数理解析研究所講究録 1445 巻 2005 年 143-153

Stability Properties for Linear Volterra Difference Equations with Convolution Kernels in a Banach Space

岡山理科大学・理学部 村上 悟 (Satoru Murakami) *
Department of Applied Mathematics,
Okayama University of Science

岡山理科大学・理学部 長渕 裕 (Yutaka Nagabuchi)
Department of Applied Science,
Okayama University of Science

1. Introduction

We consider the Volterra difference equations of convolution type

$$x(n+1)=\sum_{r=0}^{n}B(n-r)x(r), \qquad n\in\mathbb{Z}^{+}, \tag{E_0}$$

and

$$y(n+1)=\sum_{r=-\infty}^{n}B(n-r)y(r), \qquad n\in\mathbb{Z}, \tag{E_∞}$$

where $B(n)$ ($n\in\mathbb{Z}^{+}$, the nonnegative integers) are bounded linear operators on a Banach space X over the field $\mathbb{C}$. The study of Volterra difference equations has actively been done. Indeed, in the case where X is of finite dimension, the equations have extensively been treated in the book [1] and some results on stability properties and so on were obtained; for more details we refer the reader to [1, 2, 3] and the references therein. Also, in [4, 5], Volterra difference equations with infinite dimensional X were discussed in connection with some partial differential equations with piecewise continuous delays, and uniform asymptotic stability for (E_∞) was investigated in connection with the invertibility of the characteristic operator together with the summability of the fundamental solution, under additional conditions such as the mutual commutativity of the operators $B(n), n\in\mathbb{Z}^{+}$ or the exponential decay of the norm $\|B(n)\|$.

In this paper, we give a nice result on the stability properties of the zero solution of (E_0) or (E_∞) in the context above. Indeed, without the additional conditions imposed in [4, 5], we will establish an equivalence relation among the uniform asymptotic stability of the zero solution of (E_0) or (E_∞), the summability of the fundamental solution and the invertibility of the characteristic operator outside the unit circle in the complex plane.

*Partly supported by the Grant-in-Aid for Scientific Research (C), No.16540177, of the Japanese Ministry of Education, Culture, Sports, Science and Technology.

2. Notations

Let X be a (complex) Banach space with the norm $|\cdot|$. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X. Clearly, $\mathcal{L}(X)$ is a Banach space equipped with the operator norm $\|\cdot\|$, which is defined by

$$\|T\| = \sup\{|Tx| : x \in X,\ |x| = 1\}$$

for any $T \in \mathcal{L}(X)$.

For any interval $J \subset \mathbb{R}$ we use the same notaion J meaning the discrete one $J \cap \mathbb{Z}$, e.g. $[0,\ \sigma] = \{0, 1, \dots, \sigma\}$ for $\sigma \in \mathbb{Z}^+$. Also, for an X-valued function ξ on a discrete interval J, its norm is denoted by $\|\xi\|_J := \sup\{|\xi(j)| : j \in J\}$. Let $\sigma \in \mathbb{Z}^+$ and a function $\phi : [0,\ \sigma] \to X$ be given. We denote by $x(n; \sigma, \phi)$ the solution $x(n)$ of (E_0) satisfying $x(n) = \phi(n)$ on $[0,\ \sigma]$. Similarly, for $\tau \in \mathbb{Z}$ and a function $\psi : (-\infty, \tau] \to X$, we denote by $y(n; \tau, \psi)$ the solution $y(n)$ of (E_∞) satisfying $y(n) = \psi(n)$ on $(-\infty, \tau]$.

Definition 1. The zero solution of (E_0) is said to be
(i) *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\sigma \in \mathbb{Z}^+$ and ϕ is an initial function on $[0, \sigma]$ with $\|\phi\|_{[0,\sigma]} < \delta$ then $|x(n; \sigma, \phi)| < \varepsilon$ for all $n \geq \sigma$.
(ii) *uniformly asymptotically stable* if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\sigma \in \mathbb{Z}^+$ and ϕ is an initial function on $[0, \sigma]$ with $\|\phi\|_{[0,\sigma]} < \mu$ then $|x(n; \sigma, \phi)| < \varepsilon$ for all $n \geq \sigma + N$.

Definition 2. The zero solution of (E_∞) is said to be
(i) *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\tau \in \mathbb{Z}$ and ψ is an initial function on $(-\infty,\ \tau]$ with $\|\psi\|_{(-\infty,\tau]} < \delta$ then $|y(n; \tau, \psi)| < \varepsilon$ for all $n \geq \tau$.
(ii) *uniformly asymptotically stable* if it is uniformly stable, and if there exists a $\mu > 0$ such that, for any $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{Z}^+$ with the property that, if $\tau \in \mathbb{Z}$ and ψ is an initial function on $(-\infty,\ \tau]$ with $\|\psi\|_{(-\infty,\tau]} < \mu$ then $|y(n; \tau, \psi)| < \varepsilon$ for all $n \geq \tau + N$.

The fundamental solution of (E_0) is a family in $\mathcal{L}(X)$ satisfying the relation

$$R(n+1) = \sum_{j=0}^{n} B(n-j)R(j), \qquad n \in \mathbb{Z}^+$$

and $R(0) = I$. Then, for instance, the solution $y(n; \tau, \psi)$ of (E_∞) is given by the variation of constant formula as follows:

$$y(n; \tau, \psi) = R(n-\tau)\psi(\tau) + \sum_{r=\tau}^{n-1} R(n-r-1)\Big(\sum_{s=-\infty}^{\tau-1} B(r-s)\psi(s)\Big). \tag{1}$$

3. Main Rersults

In what follows, we assume that $B := \{B(n)\} \subset \mathcal{L}(X)$ is summable, that is, the condition $\sum_{n=0}^{\infty} \|B(n)\| < \infty$ holds, and study stability properties of the zero solution of Eq. (E_∞), together with those of the zero solution of Eq. (E_0). Here and subsequently, $\hat{B}(z)$ denotes the Z-transform of B; that is, $\hat{B}(z) := \sum_{n=0}^{\infty} B(n)z^{-n}$ for $|z| \geq 1$.

In [5, Theorem 2] and [4, Theorem 2], the equivalence among the uniform asymptotic stability of the zero solution of Eq. (E_∞), the summability of the fundamental solution $R = \{R(n)\}$ of Eq. (E_0), and the invertibility of the characteristic operator $zI - \hat{B}(z)$ associated with Eq. (E_0) has been established under some restrictions such as the mutual commutativity of the operators $B(n)$, $n \in \mathbb{Z}^+$ or the exponential decay of the norm $\|B(n)\|$. We will show in the following theorem that [5, Theorem 2] and [4, Theorem 2] hold true without such restrictions.

Theorem 1. *Let $B = \{B(n)\}_{n=0}^{\infty} \in l^1(\mathbb{Z}^+) := l^1(\mathbb{Z}^+; \mathcal{L}(X))$, and assume that $B(n)$, $n \in \mathbb{Z}^+$, are all compact. Then the following statements are equivalent.*

(i) *The zero solution of Eq. (E_0) is uniformly asymptotically stable.*
(ii) *The zero solution of Eq. (E_∞) is uniformly asymptotically stable.*
(iii) $R = \{R(n)\}_{n=0}^{\infty} \in l^1(\mathbb{Z}^+)$.
(iv) *For any z such that $|z| \geq 1$, the operator $zI - \hat{B}(z)$ is invertible in $\mathcal{L}(X)$.*

In order to prove the theorem, we need the following preparatory results.

Proposition 1. *Let $K = \{K(n)\}_{n=-\infty}^{\infty} \in l^1(\mathbb{Z}) := l^1(\mathbb{Z}; \mathcal{L}(X))$, and assume that $I - \tilde{K}(\rho)$ is invertible for each $\rho \in \mathbb{R}$, where $\tilde{K}(\rho) := \sum_{n=-\infty}^{\infty} K(n)e^{-i\rho n}$. Then there is an $R \in l^1(\mathbb{Z})$ such that*

$$\tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1} = \tilde{R}(\rho), \qquad \forall \rho \in \mathbb{R}.$$

Proof. (1-Step) For each (small) $\varepsilon > 0$ we define a 2π-periodic function $\tilde{\phi}_\varepsilon$ by

$$\tilde{\phi}_\varepsilon(t) = \begin{cases} 1 & (|t| \leq \varepsilon) \\ 0 & (2\varepsilon \leq |t| \leq \pi) \\ (2\varepsilon - t)/\varepsilon & (\varepsilon < t < 2\varepsilon) \\ (2\varepsilon + t)/\varepsilon & (-2\varepsilon < t < -\varepsilon). \end{cases}$$

One can easily check that Fourier coefficients of $\tilde{\phi}_\varepsilon$ are given by

$$d_l = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_\varepsilon(t) e^{ilt} dt = \begin{cases} \frac{2}{\pi \varepsilon l^2} \sin \frac{3\varepsilon l}{2} \sin \frac{\varepsilon l}{2} & (l = \pm 1, \pm 2, \dots) \\ \frac{3\varepsilon}{2\pi} & (l = 0). \end{cases}$$

Clearly, the sequence $\phi_\varepsilon := \{d_l\}_{l=-\infty}^{\infty}$ is summable, and by the Fourier expansion theorem we get

$$\sum_{l=-\infty}^{\infty} d_l e^{-ilt} = \tilde{\phi}_\varepsilon(t), \qquad \forall t \in \mathbb{R}.$$

Hence it follows that for any $t_0 \in \mathbb{R}$,

$$\tilde{\phi}_\varepsilon(t - t_0) \equiv \sum_{l=-\infty}^{\infty} c_l e^{-itl},$$

where $c := \{c_l\}_{l\in\mathbb{Z}}$ is a sequence defined by $c_l = d_l e^{it_0 l} = \phi_\varepsilon(l) e^{it_0 l}$ for $l \in \mathbb{Z}$. Notice that the sequence c is summable.

(2-Step) Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{1}{\pi x^2} \sin 3x \sin x & (x \neq 0) \\ \frac{3}{\pi} & (x = 0). \end{cases}$$

One can easily see that f is continuously differentiable. In fact, $f'(x)$ is given by

$$f'(x) = \begin{cases} -\frac{2}{\pi x^3} \sin 3x \sin x + \frac{1}{\pi x^2}(3 \cos 3x \sin x + \sin 3x \cos x) & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Since $\lim_{|x|\to\infty}(|f(x)|+|f'(x)|) = 0$, there exists a constant $H > 0$ such that $\sup_{-\infty<x<\infty}(|f(x)|+|f'(x)|) = H$. Moreover,

$$\begin{aligned} \int_0^\infty |f'(x)|dx &= \int_0^1 |f'(x)|dx + \int_1^\infty |f'(x)|dx \\ &\leq H + \int_1^\infty \frac{6}{\pi x^2} dx \\ &\leq H + C, \end{aligned}$$

where $C = 6/\pi$.

(3-Step) Put $M = \sup_{\rho\in\mathbb{R}} \|(I - \tilde{K}(\rho))^{-1}\|$, and take a positive integer N such that

$$\frac{23M}{\pi} \sum_{|\tau|\geq N+1} \|K(\tau)\| \leq \frac{1}{4}.$$

Moreover, take a positive integer k_0, $k_0 \geq 3$, such that

$$2NM(H + C)\pi \sum_{l\in\mathbb{Z}} \|K(l)\| < \frac{3k_0}{4},$$

and set $\varepsilon = \pi/(3k_0)$ and $\rho_n = 3n\varepsilon$, $n = 0, 1, \ldots$. Then $\rho_{2k_0} = 2\pi$, and the following relation holds:

$$\sum_{n=0}^{2k_0-1} \tilde{\phi}_\varepsilon(\rho - \rho_n) \equiv 1, \qquad \forall\, \rho \in \mathbb{R},$$

where $\tilde{\phi}_\varepsilon$ is the one introduced in 1-Step. Set $F(\rho) = \tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1}$ and $F_n(\rho) = \tilde{\phi}_\varepsilon(\rho - \rho_n)F(\rho)$, $\rho \in \mathbb{R}$. Then

$$F(\rho) \equiv \sum_{n=0}^{2k_0-1} F_n(\rho).$$

Therefore, in order to establish the proposition it suffices only to certify that for each n there exists an $R_n \in l^1(\mathbb{Z})$ such that $F_n(\rho) \equiv \tilde{R}_n(\rho)$. We now set

$$K_n(l) = \left[((\phi_{2\varepsilon}e^{i\rho_n\cdot}) * K)(l) - \phi_{2\varepsilon}(l)e^{i\rho_n l}\tilde{K}(\rho_n)\right](I - \tilde{K}(\rho_n))^{-1}, \qquad l \in \mathbb{Z},$$

where $*$ denotes the convolution in $l^1(\mathbb{Z})$. Then $K_n \in l^1(\mathbb{Z})$, and moreover

$$\begin{aligned}
\tilde{K}_n(\rho) &= \left[(\phi_{2\varepsilon}e^{i\rho_n\cdot})\tilde{\ }(\rho)\tilde{K}(\rho) - (\phi_{2\varepsilon}e^{i\rho_n\cdot})\tilde{\ }(\rho)\tilde{K}(\rho_n)\right](I - \tilde{K}(\rho_n))^{-1} \\
&= \tilde{\phi_{2\varepsilon}}(\rho - \rho_n)(\tilde{K}(\rho) - \tilde{K}(\rho_n))(I - \tilde{K}(\rho_n))^{-1}.
\end{aligned}$$

Observe that $\tilde{\phi}_\varepsilon(\rho - \rho_n) \neq 0$ implies $\tilde{\phi_{2\varepsilon}}(\rho - \rho_n) = 1$, and hence

$$\begin{aligned}
\tilde{K}_n(\rho) &= (\tilde{K}(\rho) - \tilde{K}(\rho_n))(I - \tilde{K}(\rho_n))^{-1} \\
&= (\tilde{K}(\rho) - I)(I - \tilde{K}(\rho_n))^{-1} + I,
\end{aligned}$$

or

$$I - \tilde{K}_n(\rho) = (I - \tilde{K}(\rho))(I - \tilde{K}(\rho_n))^{-1},$$

which implies

$$(I - \tilde{K}(\rho))^{-1} = (I - \tilde{K}(\rho_n))^{-1}(I - \tilde{K}_n(\rho))^{-1}.$$

This observation leads to

$$\begin{aligned}
F_n(\rho) &\equiv \tilde{\phi}_\varepsilon(\rho - \rho_n)\tilde{K}(\rho)(I - \tilde{K}(\rho))^{-1} \\
&\equiv \tilde{\phi}_\varepsilon(\rho - \rho_n)\tilde{K}(\rho)(I - \tilde{K}(\rho_n))^{-1}(I - \tilde{K}_n(\rho))^{-1}.
\end{aligned}$$

We claim that

$$|K_n|_1 := \sum_{l=-\infty}^{\infty} \|K_n(l)\| < \frac{1}{2}. \tag{2}$$

If the claim is true, then the series $\sum_{\tau=0}^{\infty} K_n^{*\tau} := e + K_n + K_n * K_n + K_n * K_n * K_n + \cdots$, (here e is the unit element in $l^1(\mathbb{Z})$), converges in $l^1(\mathbb{Z})$ with $(I - \tilde{K}_n(\rho))^{-1} \equiv (\sum_{\tau=0}^{\infty} K_n^{*\tau})\tilde{\ }(\rho)$, and hence we may set $R_n = (\phi_\varepsilon e^{i\rho_n\cdot}) * K * \{(I - \tilde{K}(\rho_n))^{-1}\sum_{\tau=0}^{\infty} K_n^{*\tau}\}$ to get the equality $F_n = \tilde{R}_n$ with $R_n \in l^1(\mathbb{Z})$.

In what follows we will evaluate $|K_n|_1$ to establish (2). It follows that

$$\begin{aligned}
|K_n|_1 &\leq M \sum_{l=-\infty}^{\infty} \| \sum_{\tau=-\infty}^{\infty} K(\tau)(\phi_{2\varepsilon}(l-\tau)e^{i\rho_n(l-\tau)}) - \phi_{2\varepsilon}(l)e^{i\rho_n l} \sum_{\tau=-\infty}^{\infty} K(\tau)e^{-i\rho_n \tau}\| \\
&\leq M \sum_{1\leq|\tau|\leq N} \|K(\tau)\| \sum_{l=-\infty}^{\infty} |\phi_{2\varepsilon}(l-\tau) - \phi_{2\varepsilon}(l)| \\
&\quad + M \sum_{|\tau|\geq N+1} \|K(\tau)\| \sum_{l=-\infty}^{\infty} |\phi_{2\varepsilon}(l-\tau) - \phi_{2\varepsilon}(l)| \\
&=: I_1 + I_2.
\end{aligned}$$

Noting $0 < \varepsilon < 1/2$, we get

$$\begin{aligned}
I_2 &\leq 2M \sum_{|\tau|\geq N+1} \|K(\tau)\| \times \sum_{l=-\infty}^{\infty} |\phi_{2\varepsilon}(l)| \\
&\leq 2M \sum_{|\tau|\geq N+1} \|K(\tau)\| \times \left(\frac{3\varepsilon}{\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi k^2 \varepsilon} |\sin 3\varepsilon k \sin \varepsilon k| \right) \\
&\leq 2M \sum_{|\tau|\geq N+1} \|K(\tau)\| \times \left(\frac{3\varepsilon}{\pi} + \frac{2}{\pi\varepsilon} \sum_{k=1}^{[1/\varepsilon]} \frac{1}{k^2} |\sin 3\varepsilon k \sin \varepsilon k| + \frac{2}{\pi\varepsilon} \sum_{k=[1/\varepsilon]+1}^{\infty} \frac{1}{k^2} \right) \\
&\leq 2M \sum_{|\tau|\geq N+1} \|K(\tau)\| \times \left(\frac{3\varepsilon}{\pi} + \frac{2}{\pi\varepsilon} \sum_{k=1}^{[1/\varepsilon]} \frac{3\varepsilon^2 k^2}{k^2} + \frac{2}{\pi\varepsilon} \int_{[1/\varepsilon]}^{\infty} \frac{dx}{x^2} \right) \\
&\leq 2M \sum_{|\tau|\geq N+1} \|K(\tau)\| \times \left(\frac{3\varepsilon}{\pi} + \frac{6\varepsilon}{\pi} \times [1/\varepsilon] + \frac{2}{\pi\varepsilon} \frac{1}{[1/\varepsilon]} \right) \\
&\leq \frac{23M}{\pi} \sum_{|\tau|\geq N+1} \|K(\tau)\| \\
&\leq \frac{1}{4},
\end{aligned}$$

where $[1/\varepsilon]$ denotes the largest integer which does not exceed $1/\varepsilon$. Also, using the function f introduced in 2-Step we get

$$\begin{aligned}
I_1 &\leq M|K|_1 \sup_{1\leq|\tau|\leq N} \left(\sum_{l=-\infty}^{\infty} |f((l-\tau)\varepsilon) - f(l\varepsilon)|\varepsilon \right) \\
&\leq M\varepsilon|K|_1 \sup_{1\leq|\tau|\leq N} \left(\sum_{m=0}^{|\tau|-1} \sum_{s=-\infty}^{\infty} |f(\{(s+1)|\tau| + m\}\varepsilon) - f(\{s|\tau| + m\}\varepsilon)| \right) \\
&\leq M\varepsilon|K|_1 \sup_{1\leq|\tau|\leq N} \left(\sum_{m=0}^{|\tau|-1} \sum_{s=-\infty}^{\infty} \int_{\{s|\tau|+m\}\varepsilon}^{\{(s+1)|\tau|+m\}\varepsilon} |f'(x)|dx \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq M\varepsilon|K|_1 \sup_{1\leq|\tau|\leq N}\left(\sum_{m=0}^{|\tau|-1}\int_{-\infty}^{\infty}|f'(x)|dx\right) \\
&\leq M\varepsilon|K|_1 N\int_{-\infty}^{\infty}|f'(x)|dx \\
&< \frac{2(H+C)MN\pi|K|_1}{3k_0} \\
&< \frac{1}{4}.
\end{aligned}
$$

Thus $|K_n|_1 \leq I_1 + I_2 < 1/4 + 1/4 = 1/2$, as required. □

Proposition 2. *Let $B = \{B(n)\}_{n=0}^{\infty} \in l^1(\mathbb{Z}^+)$, and assume that $I - B(0)$ is invertible and that $I - \hat{B}(z)$ is invertible for each $z \in \mathbb{C}$ with $|z| \geq 1$, where $\hat{B}(z) := \sum_{n=0}^{\infty} B(n)z^{-n}$. Then there is an $R \in l^1(\mathbb{Z}^+)$ such that*

$$\hat{B}(z)(I - \hat{B}(z))^{-1} = \hat{R}(z), \qquad \forall |z| \geq 1.$$

Proof. Consider the sequence $B' \in l^1(\mathbb{Z})$ defined by $B'(n) = B(n)$ if $n \geq 0$,and $B'(n) = 0$ if $n < 0$. Then

$$I - \tilde{B}'(\rho) = I - \sum_{n=0}^{\infty} B'(n)e^{-i\rho n} = I - \hat{B}(e^{i\rho}), \qquad \forall\ \rho \in \mathbb{R}.$$

Hence $I - \tilde{B}'(\rho)$ is invertible for each $\rho \in \mathbb{R}$, and consequently there exists a $Q \in l^1(\mathbb{Z})$ such that $\tilde{B}'(\rho)(I - \tilde{B}'(\rho))^{-1} = \tilde{Q}(\rho)$, $\forall\ \rho \in \mathbb{R}$, by Proposition 1. Define an element Q_+ in $l^1(\mathbb{Z}^+)$ by $Q_+(n) = Q(n)$ for any $n \in \mathbb{Z}^+$. The function $\hat{Q}_+(z)$ is bounded and continuous on the domain $|z| \geq 1$, and it is analytic on $|z| > 1$. Similarly, the function $\sum_{n=1}^{\infty} Q(-n)z^n$ is bounded and continuous on the domain $|z| \leq 1$, and it is analytic on $|z| < 1$. Moreover, if $|z| = 1$ with $z = e^{i\rho}$, then

$$
\begin{aligned}
\hat{B}(z)(I - \hat{B}(z))^{-1} - \hat{Q}_+(z) &= \tilde{B}'(\rho)(I - \tilde{B}'(\rho))^{-1} - \sum_{n=0}^{\infty} Q(n)e^{-i\rho n} \\
&= \tilde{Q}(\rho) - \sum_{n=0}^{\infty} Q(n)e^{-i\rho n} \\
&= \sum_{n=-\infty}^{-1} Q(n)e^{-i\rho n} \\
&= \sum_{n=1}^{\infty} Q(-n)e^{i\rho n} \\
&= \sum_{n=1}^{\infty} Q(-n)z^n.
\end{aligned}
$$

Therefore, the function $G(z)$ defined by

$$G(z) = \begin{cases} \hat{B}(z)(I - \hat{B}(z))^{-1} - \hat{Q}_+(z) & (|z| \geq 1) \\ \\ \sum_{n=1}^{\infty} Q(-n) z^n & (|z| < 1) \end{cases}$$

is analytic on the entire domain by Morera's theorem. Observe that $I - \hat{B}(z) \to I - B(0)$ in $\mathcal{L}(X)$ as $|z| \to \infty$. Since $I - B(0)$ is invertible by the assumption, it follows that $\lim_{|z|\to\infty} \|(I - \hat{B}(z))^{-1}\| = \|(I - B(0))^{-1}\|$, and consequently $\sup_{|z|\geq 1} \|(I - \hat{B}(z))^{-1}\| < \infty$. Therefore, $G(z)$ is bounded on the entire domain, and hence $G(z)$ is a constant function by Liouville's theorem. Then $G(z) \equiv G(0) = 0$, and hence it follows that $\hat{B}(z)(I - \hat{B}(z))^{-1} = \hat{Q}_+(z)$ for any z with $|z| \geq 1$. Thus we may set $Q_+ = R$ to establish the proposition. $\square$

We are now in a position to prove the theorem.

Clearly, the implication [(ii)$\Longrightarrow$ (i)] holds true. Also, the implications [(iii)$\Longrightarrow$ (ii)] and [(ii)$\Longrightarrow$ (iv)] have already been proved in [5, Theorem 2] and [4, Theorem 2]. In what follows, we will prove the implications [(iv)$\Longrightarrow$ (iii)] and [(i)$\Longrightarrow$ (ii)] .

Proof of [(iv)$\Longrightarrow$ (iii)]. Let us consider the sequence $D \in l^1(\mathbb{Z}^+)$ defined by $D(n) = B(n-1)$ if $n \geq 1$, and $D(n) = 0$ if $n = 0$. Clearly, $I - D(0)$ is invertible. For any $z \in \mathbb{C}$ with $|z| \geq 1$, we get

$$\hat{D}(z) = \sum_{n=0}^{\infty} D(n) z^{-n} = z^{-1} \hat{B}(z),$$

and hence

$$I - \hat{D}(z) = \frac{1}{z}(zI - \hat{B}(z)).$$

Thus $I - \hat{D}(z)$ is invertible for each $z \in \mathbb{C}$ with $|z| \geq 1$, and it satisfies the relation

$$(I - \hat{D}(z))^{-1} = z(zI - \hat{B}(z))^{-1}, \qquad |z| \geq 1.$$

By virtue of Proposition 2, there exists a $Q \in l^1(\mathbb{Z}^+)$ such that $\hat{D}(z)(I - \hat{D}(z))^{-1} = \hat{Q}(z)$, $|z| \geq 1$, and hence we get

$$\begin{aligned} I + \hat{Q}(z) &= I + \hat{D}(z)(I - \hat{D}(z))^{-1} \\ &= (I - \hat{D}(z))^{-1} \\ &= z(zI - \hat{B}(z))^{-1} \end{aligned}$$

for all $|z| \geq 1$. Consider the sequence $S = \{S(n)\}_{n=0}^{\infty}$ defined by

$$S(n) = \begin{cases} I + Q(0) & (n = 0) \\ Q(n) & (n \geq 1). \end{cases}$$

Then $S \in l^1(\mathbb{Z}^+)$, and $\hat{S}(z) = I + \hat{Q}(z) = z(zI - \hat{B}(z))^{-1}$ for all $|z| \geq 1$. Notice that the fundamental solution R is bounded exponentially, that is, $\sup_{n\geq 0} e^{-n\omega}\|R(n)\| < \infty$ for some constant $\omega \geq 0$. Hence the Z-transform $\sum_{n=0}^{\infty} R(n)z^{-n}$ of R converges for $|z| > e^{\omega}$. Let us consider the Z-transform of both sides in the equation $R(n+1) = \sum_{k=0}^{\infty} B(n-k)R(k)$ with $R(0) = I$ to get the relation $z(\hat{R}(z) - I) = \hat{B}(z)\hat{R}(z)$, or $(zI - \hat{B}(z))\hat{R}(z) = zI$ for $|z| > e^{\omega}$. Thus it follows that $\hat{R}(z) = z(zI - \hat{B}(z))^{-1} = \hat{S}(z)$ for all $|z| > e^{\omega}$. By the uniqueness of the Z-transform, we get $R(n) \equiv S(n)$, $n \in \mathbb{Z}^+$, which shows the summability of R, as required.

Proof of [(i)$\Longrightarrow$ (ii)]. Let $\tau \in \mathbb{Z}$, and $\phi,\ \psi : (-\infty, \tau] \to X$ be given in such a way that

$$\|\phi\|_{(-\infty,\tau]} < \delta(\varepsilon/2) \quad \text{and} \quad \|\psi\|_{(-\infty,\tau]} < \min\{\delta(1/2),\ \mu\},$$

where $\delta(\cdot)$ and μ are those in Definition 1. Let us take a sequence $\{n_j\} \subset \mathbb{Z}^+$ such that $n_j \to \infty$ $(j \to \infty)$. We may assume that $\tau + n_j > 0$ for $j = 1, 2, \ldots$. Define $\phi^j : [0, \tau + n_j] \to X$ by

$$\phi^j(n) := \phi(n - n_j), \qquad n \in [0, \tau + n_j],$$

and $x^j(n)$ by

$$x^j(n) := \begin{cases} x(n + n_j; \tau + n_j, \phi^j) & (n \geq -n_j) \\ \phi(n) & (n < -n_j) \end{cases}$$

for $j = 1, 2, \ldots$. Since $x^j(n) = \phi^j(n+n_j) = \phi(n)$ for $n \in [-n_j,\ \tau]$, the uniform asymptotic stability of the zero solution of (E_0) yields

$$|x^j(n)| < \frac{\varepsilon}{2} \qquad \text{for } n \geq \tau. \tag{3}$$

Let any $n \in \mathbb{Z}$ be given. We now assert that the sequence $\{x^j(n)\}_j$ contains a convergent subsequence. Indeed, in case of $n \leq \tau$, we get $x^j(n) = \phi(n)$, and hence the assertion clearly holds. Let us consider the case $\tau < n$. It follows that

$$\begin{aligned} x^j(n) &= \sum_{k=0}^{n_j+n-1} B(n_j + n - 1 - k)x(k; \tau + n_j, \phi^j) \\ &= \sum_{s=-n_j}^{n-1} B(n - 1 - s)x^j(s) \\ &= \sum_{s=-\infty}^{n-1} B(n-1-s)x^j(s) + \sum_{s=-\infty}^{-n_j-1} B(n-1-s)\phi(s). \end{aligned}$$

By virtue of the summability of $B = \{B(n)\}_{n=0}^{\infty}$, it is easy to certify that the term $\sum_{s=-\infty}^{-n_j-1} B(n-1-s)\phi(s)$ tends to 0 as $j \to \infty$. Moreover, since the operator $B(n-1-s)$ is compact, we see that the sequence $\{\sum_{s=-\infty}^{n-1} B(n-1-s)x^j(s)\}_j$ contains a convergent subsequence. This observation leads to that the sequence $\{x^j(n)\}_j$ contains a convergent subsequence, which completes the proof of the assertion.

Now one can select a subsequence of $\{x^j(n)\}_j$, denoted by the same notation $x^j(n)$, which converges to some $\tilde{y}(n)$ on $\mathbb{Z}$ as $j \to \infty$. Obviously $\tilde{y}(n) = \phi(n)$ for $n \in (-\infty, \tau]$. Moreover, it follows that $\lim_{j\to\infty} \sum_{s=-n_j}^{n} B(n-s)x^j(s) = \sum_{s=-\infty}^{n} B(n-s)\tilde{y}(s)$. Thus we obtain that

$$\begin{aligned}
\tilde{y}(n+1) &= \lim_{j\to\infty} x^j(n+1) \\
&= \lim_{j\to\infty} x(n+1+n_j; \tau+n_j, \phi^j) \\
&= \lim_{j\to\infty} \sum_{r=0}^{n+n_j} B(n+n_j-r)x(r; \tau+n_j, \phi^j) \\
&= \lim_{j\to\infty} \sum_{s=-n_j}^{n} B(n-s)x^j(s) \\
&= \sum_{s=-\infty}^{n} B(n-s)\tilde{y}(s),
\end{aligned}$$

which implies that $\tilde{y}(n) = y(n; \tau, \phi)$ on $\mathbb{Z}$. Letting $j \to \infty$ in (3) we get

$$|y(n; \tau, \phi)| \le \frac{\varepsilon}{2} < \varepsilon \qquad \text{for } \; n \ge \tau. \tag{4}$$

Furthermore, by the same argument we see that

$$|y(n; \tau, \psi)| < \frac{\varepsilon}{2} \qquad \text{for } \; n \ge \tau + N(\varepsilon/2), \tag{5}$$

where $N(\cdot)$ is the one in Definition 1. The inequality (4), together with (5), shows that the zero solution of (E_∞) is uniformly asymptotically stable. $\square$

Remark 1. One can see from the proof that in Theorem 1, the implications (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii)$\Longrightarrow$(i) hold true without the assumption that $B(n)$, $n \in \mathbb{Z}^+$, are compact. It is an interesting problem to ask whether or not the implication (ii)$\Longrightarrow$(iii) (or (ii)$\Longrightarrow$(iv)) holds good without the compactness condition on $B(n)$. But the problem is still open for the authors.

Remark 2. We can apply Theorem 1 to establish the existence of bounded (resp. asymptotically almost periodic) solutions for forced equations of (E_∞) with a bounded (resp. asymptotically almost periodic) forcing term, provided that the zero solution of (E_0) is uniformly asymptotically stable. Details will be discussed in a forth-coming paper [6].

References

1. S. Elaydi, *An Introduction to Difference Equations*, Springer, New-York, 1996.
2. S. Elaydi and S. Murakami, Asymptotic stability versus exponential stability in linear Volterra difference equations of convolution type, *J. Difference Equations and Appl.*, 2 (1996), 401–410.

3. S. Elaydi and S. Murakami, Uniform asymptotic stability in linear Volterra difference equations, *J. Difference Equations and Appl.*, 3 (1998), 203–218.
4. T. Furumochi, S. Murakami and Y. Nagabuchi, Volterra difference equations on a Banach space and abstract differential equations with piecewise continuous delays, *Japanese. J. Math.*, Vol. 30, No. 2 (2004), 387–412.
5. T. Furumochi, S. Murakami and Y. Nagabuchi, A generalization of Wiener's lemma and its application to Volterra difference equations on a Banach space, *J. Difference Equations and Appl.*, Vol. 10, No. 13-15 (2004), 1201–1214.
6. S. Murakami and Y. Nagabuchi, Stability properties and asymptotic almost periodicity for linear Volterra difference equations in a Banach space, *Japanese J. Math.*, Vol. 31-2 (in press).