}$ be an equilibrium of the economy. Assume that the value function V satisfies Condition 2. If $(p, q, r) \in \psi(k, \theta, \beta, D; s)$, then

$$(p, q - \Delta D, r) \in \psi(K, \theta, \beta + \theta\cdot\Delta D, D + \Delta D; s) \qquad \text{for any } \Delta D \in R^J_+.$$

Proof: By Condition 2, (1) in Definition 3.1 can be rewritten in the following.

$$p\cdot(\hat{c}(a) + \hat{k}(a)) + (q - \Delta D)\cdot\hat{\theta}(a) + \hat{\beta}(a) + \hat{\theta}(a)\cdot\Delta D$$

$$\leqq p\cdot k(a) + (q - \Delta D)\cdot\theta(a) + (1+r)(\beta(a) + \theta(a)\cdot\Delta D) + \sum_{j=1}^{J}\theta_j(a)(p\cdot y_j - r(D_j + \Delta D_j))$$

and

$$V_a(k(a), \theta(a), \beta(a) + \theta(a)\cdot\Delta D; k, \theta, \beta + \theta\cdot\Delta D, D + \Delta D; s)$$

$$= U_a(\hat{c}(a)) + \delta\int_S V_a(\hat{k}(a), \hat{\theta}(a), \hat{\beta}(a) + \hat{\theta}(a)\cdot\Delta D; \hat{k}, \hat{\theta}, \hat{\beta} + \hat{\theta}\cdot\Delta D, \hat{D} + \Delta D; \cdot)d\mu_s$$

$$\geqq U_a(x) + \delta\int_S V_a(z, e, b + e\cdot\Delta D; \hat{k}, \hat{\theta}, \hat{\beta} + \hat{\theta}\cdot\Delta D, \hat{D} + \Delta D; \cdot)d\mu_s$$

for all $(x, z, e, b) \in R^n_+ \times R^n_+ \times R^J_+ \times R$ with

$$p\cdot(x + z) + (q - \Delta D)\cdot e + (b + e\cdot\Delta D)$$

$$\leqq p\cdot k(a) + (q - \Delta D)\cdot\theta(a) + (1+r)(\beta(a) + \theta(a)\cdot\Delta D)) + \sum_{j=1}^{J}\theta_j(a)(p\cdot y_j - r(D_j + \Delta D)).$$

Also, obviously we have

$$\int_A (\hat{\beta}+\theta\cdot\Delta D)d\nu=\sum_{j=1}^{J}(\hat{D}_j+\Delta D_j).$$

This implies that $(p, q-\Delta D, r)\in\psi(k, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s)$. ■

Let $\{\psi, V\}$ be an equilibrium of the economy. If $(p, q, r)\in\psi(k, \theta, \beta, D; s)$, the value of firms are defined by

$$v=q+D.$$

Therefore, Theorem 2.2 implies that the price of firms' equities becomes $q-\Delta D$ if the amount of firms' debts change by ΔD. The value of firms after the change of D is

$$(q-\Delta D)+(D+\Delta D)=q+D=v.$$

Thus, the value of firms is unchanged and independent of amount D of firms' debts.
In addition, the price p of commodities and the interest rate r remain constant. Moreover, since

$$V_a(k(a), \theta(a), \beta(a)+\theta(a)\cdot\Delta D; k, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s)$$
$$=V_a(k(a), \theta(a), \beta(a); k, \theta, \beta, D; s) \qquad \text{for each } a\in A,$$

all the consumers can attain the same level of expected utility after the change of D. Hence, Theorem 2 means that the equilibrium of the economy is not affected by change of D, which is the assertion of the M-M theorem.

4. An Aggregate Economy

In this section we consider a simplified economy where there are many, but identical consumers and prove the existence of an equilibrium for the economy. In what follows, we assume that the consumers in the economy are all the same and identical. By virtue of this assumption, we have only to consider the behavior of a representative consumer. Such a simple model of the economy is useful for macroeconomic analysis.

The utility functions of consumers are denoted by a map $U:A\to\mathscr{U}$, which is an element of set $\mathscr{U}^A$. We assume that the utility functions of all consumers are the same, and that map U is constant, i.e., for some $u\in U$, $U(a)=u$ for all $a\in A$. Therefore, we can regard $\mathscr{U}^A$ as $\mathscr{U}$.

Moreover, we assume that consumers are all in the same situation, and that their holdings of commodities, equities, and bonds are the same. The amounts of commodities held by consumers are described by a function k: $A\to R^n_+$, which is an element of set $\mathscr{L}^n_{\infty+}$. When consumers have all the same amounts of commodities, then function k is constant, i.e., for some $z\in R^n_+$, $k(a)=z$ for all $a\in A$. Therefore, we can regard $\mathscr{L}^n_{\infty+}$ as R^n_+.

The equity holdings by consumers are denoted by a function θ: $A\to R^J_+$, which is an element of set $\mathscr{L}^J_{1+}$. Since the total of equities of each firm is assumed to be unity, when consumers have all the same amounts of equities, $\theta(a)=1$ for all $a\in A$. Thus, set $\mathscr{L}_{1+}^{J}$ can be regarded as a one point set $\{\underline{1}\}$.

The numbers of bond held by consumers are described by a function β: $A\to R$, which is an element of set $\mathscr{L}_1$. When consumers have all the same amounts of bonds, then function β is constant, i.e., for some $b\in R$, $\beta(a)=b$ for all $a\in A$. Therefore, we can regard $\mathscr{L}_1$ as R.

By the above simplification, a macro-state $(k, \theta, \beta, D, U, Y)$ of the economy can be depicted in the aggregate economy by an element $(z, \underline{1}, b, D, u, Y)\in R_+^n\times R_+^J\times R\times R_+^J\times\mathscr{U}\times\mathscr{Y}$. In the procedure, we can define the price correspondence and the value function in the following fashion.

Let $S=\mathscr{U}\times\mathscr{Y}^J$. Define a price correspondence ψ by

$$(z, e, b, D; s)\in R_+^n\times R_+^J\times R\times R^J\times S \ \to\ \psi(z, e, b, D; s)\subset R^n\times R^J\times R.$$

Also, define a value function V by

$$(z', e', b'; z, e, b, D; s)\in R_+^n\times R_+^J\times R\times R_+^n\times R_+^J\times R\times R^J\times S$$
$$\to\ V(z', e', b'; z, e, b, D; s)\in R.$$

Now we can define the equilibrium for the aggregate economy. Definition 3.1 is reduced to the following.

Definition 4.1: A pair $\{\psi, V\}$ of a price correspondence and a value function is called *then equilibrium for the aggregate economy*, if $\{\psi, V\}$ has the following property:
Let $(z, e, b, D)\in R_+^n\times R_+^J\times R\times R^J$, $s=(u, Y)\in S$, and $(p, q, r)\in R^n\times R^J\times R$ such that

$$b=\sum_{j=1}^{J} D_j,\quad e=\underline{1},\quad \text{and } (p, q, r)\in\psi(z, e, b, D; s).$$

Then, there exist $\hat{x}\in R_+^n$, $(\hat{z}, \hat{e}, \hat{b}, \hat{D})\in R_+^n\times R_+^J\times R\times R^J$, and $\hat{y}_j\in Y_j$ such that the following conditions are satisfied.

(1) Firms are maximizing their profits, i.e., for each $j=1, 2,\cdots, J$,

$$p\cdot\hat{y}_j\geqq p\cdot z\quad\text{for all } y_j\in Y_j.$$

(2) Consumers are maximizing their expected utilities subject to their budget constraints, i.e.,

$$p\cdot(\hat{x}+\hat{z})+q\cdot\hat{e}+\hat{b}\leqq p\cdot z+q\cdot e+(1+r)b+\sum_{j=1}^{J} e_j(p\cdot\hat{y}_j-rD_j),$$

and

$$V(z, e, b; z, e, b, D; s)=u(\hat{x})+\delta\int_S V(\hat{z},\hat{e},\hat{b};\hat{z},\hat{e},\hat{b},\hat{D};\cdot)d\mu_s$$
$$\geqq u(x')+\delta\int_S V(z', e', b',\hat{z},\hat{e},\hat{b},\hat{D};\cdot)d\mu_s$$

for all $(x', z', e', b') \in R^n_+ \times R^J_+ \times R \times R^J$ with

$$p \cdot (x'+z')+q \cdot e'+b' \leqq p \cdot z+q \cdot e+(1+r)b+\sum_{j=1}^{J} e_j(p \cdot \hat{y}_j - rD_j).$$

(3) All the markets are in equilibrium, i.e.,

$$\hat{x}+\hat{z}=z+\sum_{j=1}^{J}\hat{y}_j, \quad \hat{e}=\underline{1}, \quad \text{and} \quad \hat{b}=\sum_{j=1}^{J}\hat{D}_j .$$

In what follows we state the conditions that insure the existence of the equilibrium for the aggregate economy which is defined in the above.

For the set $\mathscr{U}$ of utility functions and the family $\mathscr{Y}$ of production sets, we assume the following.

Assumption 1: Let $u \in \mathscr{U}$. Then, u has the following properties.

(1) u is a continuous and concave function.

(2) u is a monotone-increasing function, i.e., if $c \geqq c'$ and $c \neq c'$, then $u(c) > u(c')$.

(3) $u(\underline{0})=0$.

(4) There exists a number $\varepsilon>0$ such that $c \in R^n_+$ implies $|u(c)| \leqq \varepsilon$.

Assumption 2: Let $Y=(Y_1, Y_2, \cdots, Y_J) \in \mathscr{Y}^J$. Then, Y has the following properties.

(1) Y_j is a closed and convex subset of R^n.

(2) $Y_j \cap R^n_+ = \{\underline{0}\}$.

(3) There exists a number $\varepsilon>0$ such that $y \in Y_j$ implies $\| y \| \leqq \varepsilon$.

Under the above assumptions, we have the following theorem on the existence of the rational expectations equilibrium.

Theorem 4.1: Under Assumptions 1 and 2, there exists an equilibrium $\{\psi, V\}$ for the aggregate economy that has the following properties.

(1) The value function V is continuous and bounded and $V(z, e, b, D; s)$ is monotone non-decreasing and concave in (z, e, b).

(2) The value function V satisfies Condition 2, i.e., If $(z, e, b, D; s) \in R^n_+ \times R^J_+ \times R \times R^J \times S$, then

$$V(z, e, b+e \cdot \Delta D, D+\Delta D; s)=V(z, e, b, D; s) \quad \text{for all } \Delta D \in R^J_+ .$$

5. Proof of the Theorem 4.1

In this section the outline of the proof of Theorem 4.1 is shown, while all the lemmas for the theorem will be omitted.

Let $\mathscr{C}_*$ be the space of all bounded continuous functions defined on $R_+^n \times R_+^J \times S$. For each $W \in \mathscr{C}_*$, define a function MW on $R_+^n \times R_+^J \times S$ by

$$MW(z, e; s) = \sup \{u(x') + \delta \int_S W(z', e; \cdot) d\mu_s \mid x' \in R_+^n,\ z' \in R_+^n,$$

$$y_j \in Y_j\ (j = 1, 2, \cdots, J),\ x' + z' = z + \sum_{j=1}^{J} e_j y_j \},$$

where $s = (u, Y)$, $Y = (Y_1, \cdots, Y_J)$.

For the map M defined above, the following lemma holds.

Lemma 5.1: For any $W \in \mathscr{C}_*$, MW is a function that has the following properties.

(1) $MW \in \mathscr{C}_*$, i.e., MW is a continuous and bounded function.

(2) If $W(z, e; s)$ is monotone non-decreasing and concave in (z, e), then so is $MW(z, e; s)$ in (z, e).

(3) If $W(\underline{0}, e; s) = 0$ for all $(e; s)$, then $MW(\underline{0}, e; s) = 0$ for all $(e; s)$.

By (1) of the above lemma, we have a map,

$$W \in \mathscr{C}_* \rightarrow MW \in \mathscr{C}_*,$$

which is denoted by $M: \mathscr{C}_* \rightarrow \mathscr{C}_*$. This map has the following property.

Lemma 5.2: There exists a unique a function $W_0 \in \mathscr{C}_*$ that has the following properties.

(1) W_0 is a fixed-point of map M, i.e., $W_0 = MW_0$.

(2) For each D and s, $W_0(z, e; s)$ is monotone non-decreasing and concave in (z, e).

(3) $W_0(\underline{0}, e; s) = 0$ for all e and s.

Let $(z; s) \in R_+^n \times S$. Since $W_0 = MW_0$, by Assumptions 1 and 2, there exist $\hat{x} \in R_+^n$, $\hat{z} \in R_+^n$, and $\hat{y}_j \in Y_j$ $(j = 1, \cdots, J)$ such that

$$W_0(z, \underline{1}; s) = u(\hat{x}) + \delta \int_S W_0(\hat{z}, \underline{1}; \cdot) d\mu_s \quad \text{and} \quad \hat{x} + \hat{z} = z + \sum_{j=1}^{J} \hat{y}_j .$$

Now, let us define a subset $\Phi(z, b, D; s)$ of $R^n \times R^J \times R$ by

$$\Phi(z, b, D; s) = \{(p, q, r) \in R^n \times R^J \times R \mid$$

$$W_0(z, \underline{1}; s) + b - \underline{1} \cdot D \geqq u(c') + \delta \int_S (W_0(z', e'; \cdot) + b' - e' \cdot D') d\mu_s$$

for all $(x', z', e', b', D') \in R_+^n \times R_+^J \times R \times R^J$ with

$$p \cdot (x' + z') + q \cdot e' + b' \leqq p \cdot z + q \cdot \underline{1} + (1 + r) b + \sum_{j=1}^{J} (\sup p \cdot Y_j - r D_j) \},$$

where $s=(u, Y)$, $Y=(Y_1,\cdots, Y_J)$.

Lemma 5.3: For all $(z, b, D; s)\in R_+^n\times R\times R^J\times S$, $\Phi(z, b, D; s)\neq \phi$.

By this lemma we can define a correspondence,

$$(z, b, D; s)\in R_+^n\times R\times R^J\times S \quad\rightarrow\quad \Phi(z, b, D; s)\subset R^n\times R^J\times R,$$

which is denoted by $\Phi: R_+^n\times R\times R^J\times S\rightarrow R^n\times R^J\times R$.

Define a correspondence $\psi: R_+^n\times R_+^J\times R\times R^J\times S\rightarrow R^n\times R^J\times R$ by

$$\psi(z, e, b, D; s)=\Phi(z, b, D; s).$$

Also, let us define a function $V: R_+^n\times R_+^J\times R\times R_+^n\times R_+^J\times R\times R^J\times S\rightarrow R$ by

$$V(z', e', b'; z, e, b, D; s)=W_0(z', e'; s)+b'-e\cdot D.$$

Then, obviously, V is continuous and bounded. Also, by Lemma 5.2, we can easily check that function V has properties (1) and (2) of Theorem 4.1. It remains to show that $\{\psi, V\}$ is an equilibrium for the aggregate economy in the sense of Definition 4.1, which can be followed by the above definition of ψ and V.

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