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Optimal control problems for the viscoelastic equations with long nonlinear memory

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1 Introduction

Let Ω be an open and bounded set in $\mathbf{R}^n$ with sufficiently smooth boundary Γ. We will study optimal control problems (cf. Lions [6]) for the second order integro-differential equation with a nonlinear kernel. This may give just abstract meaning, so we will represent more concretive model which is the following simplified viscoelastic system with long memory:

$$\begin{cases} \dfrac{\partial^2 y(v)}{\partial t^2} - \alpha\Delta y(v) - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(v;s)}{\sqrt{1+|\nabla y(v;s)|^2}}\Big)ds = Bv + f \quad \text{in} \quad Q = (0,T)\times\Omega, \\ y(v) = 0 \qquad \text{on} \quad \Sigma = (0,T)\times\Gamma, \\ y(v;0) = y_0 \in H_0^1(\Omega), \quad \dfrac{\partial y}{\partial t}(v;0) = y_1 \in L^2(\Omega) \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\alpha > 0, f \in L^2(0,T;L^2(\Omega))$ are fixed, integral kernel $k(\cdot) \in C^1[0,T]$ and B is a controller such that

$$B \in \mathcal{L}(\mathcal{U}; L^2(0,T;L^2(\Omega))),$$

where $\mathcal{U}$ is a Hilbert space of control variables. Especially it is meaningful that α represents the velocity of deformation. The integral kernel $k(\cdot)$ in (1.1) represents the fading rate of memory effect. And for the background of the nonlinear term of (1.1), we refer to [2] and [5]. The purpose of this paper is to solve the quadratic cost optimal control problems for (1.1) by giving the existence and the necessary conditions on optimal controls.

The well posedness of less regular solutions, called the weak solutions of (1.1) is proved in the framework of variational method in Dautray and Lions [1] under Dirichlet boundary conditions. This result enables us to study the optimal control problems associated with (1.1) in the standard manner due to the theory of Lions [6]. The main contribution of this paper is to establish the necessary conditions of optimality for distributive and terminal value observation cases by transposition method. For this we prove the Gâteaux differentiability of the nonlinear mapping $v \to y(v)$, which is used to define the associate adjoint system.

2 Solutions of viscoelastic system with long nonlinear memory

We consider the following Dirichlet boundary value problem for the equation of viscoelastic equation with long nonlinear memory:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \alpha\Delta y - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(s)}{\sqrt{1+|\nabla y(s)|^2}}\Big)ds = f \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \Sigma, \\ y(0) = y_0, \qquad \dfrac{\partial y(0)}{\partial t} = y_1 \quad \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $\alpha > 0$ and f is a external forcing term. We will state the notations used in this paper. We give scalar products and norms on $L^2(\Omega)$, $H_0^1(\Omega)$ as $(\phi,\psi)_{L^2(\Omega)} = (\phi,\psi) = \int_\Omega \phi\psi dx$, $|\phi| = (\phi,\psi)^{\frac{1}{2}}$, $(\phi,\psi)_{H_0^1(\Omega)} = (\nabla\phi,\nabla\psi)$, $\|\phi\|_{H_0^1(\Omega)} = |\nabla\phi|$. Related to the nonlinear term in (2.1), we define the function $G : \mathbf{R}^n \to \mathbf{R}^n$ by $G(x) = \frac{x}{\sqrt{1+|x|^2}}$, $x \in \mathbf{R}^n$. Then it is vertified that

$$|G(x) - G(y)| \le 2|x-y|, \quad \forall x, y \in \mathbf{R}^n. \tag{2.2}$$

The nonlinear operator $G(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$ is introduced by

$$G(\nabla\phi)(x) = \frac{\nabla\phi}{\sqrt{1+|\nabla\phi(x)|^2}}, \quad \text{a.e. } x \in \Omega, \quad \forall\phi \in H_0^1(\Omega). \tag{2.3}$$

We have the following properties on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi) - G(\nabla\psi)| \le 2|\nabla\phi - \nabla\psi|, \quad \forall\phi,\psi \in H_0^1(\Omega). \tag{2.4}$$

Definition 2.1 A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot),\phi\rangle + \alpha(\nabla y(\cdot),\nabla\phi) + (k * G(\nabla y)(\cdot),\nabla\phi) = (f(\cdot),\phi) \\ \qquad\qquad\qquad\qquad \text{for all } \phi \in H_0^1(\Omega) \text{ in the sense of } \mathcal{D}'(0,T) \\ y(0) = y_0 \in H_0^1(\Omega), \qquad y'(0) = y_1 \in L^2(\Omega), \end{cases} \tag{2.5}$$

where

$$W(0,T) = \{g | g \in L^2(0,T;H_0^1(\Omega)), g' \in L^2(0,T;L^2(\Omega)), g'' \in L^2(0,T;H^{-1}(\Omega))\}$$

with norm

$$\|g\|_{W(0,T)} = (\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;L^2(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))})^{\frac{1}{2}},$$

where g' and g'' denote the first and second order distributive derivatives of g.

Theorem 2.1 Assume that $f \in L^2(0,T;L^2(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Then the problem (2.1) has a unique weak solution y in $W(0,T)$. Further we have the regularity $y \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$.

Proof. For the proof of this theorem we use the Galerkin finite approximation method and the technique of regularization based on the energy equality. Since $H_0^1(\Omega)$ is separable, there exists a basis $\{w_m\}_{m=1}^\infty$ in $H_0^1(\Omega)$ such that

(i) $\{w_m\}_m^\infty$ is a complete orthonomal system in $L^2(\Omega)$,

(ii) $\{w_m\}_{m=1}^\infty$ is free and total in $H_0^1(\Omega)$.

For each $m = 1, 2, \cdots$ we define an approximate solution of the equation (2.1) by

$$y_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j,$$

where $y_m(t)$ satisfies

$$\begin{cases} (y_m''(t), w_j) + \alpha(\nabla y_m(t), \nabla w_j) + (k * G(\nabla y_m)(t), \nabla w_j) \\ = (f(t), w_j), \quad t \in [0, T], \quad 1 \le j \le m, \\ y_m(0) = y_{0m}, \qquad y_m'(0) = y_{1m}. \end{cases} \tag{2.6}$$

By (i) and (ii) we can deduce for $i = 1, 2, \cdots, m, m \in N$ such that

$$y_{0m} = \sum_{i=1}^{m} (y_0, w_i) w_i \to y_0 \quad \text{in } H_0^1(\Omega) \quad \text{as } m \to \infty, \tag{2.7}$$

$$y_{1m} = \sum_{i=1}^{m} (y_1, w_i) w_i \to y_1 \quad \text{in } L^2(\Omega) \quad \text{as } m \to \infty. \tag{2.8}$$

To derive a priori estimates of $y_m(t)$. We multiply both sides of the equation (2.6) by $g_{jm}'(t)$ and sum over j to have

$$\begin{cases} (y_m''(t), y_m'(t)) + \alpha(\nabla y_m(t), \nabla y_m'(t)) \\ +(k * G(\nabla y_m)(t), \nabla y_m'(t)) = (f(t), y_m'(t)). \end{cases} \tag{2.9}$$

Using

$$\begin{cases} (y_m''(t), y_m'(t)) = \dfrac{1}{2}\dfrac{d}{dt}|y_m'(t)|^2, \\ \alpha(\nabla y_m(t), \nabla y_m'(t)) = \dfrac{1}{2}\dfrac{d}{dt}\alpha(\nabla y_m(t), \nabla y_m(t)), \\ (k * G(\nabla y_m)(t), \nabla y_m'(t)) = \dfrac{d}{dt}(k * G(\nabla y_m)(t), \nabla y_m(t)) \\ -k(0)(G(\nabla y_m(t)), \nabla y_m(t)) - (k' * G(\nabla y_m)(t), \nabla y_m(t)), \end{cases} \tag{2.10}$$

(2.9) can be written as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\alpha(\nabla y_m(t), \nabla y_m(t)) + |y_m'(t)|^2 + 2(k * G(\nabla y_m)(t), \nabla y_m(t))] \\ &= (f(t), y_m'(t)) + k(0)(G(\nabla y_m(t)), \nabla y_m(t)) \\ &+(k' * G(\nabla y_m)(t), \nabla y_m(t)). \end{aligned} \tag{2.11}$$

Let us integrate it on $[0, t)$ then we have

$$\begin{aligned} &\alpha|\nabla y_m(t)|^2 + |y_m'(t)|^2 \\ = \ &\alpha|\nabla y_{0m}|^2 + |y_{1m}|^2 - 2(k * G(\nabla y_m)(t), \nabla y_m(t)) \\ &+2\int_0^t (f(s), y_m'(s)) ds + 2\int_0^t k(0)(G(\nabla y_m(s)), \nabla y_m(s)) ds \\ &+2\int_0^t (k' * G(\nabla y_m)(s), \nabla y_m(s)) ds, \end{aligned} \tag{2.12}$$

and estimate it to obtain a priori estimates of $\{y_m\}$. Let $\epsilon > 0$ be an arbitrary positive real number. First we have

$$\begin{aligned}\Big|(2k * G(\nabla y_m)(t), \nabla y_m(t))\Big| &\leq 2k_0|\nabla y_m(t)| \int_0^t |\nabla y_m(s)|ds \\ &\leq \epsilon|\nabla y_m(t)|^2 + \frac{4k_0^2 T}{\epsilon} \int_0^t |\nabla y_m(s)|^2 ds. \end{aligned} \tag{2.13}$$

We can deduce:

$$\begin{cases} \Big|\int_0^t (2k' * G(\nabla y_m)(s), \nabla y_m(s))ds\Big| \leq 2k_1\Big(\int_0^t |\nabla y_m(s)|ds\Big)^2, \\ \Big|\int_0^t 2k(0)(G(\nabla y_m(s)), \nabla y_m(s))ds\Big| \leq 2k_0 \int_0^t |\nabla y_m(s)|^2 ds. \end{cases} \tag{2.14}$$

Therefore by using the above inequalities (2.13) and (2.14), we can obtain the following

$$\begin{aligned}|y_m'(t)|^2 + \alpha|\nabla y_m(t)|^2 &\leq |y_{1m}|^2 + \alpha|\nabla y_{0m}|^2 + \epsilon|\nabla y_m(t)|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ (2k_0 + \frac{4k_0^2 T}{\epsilon} + 2k_1 T) \int_0^t |\nabla y_m(s)|^2 ds \\ &+ 2\int_0^t |y_m'(s)|^2 ds. \end{aligned} \tag{2.15}$$

Thus it follows by the Bellman-Gronwall's inequality that

$$|\nabla y_m(t)|^2 + |y_m'(t)|^2 \leq K, \tag{2.16}$$

for some positive constant $K > 0$.

Hence by the extraction theorem of Rellich's, we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find $z \in L^\infty(0, T; H_0^1(\Omega))$, $z' \in L^\infty(0, T; L^2(\Omega))$ and $F(\cdot) \in L^\infty(0, t; [L^2(\Omega)]^n)$ such that

$$\begin{aligned} &\text{i)}\quad y_{m_k} \to z \ \text{ weakly-star in } L^\infty(0, T; H_0^1(\Omega)), \\ &\qquad \text{and weakly in } L^2(0, T; H_0^1(\Omega)), \\ &\text{ii)}\quad y_{m_k}' \to z' \ \text{ weakly-star in } L^\infty(0, T; L^2(\Omega)), \\ &\qquad \text{and weakly in } L^2(0, T; L^2(\Omega)), \\ &\text{iii)}\quad \alpha\Delta y_{m_k} \to \alpha\Delta z \ \text{ weakly in } L^2(0, T; H^{-1}(\Omega)), \\ &\qquad \text{weakly-star in } L^\infty(0, t; H^{-1}(\Omega)), \\ &\text{iv)}\quad G(\nabla y_m) \to F(\cdot) \text{ weakly-star in } L^\infty(0, t; [L^2(\Omega)]^n), \\ &\qquad \text{and weakly in } L^2(0, t; [L^2(\Omega)]^n), \end{aligned} \tag{2.17}$$

as $k \to \infty$. Therefore by the same manipulations of Dautray and Lions [1], z is a weak solution satisfying

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \alpha\Delta z - \displaystyle\int_0^t k(t-s)\mathrm{div}F(s)ds = f \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0) = y_0 \in H_0^1(\Omega), \qquad \dfrac{\partial z(0)}{\partial t} = y_1 \in L^2(\Omega) \quad \text{in } \Omega. \end{cases} \tag{2.18}$$

The main difficulty is showing $F(\cdot) = G(\nabla z)$. We shall show

$$\begin{aligned} y_m(t) &\to z(t) \quad \text{strongly in } H_0^1(\Omega) && (2.19)\\ y'_m(t) &\to z'(t) \quad \text{strongly in } L^2(\Omega). && (2.20) \end{aligned}$$

Integrating an approximate equation (2.11) on $[0,t]$, we obtain

$$\begin{aligned} &\alpha|\nabla y_m(t)|^2 + |y'_m(t)|^2 \\ &= \alpha|\nabla y_{0m}|^2 + |y_{1m}|^2 - 2(k * G(\nabla y_m)(t), \nabla y_m(t)) \\ &+2\int_0^t k(0)(G(\nabla y_m(s)), \nabla y_m(s))ds \\ &+2\int_0^t (k' * G(\nabla y_m)(s), \nabla y_m(s))ds + 2\int_0^t (f(s), y'_m(s))ds. \qquad (2.21) \end{aligned}$$

As shown in the [4], for the weak solution z of (2.18), we can show the following energy equality

$$\begin{aligned} &\alpha|\nabla z(t)|^2 + |z'(t)|^2 \\ &= \alpha|\nabla y_0|^2 + |y_1|^2 - 2(k * F(t), \nabla z(t)) + 2\int_0^t (k(0)F(s), \nabla z(s))ds \\ &+2\int_0^t (k' * F(s), \nabla z(s))ds + 2\int_0^t (f(s), z'(s))ds. \qquad (2.22) \end{aligned}$$

If we sum the above equalities, then we have

$$\begin{aligned} &\alpha|\nabla(y_m(t) - z(t))|^2 + |y'_m(t) - z'(t)|^2 \qquad (2.23)\\ &= \sum_{i=1}^{6} \Phi_m^i(t) + \alpha|\nabla(y_{0m} - y_0)|^2 + |y_{1m} - y_1|^2 \\ &-2(k * \Big(G(\nabla y_m) - G(\nabla z)\Big)(t), \nabla(y_m(t) - z(t))) \\ &+2\int_0^t k(0)(G(\nabla y_m(s)) - G(\nabla z(s)), \nabla(y_m(s) - z(s)))ds \\ &+2\int_0^t (k' * \Big(G(\nabla y_m) - G(\nabla z)\Big)(s), \nabla(y_m(s) - z(s)))ds, \qquad (2.24) \end{aligned}$$

where

$$\begin{aligned} \Phi_m^1 &= 2\alpha(\nabla y_{0m}, \nabla y_0) + 2(y_{1m}, y_1), && (2.25)\\ \Phi_m^2 &= -2\alpha(\nabla y_m(t), \nabla z(t)) - 2(y'_m(t), z'(t)), && (2.26)\\ \Phi_m^3 &= 2\int_0^t (f(s), z'(s))ds + 2\int_0^t (f(s), y'_m(s))ds, && (2.27)\\ \Phi_m^4 &= -2(k * G(\nabla y_m)(t), \nabla z(t)) \\ &\quad -2(k * G(\nabla z)(t), \nabla(y_m(t) - z(t))) - 2(k * F(t), \nabla z(t)), && (2.28)\\ \Phi_m^5 &= 2\int_0^t k(0)(G(\nabla y_m(s)), \nabla z(s))ds \\ &\quad +2\int_0^t k(0)(G(\nabla z(s)), \nabla(y_m(s) - z(s)))ds \\ &\quad +2\int_0^t k(0)(F(s), \nabla z(s))ds, && (2.29) \end{aligned}$$

$$\Phi_m^6 = 2\int_0^t (k' * G(\nabla y_m)(s), \nabla z(s))ds$$
$$+2\int_0^t (k' * G(\nabla z)(s), \nabla(y_m(s) - z(s)))ds + 2\int_0^t (k' * F(s), \nabla z(s))ds. \quad (2.30)$$

For simplicity we set

$$\Phi_m(t) = \sum_{i=1}^{6} \Phi_m^i(t).$$

Then we can derive the following estimation

$$\begin{aligned} & |\nabla(y_m(t) - z(t))|^2 + |y_m'(t) - z'(t)|^2 \\ \leq \;& C\Phi_m(t) + C(|\nabla(y_{0m} - y_0)|^2 + |y_{1m} - y_1|^2) \\ & +C\int_0^t (|\nabla(y_m(s) - z(s))|^2 + |y_m'(s) - z'(s)|^2)ds, \end{aligned} \quad (2.31)$$

where C is some positive constant. It is followed from (2.25) to (2.30) and ultimately from (2.22) that

$$\Phi_m(t) \to 0 \quad \text{when} \quad m \to \infty \quad \text{for all} \quad t \in [0, T].$$

Therefore by applying the Bellmann Gronwall's lemma to (2.31), we can obtain

$$y_m(t) \;\to\; z(t) \;\text{ strongly in } H_0^1(\Omega) \;\text{ for all }\; t \in [0, T], \quad (2.32)$$
$$y_m'(t) \;\to\; z'(t) \;\text{ strongly in } L^2(\Omega) \;\text{ for all }\; t \in [0, T]. \quad (2.33)$$

Moreover by (2.32) and (2.4), it is readily followed that

$$F(\cdot) = G(\nabla z). \quad (2.34)$$

This proves that z is a weak solution of (2.1). The uniqueness of weak solutions follows by the standard manner using the energy equality (2.22) with (2.34).

3 Quadratic cost optimal control problems

The observation of the state is assumed to be given by $z(v) = Cy(v)$, $C \in \mathcal{L}(W(0,T), M)$, where C is an operator called the observer, and M is a Hilbert space of observation variables. The quadratic cost function associated with the control system (1.1) is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}} \quad \text{for } v \in \mathcal{U}, \quad (3.1)$$

where $z_d \in M$ is a desired value of $z(v)$ and $R \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is a regulator satisfying the symmetry and positivity. Assume that an admissible subset $\mathcal{U}_{ad}$ of $\mathcal{U}$ is convex and closed. The optimal control problems for (1.1) subject to the cost (3.1) are the following the existence and characterizations of them.

i) Find an element $u \in \mathcal{U}_{ad}$ such that

$$\inf_{v \in \mathcal{U}_{ad}} J(v) = J(u). \quad (3.2)$$

ii) Give a characterization of such the u.

We shall call such the u the optimal control and $y(u)$ the optimal state.

3.1 Gâteaux differentiability of solution map

We assume the existence of an optimal control u for the cost (3.1). For example, if $B \in \mathcal{L}(\mathcal{U}, L^2(0,T;L^2(\Omega))$ is a compact operator, then there exists at least one optimal control u.

In order to solve the characterization problem ii) we need to show that the map $v \to y(v)$ of $\mathcal{U} \to W(0,T)$ is Gâteaux differentiable at $v = u$. The solution map $v \to y(v)$ of $\mathcal{U} \to W(0,T)$ is said to be Gâteaux differentiable if any $w \in \mathcal{U}$ there exists a $Dy(w) \in \mathcal{L}(\mathcal{U}, W(0,T))$ such that

$$\left\| (y(w + \lambda(v-w)) - y(w)) - Dy(w)(v-w) \right\|_{W(0,T)} \to 0 \quad \text{as} \quad \lambda \to 0.$$

The operator $Dy(w)$ is called the Gâteaux derivative of $y(w)$ at $v = w$ and the function $Dy(w)(v-w) \in W(0,T)$ is called the Gâteaux derivative in the direction $v - w \in \mathcal{U}$.

Theorem 3.1 The solution map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$ is Gâteaux differentiable at $v = u$ and such the Gâteaux derivative of $y(v)$ at $v = u$ in the direction $v - u \in \mathcal{U}$, say $z = Dy(u)(v-u)$ is a unique weak solution satisfying the following equation

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \alpha\Delta z - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla z}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla z}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = B(v-u) \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(0) = 0, \quad \dfrac{\partial z}{\partial t}(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.3}$$

Theorem 3.1 means that the cost $J(v)$ is Gâteaux differentiable at u in the direction $v - u$ and the optimality condition is rewritten by

$$\begin{aligned} & (Cy(u) - z_d, C(Dy(u)(v-u)))_M + (Ru, v-u)_{\mathcal{U}} \\ = \; & \langle C^*\Lambda_M(Cy(u) - z_d), Dy(u)(v-u)\rangle_{W(0,T)', W(0,T)} \\ & + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \end{aligned} \tag{3.4}$$

where Λ_M is the canonical isomorphism M onto M'.

In deriving the optimality condition by formal calculations, we will derive some adjoint system for the above observation. In this case, the formal adjoint system must have the term of

$$\int_t^T k(s-t)\mathrm{div}\Big(\frac{\nabla p(u;s)}{\sqrt{1+|\nabla y(u;t)|^2}} - \nabla y(u;t)\frac{\nabla y(u;t)\cdot\nabla p(u;s)}{(1+|\nabla y(u;t)|^2)^{\frac{3}{2}}}\Big)ds,$$

where the integral kernel is not differentiable in t. So that we can not verify the existence of a weak solution for the formal adjoint system. To overcome this difficulty, we will introduce transposition method to represent a proper adjoint system.

3.2 Transposition method

Let $g \in L^2(0,T;L^2(\Omega))$. Then we have a unique weak solution $\psi \in W(0,T)$ of the following equation

$$
\begin{cases}
\dfrac{\partial^2 \psi}{\partial t^2} - \alpha\Delta\psi - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla\psi}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla\psi}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = g \quad \text{in } Q, \\
\psi = 0 \quad \text{on } \Sigma, \\
\psi(0) = 0, \quad \dfrac{\partial\psi}{\partial t}(0) = 0 \quad \text{in } \Omega.
\end{cases}
\tag{3.5}
$$

Therefore we can define the space

$$X \equiv \{\psi | \psi \text{ satisfies (3.5) with } g \in L^2(0,T;L^2(\Omega))\}.$$

It is seen in Theorem 2.1 that $X \subset W(0,T) \cap C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. We give a inner product $(\cdot,\cdot)_X$ on X by

$$(\psi_1,\psi_2)_X = (g_1,g_2)_{L^2(0,T;L^2(\Omega))}, \tag{3.6}$$

where ψ_1,ψ_2 are the weak solutions of (3.5) for given $g = g_1, g_2 \in L^2(0,T;L^2(\Omega))$, respectively. We can know that $(X,(\cdot,\cdot)_X)$ is a Hilbert space. And we can see that the map

$$\mathcal{T} : \psi \to \frac{\partial^2\psi}{\partial t^2} - \alpha\Delta\psi - \int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla\psi}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla\psi}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \tag{3.7}$$

of X onto $L^2(0,T;L^2(\Omega))$ is an isomorphism. Hence for each continuous linear functional $L : X \to \mathbf{R}$, there exists uniquely a $p = p_L \in L^2(0,T;L^2(\Omega))$ such that

$$\int_0^T (p(t),\mathcal{T}\psi(t))dt = L(\psi), \quad \forall\psi \in X. \tag{3.8}$$

For $g \in L^1(0,T;H^{-1}(\Omega))$, $p_0 \in L^2(\Omega)$ and $p_1 \in H^{-1}(\Omega)$, let us define the functional $L = L(g,p_0,p_1)$ by

$$L(\psi) = \int_0^T \langle g(t),\psi(t)\rangle dt + \langle p_1,\psi(T)\rangle - (p_0,\psi'(T)). \tag{3.9}$$

Then this L is linear on X. Next we shall show the boundedness of L. It is easily checked from the fact $\psi \in X \subset C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ that

$$|L(\psi)| \le (\|g\|_{L^1(0,T;H^{-1}(\Omega))} + \|p_1\|_{H^{-1}(\Omega)} + |p_0|)(\|\psi\|_{C([0,T];H_0^1(\Omega))} + |\nabla\psi(T)| + |\psi'(T)|).$$

Proposition 3.1 For $g \in L^1(0,T;H^{-1}(\Omega))$, $p_0 \in L^2(\Omega)$ and $p_1 \in H^{-1}(\Omega)$, there is a unique solution $p \in L^2(0,T;L^2(\Omega))$ such that

$$
\begin{cases}
\displaystyle\int_0^T (p(t),\mathcal{T}\psi(t))dt \\
\displaystyle = \int_0^T \langle g(t),\psi(t)\rangle dt + \langle p_1,\psi(T)\rangle - (p_0,\psi'(T)), \quad \forall\psi \in X.
\end{cases}
$$

3.3 Necessary optimality conditions

We consider the following type of distributive and terminal value observations. For simplicity, we regard identity operator I_d as observation operator and $\mathcal{U}_{ad} \subset \mathcal{U}$. We take

$$I_d \in \mathcal{L}(L^2(0,T;L^2(\Omega)) \times L^2(\Omega), L^2(0,T;L^2(\Omega)) \times L^2(\Omega))$$
$$\text{and observe } z(v) = (y(v), y(v;T)).$$

Since $y \in W(0,T) \cap C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, the above observation is meaningful. In this case the cost functional is expressed by

$$J(v) = \|y(v) - z_{d1}\|^2_{L^2(0,T;L^2(\Omega))} + \|y(v;T) - z_{d2}\|^2_{L^2(\Omega)} + (Rv, v)_{\mathcal{U}}, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.10}$$

where $(z_{d1}, z_{d2}) \in L^2(0,T;L^2(\Omega)) \times L^2(\Omega)$ are desired values. Let u be the optimal control for the cost (3.10). Then the optimality condition is rewritten as

$$\int_0^T ((y(u) - z_{d1})(t), z(t))dt + (y(u;T) - z_{d2}, z(T)) + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.11}$$

where z is the weak solution of the equation (3.3). Now we will formulate the adjoint system to describe the optimality condition by applying Proposition 3.1. Since $y(u) - z_{d1} \in L^2(0,T;L^2(\Omega)) \subset L^1(0,T;H^{-1}(\Omega))$ and $y(u;T) - z_{d2} \in L^2(\Omega)$. There exists a $p(u) \in L^2(0,T;L^2(\Omega))$ satisfying

$$\begin{cases} \displaystyle\int_0^T \Big(p(u;t), \mathcal{T}\psi(t)\Big)dt = \int_0^T ((y(u) - z_{d1})(t), \psi(t))dt + (y(u;T) - z_{d2}, \psi(T)), \\ \forall \psi \ \text{ such that } \ \mathcal{T}\psi \in L^2(0,T;L^2(\Omega)), \\ \psi = 0 \quad \text{ on } \Sigma, \\ \psi(0) = 0, \quad \dfrac{\partial \psi}{\partial t}(0) = 0 \quad \text{ in } \Omega. \end{cases}$$

In fact the Gâteaux derivative $\psi = z = Dy(u)(v-u)$ satisfies

$$\begin{cases} \mathcal{T}\psi = \dfrac{\partial^2 \psi}{\partial t^2} - \alpha \Delta \psi - \displaystyle\int_0^t k(t-s) \mathrm{div}\Big(\frac{\nabla \psi}{\sqrt{1 + |\nabla y(u)|^2}} - \nabla y(u) \frac{\nabla y(u) \cdot \nabla \psi}{(1 + |\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \\ \quad = B(v-u) \in L^2(0,T;L^2(\Omega)), \\ \psi(0) = 0, \quad \dfrac{\partial \psi}{\partial t}(0) = 0 \quad \text{ in } \Omega. \end{cases} \tag{3.12}$$

Therefore, if we taking $\psi = z = Dy(u)(v-u)$, then we have

$$\begin{aligned} & (y(u) - z_{d1}, z)_{L^2(0,T;L^2(\Omega))} + (y(u;T) - z_{d2}, z(T)) \\ = \ & \int_0^T ((y(u) - z_{d1})(t), z(t))dt + (y(u;T) - z_{d2}, z(T)) \\ = \ & \int_0^T \Big(p(u;t), \mathcal{T}\psi(t)\Big)dt \\ = \ & \int_0^T (p(u;t), B(v-u)(t))\, dt = (\Lambda_{\mathcal{U}}^{-1} B^* p(u), v-u)_{\mathcal{U}}. \end{aligned}$$

Therefore we can conclude that the optimality condition is equivalent to

$$(\Lambda_{\mathcal{U}}^{-1} B^* p(u) + Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

Hence we can show the following theorem.

Theorem 3.2 The optimal control u for the cost (3.10) is characterized by the following system of equations and inequality:

$$\begin{cases} \dfrac{\partial^2 y(u)}{\partial t^2} - \alpha\Delta y(u) - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(u)}{\sqrt{1+|\nabla y(u)|^2}}\Big)ds = f + Bu \quad \text{in } Q, \\ y(u) = 0 \quad \text{on } \Sigma, \\ y(u;0) = y_0(u), \quad \dfrac{\partial y}{\partial t}(u;0) = y_1(u) \quad \text{in } \Omega. \end{cases}$$

$$\begin{cases} \displaystyle\int_Q p(u)\cdot\Big(\frac{\partial^2\psi}{\partial t^2} - \alpha\Delta\psi \\ \qquad - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla\psi}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla\psi}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds\Big)dxdt \\ = \displaystyle\int_Q (y(u)-z_{d1})\cdot z dxdt + \int_\Omega (y(u;T)-z_{d2})\cdot z(T)dx, \\ \forall\psi \text{ such that} \\ \dfrac{\partial^2\psi}{\partial t^2} - \alpha\Delta\psi - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla\psi}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla\psi}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \\ \quad \in L^2(Q), \\ \psi = 0 \quad \text{on } \Sigma, \\ \psi(0) = 0, \quad \dfrac{\partial\psi}{\partial t}(0) = 0 \quad \text{in } \Omega. \end{cases}$$

$$(\Lambda_{\mathcal{U}}^{-1}B^*p(u) + Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

Finally we note that the adjoint state p satisfies formally

$$\begin{cases} \dfrac{\partial^2 p(u)}{\partial t^2} - \alpha\Delta p(u) \\ - \displaystyle\int_t^T k(s-t)\mathrm{div}\Big(\frac{\nabla p(u;s)}{\sqrt{1+|\nabla y(u;t)|^2}} - \nabla y(u;t)\frac{\nabla y(u;t)\cdot\nabla p(u;s)}{(1+|\nabla y(u;t)|^2)^{\frac{3}{2}}}\Big)ds \\ \qquad\qquad = y(u) - z_{d1} \quad \text{in } Q, \\ p(u) = 0 \quad \text{on } \Sigma, \\ p(u;T) = 0, \quad \dfrac{\partial p}{\partial t}(u;T) = -y(u;T) + z_{d2} \quad \text{in } \Omega. \end{cases}$$

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