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Existence of positive solutions of some singular two-point boundary value problems

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In this talk we treat the two-point boundary value problem of the form

$$(|y'|^{\alpha-1}y')' + f(t,y) = 0, \quad 0 < t < 1; \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $\alpha \geq 1$ is a constant. As will be seen below, $f(t,y)$ is allowed to take ∞ at $t = 0, 1$ and at $y = 0$. In this sense this problem may be called singular problem. The main objective here is to show the existence of positive solutions of BVP(1)-(2) under suitable assumptions. When $\alpha = 1$, such problems have been studied by [1],[2],[4]. In the case of $\alpha > 1$, a sufficient condition for the existence of positive solutions has been given by [6] from different point of view from ours presented here.

We always assume the following conditions throughout the talk:

(C1) $f \in C((0,1) \times (0,\infty); (0,\infty))$;

(C2) there are functions $a \in C((0,1);(0,\infty)), g \in C((0,\infty);(0,\infty))$ and $h \in C((0,\infty);(0,\infty))$ such that

(a) $0 < f(t,y) \leq a(t)g(y)h(y)$ in $(0,1) \times (0,\infty)$; and

(b) g is nonincreasing, while $yg(y)$ is nondecreasing;

(C3) there are constants $p, q > 0$ with $1/p + 1/q = 1$ such that

$$\int_0 h(u)^p du < \infty, \quad \int_0^1 a(t)^q dt < \infty.$$

(C4) for every constants $c_1, c_2 > 0$, the set

$$\left\{ z > 0 \,\middle|\, \int_0^z g(u)^{-\frac{p}{p\alpha+1}} du \leq c_1 \left(\int_0^z h(u)^p \right)^{\frac{1}{p\alpha+1}} + c_2 \right\}$$

is bounded.

(C5) for every constant $M > 0$, there is a function $\psi_M \in C((0,1);(0,\infty))$ satisfying

$$f(t,y) \geq \psi_M(t) \quad \text{in } (0,1) \times (0,M).$$

□

Our result is as follows:

Theorem 1. Under assumptions $(C1) - (C5)$ BVP (1)-(2) has a positive solution y satisfying $y \in C[0,1] \cap C^1(0,1)$, $|y'|^{\alpha-1}y' \in C^1(0,1)$.

Example 1. Let us consider the BVP

$$(|y'|^{\alpha-1}y')' + a_1(t)y^{\beta} + a_2(t)y^{-\gamma} = 0,$$

$$y(0) = y(1) = 0$$

where $\beta, \gamma > 0$ are constants and $a_i \in C((0,1);(0,\infty)), i = 1,2$. Put $a(t) = \max\{a_1(t), a_2(t)\}$. If $0 < \beta < \alpha$, $0 < \gamma < 1 + 1/p$ and

$$\int_0^1 a(t)^q dt < \infty, \quad q = \frac{p}{p-1}$$

for some $p > 1$, then this problem has a positive solution. To see this it suffices to apply our Theorem by taking $g(y) = y^{-1}$ and $h(y) = y^{\beta+1} + y^{1-\gamma}$. □

To prove Theorem 1 we consider the modified BVP

$$(|y'|^{\alpha-1}y')' + f(t,y) = 0, \quad 0 < t < 1;$$

$$y(0) = y(1) = 1/n, \tag{3}$$

for $n \in \mathbf{N} = \{1, 2, \cdots\}$.We firstly prove the existence of a positive solution y_n to this BVP. Then, a desired solution to BVP (1)-(2) will be obtained as a limit function as $n \to \infty$ of a subsequence of $\{y_n\}$. The existence of y_n, a positive solution of BVP (1)-(3), is proved by applying the topological transversality theorem initiated by [2]. For this purpose we show that

Lemma 1. Let $n \in \mathbf{N}$ and $\lambda \in [0,1]$. Suppose that $y = y_{n,\lambda}$ be a positive solution of BVP

$$(|y'|^{\alpha-1}y')' + \lambda f(t,y) = 0, \quad 0 < t < 1; \tag{4}$$

$$y(0) = y(1) = 1/n.$$

Then, there are constants $M_0(n)$ and $M_1(n)$ independent of λ such that

$$\frac{1}{n} \leq y_{n,\lambda}(t) \leq M_0(n), \quad 0 \leq t \leq 1, \tag{5}$$

$$|y'_{n,\lambda}(t)| \leq M_1(n), \quad 0 \leq t \leq 1,$$

□

In what follows the functional spaces $C[0,1]$ and $C^1[0,1]$ are regarded as Banach spaces equipped with the usual sup-norms. We will employ the notation

$$x^{\rho *} = |x|^{\rho-1}x$$

for $\rho > 0$ and $x \in \mathbf{R}$.

Lemma 2. (a) For every positive function $u \in C[0,1]$ we can find a unique constant $\xi(u)$ satisfying

$$\int_0^1 \left(\xi(u) - \int_0^s f(r,u(r))\,dr\right)^{\frac{1}{\alpha}*} ds = 0.$$

(b) The functional ξ is continuous on the set $\{u \in C[0,1] : u(t) > 0 \text{ on } [0,1]\}$.

Proof of Lemma 1. For simplicity we denote $y_{n,\lambda} = y$. The concavity of y implies the validity of the first inequality in (5). There is a unique point $T, 0 < T < 1$, such that $y(T) = \max_{[0,1]} y$. Let $t \in [0,T]$. We find by assumption (C2) that

$$-(y')^{1/p}((y')^\alpha)' \le \lambda a(t) g(y) h(y) (y')^{1/p},$$

because $y' \ge 0$ there. Integrating both sides on $[t,T]$, $0 \le t \le T$, we have for $t \in [0,T]$

$$\frac{\alpha}{p\alpha+1}[y'(t)]^{\frac{p\alpha+1}{p}} \le g(y(t))\left(\int_{1/n}^{y(T)} h(u)^p\,du\right)^{1/p}\left(\int_0^1 a(t)^q\,dt\right)^{1/q}.$$

Therefore we obtain

$$\frac{y'(t)}{g(y(t))^{\frac{p}{p\alpha+1}}} \le c_1\left(\int_0^{y(T)} h(u)^p\,du\right)^{\frac{1}{p\alpha+1}}, \quad 0 \le t \le T,$$

where $c_1 > 0$ is a constant independent of λ. One more integration on $[0,T]$ gives

$$\int_0^{y(T)} \frac{du}{g(u)^{\frac{p}{p\alpha+1}}} \le c_1\left(\int_0^{y(T)} h(u)^p\,du\right)^{\frac{1}{p\alpha+1}} + \int_0^1 \frac{du}{g(u)^{\frac{p}{p\alpha+1}}}$$

By assumption (C4) we find that $y(T) \le M_0(n)$ for some constant $M_0(n) > 0$ independent of λ.

To see the existence of $M_1(n)$ it suffices to integrate the inequality

$$-((y')^\alpha)' \le a(t) g(1/n) \max_{[1/n, M_0(n)]} h(u).$$

Proof of Lemma 2. Let a positive function $u \in C[0,1]$ be fixed, and consider the function $\Phi_u : \mathbf{R} \to \mathbf{R}$ defined by

$$\Phi_u(z) = \int_0^1 \left(z - \int_0^s f(r,u(r))\,dr\right)^{\frac{1}{\alpha}*} ds.$$

By assumption (C2), Φ_u is well-defined, and obviously it is a strictly increasing continuous function. Since $\Phi_u(0) < 0$ and

$$\Phi_u(z) \geq \left(z - \int_0^1 |f(r,u(r))|\, dr\right)^{\frac{1}{\alpha}*},$$

there is a unique constant $\xi(u)$ satisfying $\Phi_u(\xi(u))=0$.

(b)This can be proved by the same method as in [3, Lemma 5.3].

Proposition 1. Let $n \in \mathbf{N}$. Then BVP (1)-(3) has a positive solution $y = y_n$ such that $y \in C^1[0,1]$ and $|y'|^{\alpha-1}y' \in C^1(0,1)$.

Proof. We employ the topological transvarsality theorem formulated in [3]. Put

$$C^1[0,1] \supset K = \{u \in C^1[0,1] : u(0) = u(1) = 1/n\};$$

and

$$K \supset U = \{u \in K : \frac{1}{2n} < u(t) < M_0(n) + 1,\ |u'(t)| < M_1(n) + 1 \text{ on } [0,1]\},$$

where $M_0(n)$ and $M_1(n)$ are constants appearing in Lemma 1. It is easy to see that a positive function y is a solution of BVP (1)-(3) if and only if y satisfies the integral equation

$$y(t) = \frac{1}{n} + \int_0^t \left(\xi(y) - \int_0^s f(r,y(r))\, dr\right)^{\frac{1}{\alpha}*} ds, \quad 0 \leq t \leq 1.$$

where ξ is the functional appearing in Lemma 2. Let us consider the mapping $H(\cdot,\cdot) : \overline{U} \times [0,1] \to K$ defined by

$$H(u,\lambda)(t) = \frac{1}{n} + \lambda^{1/\alpha} \int_0^t \left(\xi(u) - \int_0^s f(r,u(r))\, dr\right)^{\frac{1}{\alpha}*} ds, \quad 0 \leq t \leq 1.$$

It suffices to show that $H(\cdot,1)$ has a fixed point in U. We can prove that

(a) $H(\cdot,\lambda) : \overline{U} \to K$ is continuous for every $\lambda \in [0,1]$;

(b) $H(\cdot,\cdot) : \overline{U} \times [0,1] \to K$ is compact;

(c) $H(\cdot,\lambda)$ does not have fixed points on ∂U for $\lambda \in [0,1]$.

Since $H(\cdot,0)$ is a constant mapping $H(\cdot,0) = 1/n$, and $1/n \in U$, we know from [2,Theorem 2.5] that $H(\cdot,1)$ has a fixed point in U. □

Lemma 3. Let $n \in \mathbf{N}$ and y_n be a positive solution of BVP (1)-(3). Then, there are positive constants M_0 and M_1 independent of n satisfying

$$|y_n(t)| \leq M_0 \quad \text{on} \quad [0,1];$$

and

$$||y_n'||_{L^\theta(0,1)} \leq M_1, \quad \theta = 1/p + \alpha + 1.$$

Proof. A close look at the proof of Lemma 1 shows that the constant $M_0(n)$ appearing there really does not depend on n. So we can choose $M_0 = M_0(n)$. To find out M_1 let T_n be such $y_n(T_n) = \max_{[0,1]} y_n$. For $t \in (0, T_n]$ we have

$$-((y_n')^\alpha)' \le a(t)g(y_n)h(y_n);$$

that is

$$\begin{aligned} -y_n(y_n')^{1/p}((y_n')^\alpha)' &\le a(t)y_n g(y_n)h(y_n)(y_n')^{1/p} \\ &\le M_0 g(M_0)a(t)h(y_n)(y_n')^{1/p}. \end{aligned}$$

An integration on $[0, T_n]$ gives

$$\begin{aligned} &\frac{p\alpha}{n(1+p\alpha)}(y_n'(0))^{\alpha+1/p} + \frac{p\alpha}{1+p\alpha}\int_0^{T_n} |y_n'(s)|^{1+\alpha+1/p}\,ds \\ \le\ & M_0 g(M_0)\Big(\int_{1/n}^{y_n(T_n)} h(u)^p\,du\Big)^{1/p}\Big(\int_0^{T_n} a(t)^q\,dt\Big)^{1/q} \\ \le\ & M_0 g(M_0)\|a\|_{L^q(0,1)}\Big(\int_0^{M_0} h(u)^p\,du\Big)^{1/p}. \end{aligned}$$

Here we have employed the nondecreasing nature of $y \mapsto yg(y)$. This implies that $\int_0^{T_n} |y_n'|^\theta\,dt$ is bounded by a constant independent of n. Similarly we can show that $\int_{T_n}^1 |y_n'|^\theta\,dt$ is bounded.

□

Proof of Theorem 1. Let $y_n, n \in \mathbf{N}$, be the solution of BVP (1)-(3) introduced in Proposition 1. Since for $t_1, t_2 \in [0,1], t_1 < t_2$, we have from Lemma 3

$$\begin{aligned} |y_n(t_1) - y_n(t_2)| &\le \int_{t_1}^{t_2} |y_n'(s)|\,ds \\ &\le \Big(\int_{t_1}^{t_2} |y_n'(s)|^\theta\,ds\Big)^{1/\theta}\Big(\int_{t_1}^{t_2} ds\Big)^{\frac{\theta-1}{\theta}} \\ &\le M_3|t_2 - t_1|^{\frac{\theta-1}{\theta}}, \end{aligned}$$

the sequence $\{y_n\}$ is equicontinuous. By the Ascoli-Arzelà theorem we can choose a subsequence $\{y_{n'}\} \subset \{y_n\}$ and a continuous function $\widetilde{y}$ such that $\{y_{n'}\}$ converges to $\widetilde{y}$ uniformly on $[0,1]$.

We show that $\widetilde{y}$ is a desired solution of BVP (1)-(2). We obviously have $\widetilde{y}(0) = \widetilde{y}(1) = 0$. We firstly show that $\widetilde{y} > 0$ in $(0,1)$. Let $T_n \in (0,1)$ be such that $y_n(T_n) = \max_{[0,1]} y_n$. Since the sequence $\{T_n\}$ is bounded, it contains a subsequence converging to some point $T \in [0,1]$. We may assume that $\lim\limits_{n'\to\infty} T_{n'} = T$. By assumption (C5), we have $((y_{n'}'(t))^{\alpha*})' + \psi_{M_0}(t) \le 0$ in $(0,1)$. Integrating twice this inequality we obtain

$$y_{n'}(t) \ge \frac{1}{n'} + \int_0^t \Big(\int_s^{T_{n'}} \psi_{M_0}(r)\,dr\Big)^{1/\alpha} ds, \quad 0 \le t \le T_{n'}; \tag{6}$$

and

$$y_{n'}(t) \geq \frac{1}{n'} + \int_t^1 \left(\int_{T_{n'}}^s \psi_{M_0}(r)\, dr \right)^{1/\alpha} ds, \quad T_{n'} \leq t \leq 1. \tag{7}$$

Since $0 \leq y_{n'}(t) \leq y_{n'}(T_{n'})$ by definition, we find that

$$0 \leq \widetilde{y}(t) \leq \widetilde{y}(T), \quad 0 \leq t \leq 1 \tag{8}$$

To see $0 < T < 1$, suppose the contrary that $T = 0, 1$. We may suppose that $T = 1$. Then, it follows from (8) that $\widetilde{y} \equiv 0$ on $[0, 1]$. On the other hand by letting $n' \to \infty$ in (6) we have

$$y(t) \geq \int_0^t \left(\int_s^1 \psi_{M_0}(r)\, dr \right)^{1/\alpha} ds > 0, \quad 0 \leq t < 1,$$

which is a contradiction. Hence $0 < T < 1$. Letting $n' \to \infty$ in (6) and (7), we have

$$\widetilde{y}(t) \geq \int_0^t \left(\int_s^T \psi_{M_0}(r)\, dr \right)^{1/\alpha} ds, \quad 0 \leq t < T;$$

and

$$\widetilde{y}(t) \geq \int_t^1 \left(\int_T^s \psi_{M_0}(r)\, dr \right)^{1/\alpha} ds, \quad T < t \leq 1,$$

respectively, which imply that $\widetilde{y} > 0$ in $(0, 1)$.

Finally letting $n' \to \infty$ in the formula

$$y_{n'}(t) = y_{n'}(T) - \int_t^T \left(\int_s^{T_{n'}} f(r, y_{n'}(r))\, dr \right)^{\frac{1}{\alpha}*} ds, \quad 0 < t < 1,$$

we have

$$\widetilde{y}(t) = \widetilde{y}(T) - \int_t^T \left(\int_s^T f(r, \widetilde{y}(r))\, dr \right)^{\frac{1}{\alpha}*} ds, \quad 0 < t < 1.$$

By differentiating, we find that $\widetilde{y}$ is a positive solution of equation (1). This completes the proof of Theorem 1.

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