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# On the uniqueness of nodal radial solutions of sublinear elliptic equations in a ball

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## 1. Introduction

We consider the second order ordinary differential equation

$$u'' + \frac{N-1}{r}u' + K(r)f(u) = 0, \quad 0 < r < 1, \tag{1.1}$$

with the boundary condition

$$u'(0) = u(1) = 0, \tag{1.2}$$

where $N \geq 2$, $K \in C^2[0,1]$, $K(r) > 0$ for $0 \leq r \leq 1$, $f \in C^1(\mathbf{R})$, $sf(s) > 0$ for $s \neq 0$. Assume moreover that the following sublinear condition is satisfied:

$$\frac{f(s)}{s} > f'(s) \quad \text{for } s \neq 0. \tag{1.3}$$

Note that a solution of problem (1.1)–(1.2) is a radial solution $u(r)$ $(r = |x|)$ of the Dirichlet problem of

$$\begin{cases} \Delta u + K(|x|)f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $B = \{x \in \mathbf{R}^N : |x| < 1\}$.

We consider solutions $u$ of problem (1.1)–(1.2) satisfying $u(0) > 0$ only. If $u$ is a solution of problem (1.1)–(1.2) with $u(0) < 0$, then it can be treated similarly as in the case where $u(0) > 0$, since $v \equiv -u$ satisfies $v(0) > 0$ and is a solution of

$$\begin{cases} v'' + \dfrac{N-1}{r}v' + K(r)f_0(v) = 0, \quad 0 < r < 1, \\ v'(0) = v(1) = 0, \end{cases}$$

where $f_0(s) = -f(-s)$.

In this paper we study the uniqueness of solutions of the problem (1.1)–(1.2) having exactly $k-1$ zeros in $(0,1)$, where $k \in \mathbf{N}$.

Hence we consider the following problem:

$$\begin{cases} u'' + \dfrac{N-1}{r}u' + K(r)f(u) = 0, \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \quad u(0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0,1). \end{cases} \tag{P$_k$}$$

It is known that there exists at least one solution of $(\mathrm{P}_k)$ under a certain condition. For example, in the case where $f(u) = |u|^{p-1}u$, $p > 0$, $p \neq 1$ and $N \geq 3$, the existence results of solutions of $(\mathrm{P}_k)$ were obtained by Y. Naito [4]. Assume that there exists limits $f_0$ and $f_\infty$ such that

$$f_0 = \lim_{u\to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u} \quad (0 \leq f_0,\ f_\infty \leq \infty).$$

In the case where there is a sufficiently large gap between $f_0$ and $f_\infty$, the existence of solutions of $(\mathrm{P}_k)$ was established by Dambrosio [1].

Now we consider the uniqness of solutions of $(\mathrm{P}_k)$. For the superlinear case $f(u) = |u|^{p-1}u$ $(p > 1)$, Yanagida [6] showed that, for each $k \in \mathbf{N}$, $(\mathrm{P}_k)$ has at most one solution if $rK'(r)/K(r)$ is nonincreasing and $N \geq 3$. For the subliear case where (3), $f_0 = \infty$ and $f_\infty = 0$, Kajikiya [2] proved that, for each $k \in \mathbf{N}$, the solution of $(\mathrm{P}_k)$ exists and is unique if $K(r) \equiv 1$. However very little is known about the uniqness of solutions of $(\mathrm{P}_k)$ for the sublinear case and $K(r) \not\equiv 1$.

The main result of this paper is as follows.

**Theorem 1.1.** *Suppose that* (1.3) *holds. If*

$$3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2 \geq 0, \quad 0 \leq r \leq 1, \tag{1.4}$$

*then, for each* $k \in \mathbf{N}$, $(\mathrm{P}_k)$ *has at most one solution.*

In view of the following equality

$$\begin{aligned} &3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2 \\ &\qquad = K^2\left[\left(\frac{rK'}{K} + 2\right)\left(\frac{rK'}{K} + 2(N-1)\right) - 2r\left(\frac{rK'}{K}\right)'\right], \end{aligned}$$

we have the following corollary of Theorem 1.1.

**Corollary 1.1.** *Suppose that* (1.3) *holds. Assume moreover that one of the following* (1.5)–(1.7) *is satisfied:*

$$K'' \leq 0, \quad K' \geq 0 \quad \textit{for } 0 \leq r \leq 1, \tag{1.5}$$

$$N = 2, \quad \left(\frac{rK'}{K}\right)' \leq 0 \quad \textit{for } 0 \leq r \leq 1, \tag{1.6}$$

$$N > 2, \quad \frac{rK'}{K} \geq -2, \quad \left(\frac{rK'}{K}\right)' \leq 0 \quad \textit{for } 0 \leq r \leq 1. \tag{1.7}$$

*Then, for each* $k \in \mathbf{N}$, $(\mathrm{P}_k)$ *has at most one solution.*

## 2. Lemmas

In this section we give several lemmas.

First we note that (1.1) can be rewritten as follows:

$$(r^{N-1}u')' + r^{N-1}K(r)f(u) = 0, \quad 0 < r < 1. \tag{2.1}$$

The proof of Theorem 1.1 is based on the method of Kolodner [3]. Namely we consider the solution $u(r,\alpha)$ of (1.1) satisfying the initial condition

$$u(0)=\alpha>0, \quad u'(0)=0, \tag{2.2}$$

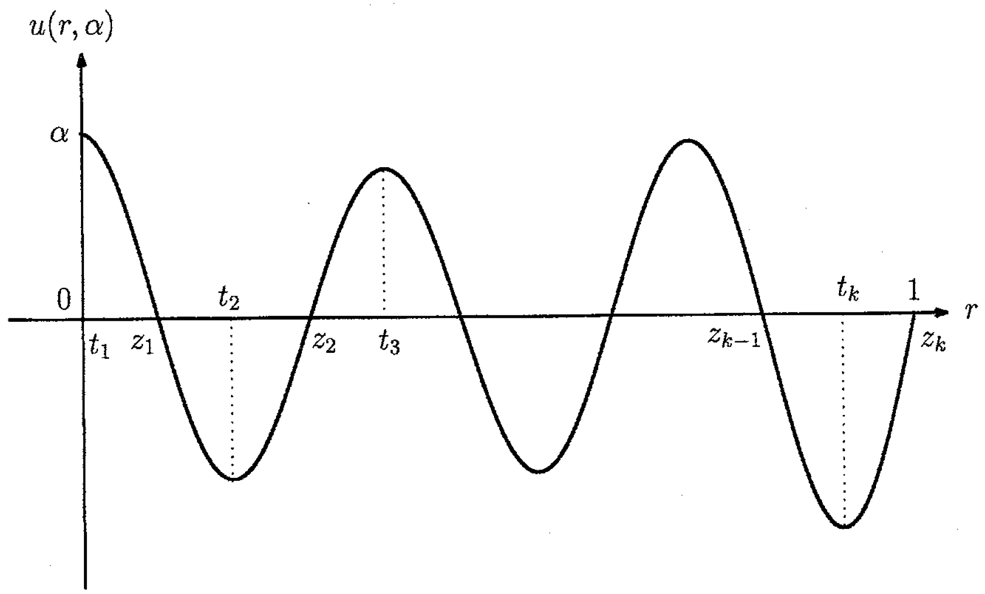
where $\alpha>0$ is a parameter. Since $K\in C^2[0,1]$ and $f\in C^1(\mathbf{R})$, we see that $u(r,\alpha)$ exists on $[0,1]$ is unique and satisfies $u$, $u'\in C^1([0,1]\times(0,\infty))$, and that $u_\alpha(r,\alpha)$ is a solution of linearized problem

$$\begin{cases} (r^{N-1}w')'+r^{N-1}K(r)f'(u(r,\alpha))w=0, & r\in(0,1], \\ w(0)=1, \quad w'(0)=0. \end{cases} \tag{2.3}$$

(See, for example, [5, §6 and 13].)

Hereafter we assume that $u(r,\alpha)$ is a solution of $(\mathrm{P}_k)$. Let $z_i$ be the $i$-th zero of $u(r,\alpha)$. Let $t_1=0$. For each $i\in\{2,3,\dots,k\}$, there exists $t_i\in(z_{i-1},z_i)$ such that $u'(t_i,\alpha)=0$, since $u(r,\alpha)(r^{N-1}u'(r,\alpha))'<0$ for $r\in(z_i,z_{i+1})$. Therefore we find that

$$0=t_1<z_1<t_2<z_2<\cdots<t_{k-1}<z_{k-1}<t_k<z_k=1,$$
$$u(z_i,\alpha)=0, \quad u'(t_i,\alpha)=0, \quad i=1,2,\dots,k,$$
$$u(r,\alpha)>0 \quad \text{for } r\in[t_1,z_1),$$
$$(-1)^i u(r,\alpha)>0 \quad \text{for } r\in(z_i,z_{i+1}), \quad i=1,2,\dots,k-1, \tag{2.4}$$
$$(-1)^i u'(r,\alpha)>0 \quad \text{for } r\in(t_i,t_{i+1}), \quad i=1,2,\dots,k-1, \tag{2.5}$$
$$(-1)^k u'(r,\alpha)>0 \quad \text{for } r\in(t_k,z_k]. \tag{2.6}$$




**Lemma 2.1.** *Assume that* (1.3) *holds. Let $w$ be the solution of* (2.3). *Then $w(r)>0$ for $x\in[0,z_1]$.*

*Proof.* Note that $w(0)=1$ and $w'(0)=0$. Assume to the contrary that there exists a number $r_1\in(0,z_1]$ such that $w(r)>0$ for $r\in[0,r_1)$ and $w(r_1)=0$. Then we see that $w'(r_1) < 0$. Let $u \equiv u(r,\alpha)$. An easy computation shows that

$$[r^{N-1}(w'u - wu')]' = r^{N-1}K(r)[f(u) - f'(u)u]w. \tag{2.7}$$

Recall that $u(r) > 0$ for $r \in [0, z_1)$. Integrating of (2.7) over $[0, r_1]$ and using (1.3), we have

$$r_1^{N-1}w'(r_1)u(r_1) = \int_0^{r_1} r^{N-1}K(r)[f(u) - f'(u)u]w\,dr > 0,$$

which implies $w'(r_1) > 0$. This is a contradiction. Consequently we find that $w(r) > 0$ for $r \in (0, z_1]$.

**Lemma 2.2.** *Assume that* (1.3) *holds. For each* $i \in \{1, 2, \dots, k-1\}$, *the solution* $w$ *of* (2.3) *has at most one zero in* $[z_i, z_{i+1}]$.

*Proof.* Note that $u \equiv u(r,\alpha)$ is a solution of

$$(r^{N-1}u')' + r^{N-1}K(r)\frac{f(u)}{u}u = 0, \quad r \in (z_i, z_{i+1})$$

and satisfies $u(z_i) = u(z_{i+1}) = 0$ and $u(r) \neq 0$ for $r \in (z_i, z_{i+1})$. From (1.3) it follows that

$$r^{N-1}K(r)f'(u) < r^{N-1}K(r)\frac{f(u)}{u}, \quad r \in (z_i, z_{i+1}).$$

Assume to the contrary that there exist numbers $r_0$ and $r_1$ such that $z_i \leq r_0 < r_1 \leq z_{i+1}$ and $w(r_0) = w(r_1) = 0$. Then Sturm's comparison theorem implies that $u$ has at least one zero in $(r_0, r_1)$. This is a contradiction. The proof is complete.

The following identity plays a crucial part in the proof of Theorem 1.1.

**Lemma 2.3.** *Let* $u \equiv u(r,\alpha)$ *and let* $w$ *be the solution of* (2.3). *Then*

$$\begin{aligned}\Big[r^{N-1}K^{-\frac{1}{2}}\big[w'u' - wu''\big] - r^{N-1}(K^{-\frac{1}{2}})'wu'\Big]' \\ = -\frac{r^{N-2}}{4K^{\frac{5}{2}}}\Big[3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2\Big]w\frac{u'}{r}.\end{aligned} \tag{2.8}$$

*for* $0 < r \leq 1$.

*Proof.* A direct calculation shows that (2.8) follows immediately.

**Remark 2.1.** We note that

$$u''(0,\alpha) = \lim_{r\to+0}\frac{u'(r,\alpha)}{r} = -\frac{K(0)f(\alpha)}{N}, \tag{2.9}$$

and hence, the right side of (2.8) is continuous for $0 \leq r \leq 1$. In fact, by integrating (2.1) over $[0, r]$, we see that

$$u'(r,\alpha) = -r^{-(N-1)}\int_0^r t^{N-1}K(t)f(u(t,\alpha))dt, \quad r \in [0,1],$$

so that

$$-\frac{r}{N}\max_{t\in[0,r]}K(t)f(u(t,\alpha)) \leq u'(r,\alpha) \leq -\frac{r}{N}\min_{t\in[0,r]}K(t)f(u(t,\alpha)), \quad r \in [0,1].$$

Then we obtain (2.9).

**Lemma 2.4.** *Assume that* (1.4) *holds. Then the solution $w$ of* (2.3) *has at least one zero in $(t_i, t_{i+1}]$ for each $i \in \{1, 2, \ldots, k-1\}$.*

*Proof.* Suppose that $w(r) \neq 0$ for $r \in (t_i, t_{i+1}]$. We may assume that $w(r) > 0$ for $r \in (t_i, t_{i+1}]$, since the case where $w(r) < 0$ for $r \in (t_i, t_{i+1}]$ can be treated similarly. Then we have $w(t_i) \geq 0$, $w(t_{i+1}) > 0$. In view of (1.1) we have

$$u''(t_j) = -K(t_j)f(u(t_j)), \quad j = 2, 3, \ldots, k.$$

From (2.4) and (2.9) it follows that $(-1)^j u''(t_j) > 0$ for $j = 1, 2, \ldots, k$. Consequently we have

$$(-1)^i\big(-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)\big) > 0,$$

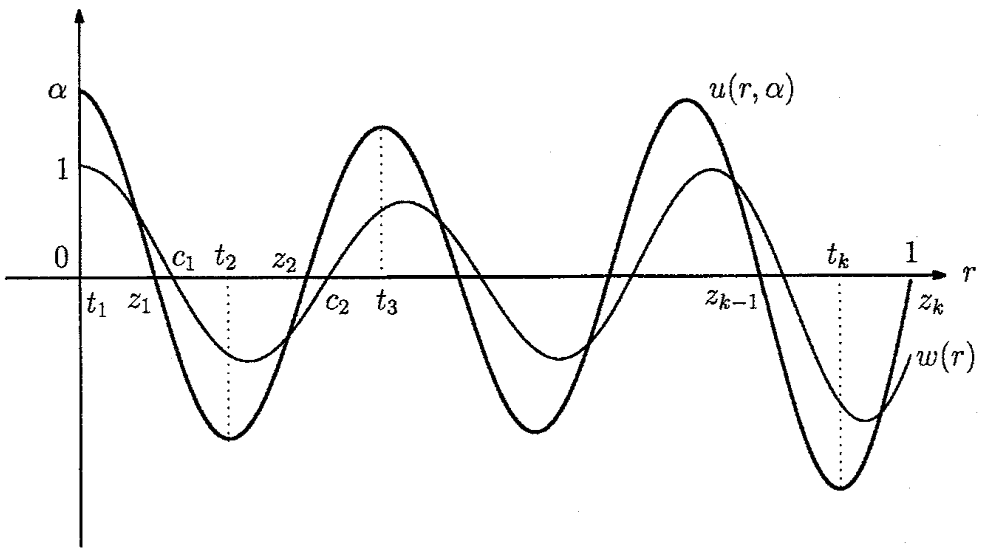
where $g(r) = r^{N-1}[K(r)]^{-\frac{1}{2}}$. On the other hand, integrating (2.8) over $[t_i, t_{i+1}]$ and using (1.4) and (2.5), we find that

$$(-1)^i\big(-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)\big) \leq 0.$$

This is a contradiction. The proof is complete.

**Lemma 2.5.** *Let $w$ be the solution of* (2.3). *Assume that* (1.3) *and* (1.4) *hold. Then $(-1)^i w(z_i) < 0$ for $i = 1, 2, \ldots, k$.*

*Proof.* Lemma 2.1 implies that $w(z_1) > 0$. By Lemmas 2.1 and 2.4, there exists a number $c_1 \in (z_1, t_2]$ such that $w(r) > 0$ for $r \in [0, c_1)$ and $w(c_1) = 0$. Then Lemma 2.2 implies that $w(r) < 0$ for $r \in (c_1, z_2]$. Hence we have $w(z_2) < 0$. From Lemma 2.4 it follows that there exists a number $c_2 \in (z_2, t_3]$ such that $w(r) < 0$ for $r \in (c_1, c_2)$ and $w(c_2) = 0$. By Lemma 2.2 we see that $w(r) > 0$ for $r \in (c_2, z_3]$, so that $w(z_3) > 0$. By continuing this process, we conclude that $(-1)^i w(z_i) < 0$ for $i = 1, 2, \ldots, k$. The proof is complete.



## 3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. To this end we employ the Prüfer transformation for the solution $u(r,\alpha)$ of problem (1.1)–(2.2). For the solution $u(r,\alpha)$ with $\alpha > 0$, we define the functions $\rho(r,\alpha)$ and $\theta(r,\alpha)$ by

$$u(r,\alpha) = \rho(r,\alpha)\sin\theta(r,\alpha),$$
$$r^{N-1}u'(r,\alpha) = \rho(r,\alpha)\cos\theta(r,\alpha),$$

where $' = d/dx$. Since $u(r,\alpha)$ and $u'(r,\alpha)$ cannot vanish simultaneously, $\rho(r,\alpha)$ and $\theta(r,\alpha)$ are written in the forms

$$\rho(r,\alpha) = \left([u(r,\alpha)]^2 + r^{2(N-1)}[u'(r,\alpha)]^2\right)^{\frac{1}{2}} > 0$$

and

$$\theta(r,\alpha) = \arctan\frac{u(r,\alpha)}{r^{N-1}u'(r,\alpha)},$$

respectively. Therefore, since $u$, $u' \in C^1([0,1]\times(0,\infty))$, we find that $\rho$, $\theta \in C^1([0,1]\times(0,\infty))$. From the initial condition (2.2) it follows that $\rho(0,\alpha) = \alpha$ and $\theta(0,\alpha) \equiv \pi/2 \pmod{2\pi}$. For simplicity we take $\theta(0,\alpha) = \pi/2$. By a simple calculation we see that

$$\theta'(r,\alpha) = \frac{1}{r^{N-1}}\cos^2\theta(r,\alpha) + r^{N-1}K(r)\frac{\sin\theta(r,\alpha)f(\rho(r,\alpha)\sin\theta(r,\alpha))}{\rho(r,\alpha)} > 0$$

for $r \in (0,1]$, which shows that $\theta(r,\alpha)$ is strictly increasing in $r \in (0,1]$ for each fixed $\alpha > 0$. It is easy to see that $u(r,\alpha)$ is a solution of $(\mathrm{P}_k)$ if and only if

$$\theta(1,\alpha) = k\pi, \tag{3.1}$$

Hence the number of solutions of $(\mathrm{P}_k)$ is equal to the number of roots $\alpha > 0$ of (3.1).

**Proposition 3.1.** *Let $k \in \mathbf{N}$ and let $u(r,\alpha_0)$ be a solution of $(\mathrm{P}_k)$ for some $\alpha_0 > 0$. Suppose that (1.3) and (1.4) hold. Then $\theta_\alpha(1,\alpha_0) < 0$.*

*Proof.* Observe that

$$\theta_\alpha(r,\alpha) = \frac{u_\alpha(r,\alpha)r^{N-1}u'(r,\alpha) - u(r,\alpha)r^{N-1}u'_\alpha(r,\alpha)}{[u(r,\alpha)]^2 + [u'(r,\alpha)]^2}.$$

Since $u(1,\alpha_0) = 0$ and $z_k = 1$, we obtain

$$\theta_\alpha(1,\alpha_0) = \frac{u_\alpha(z_k,\alpha_0)}{u'(z_k,\alpha_0)}.$$

Note that $(-1)^k u'(z_k,\alpha_0) > 0$, because of (2.6). From Lemma 2.5, it follows that $(-1)^k u_\alpha(z_k,\alpha_0) < 0$, which implies that $\theta_\alpha(1,\alpha_0) < 0$. The proof is complete.

*Proof of Theorem 1.1.* Assume to the contrary that there exist numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $u(r,\alpha_1)$ and $u(r,\alpha_2)$ are solutions of $(\mathrm{P}_k)$ and $\alpha_1 \neq \alpha_2$. Then $\theta(1,\alpha_1) = \theta(1,\alpha_2) = k\pi$. We may assume without loss of generality that $0 < \alpha_1 < \alpha_2$ and $\theta(1,\alpha) \neq k\pi$ for $\alpha \in (\alpha_1,\alpha_2)$. In view of Proposition 3.1, we conclude that $\theta_\alpha(1,\alpha_1) < 0$ and $\theta_\alpha(1,\alpha_2) < 0$. The intermediate value theorem implies that there is a number $\alpha_0 \in (\alpha_1,\alpha_2)$ such that $\theta(1,\alpha_0) = k\pi$. This is a contradiction. Consequently, $(\mathrm{P}_k)$ has at most one solution. The proof of Theorem 1.1 is complete.

## References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