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CONSTRUCTION OF DOUBLY-CONNECTED WANDERING DOMAINS

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ABSTRACT. We investigate the connectivity $\operatorname{conn}(D)$ of a wandering domain D of a transcendental entire function f. First we show that $\operatorname{conn}(f^n(D))$ is constant for large n and it is either $1, 2$ or ∞ (Theorem A). Next we construct an example of an f with doubly connected wandering domain (Theorem B), which is the main result of this paper. For this purpose we establish a slightly different version of quasiconformal surgery (Theorem 3.1). Also we construct following examples by the similar method:

- An entire function f having a wandering domain D with $\operatorname{conn}(D) = p$ for a given $p \in \mathbb{N}$ (Theorem C).
- An entire function f having a doubly connected wandering domain and all its singular values are contained in preperiodic Fatou components (Theorem D).
- An entire function f such that the set $\overline{f(\operatorname{sing}(f^{-1}))}$ is equal to the whole plane $\mathbb{C}$ but f has a wandering domain, hence $J_f \neq \mathbb{C}$ (Theorem E).
- An entire function f with infinitely many grand orbits of wandering domains. Furthermore, this f can be constructed so that the Lebesgue measure of the Julia set J_f is positive (Theorem F).

1. INTRODUCTION

Let f be a transcendental entire function and f^n denote the n-th iterate of f. Recall that the *Fatou set* F_f and the *Julia set* J_f of f are defined as follows:

$$\begin{aligned} F_f &= \{z \in \mathbb{C} \mid \{f^n\}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z\}, \\ J_f &= \mathbb{C} \setminus F_f. \end{aligned}$$

A connected component D of F_f is called a *Fatou component of* f. A Fatou component D is called a *wandering domain* if $f^m(D) \cap f^n(D) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$. If there exists a $p \in \mathbb{N}$ with $f^p(D) \subseteq D$, then D is called a *periodic component of period* p and it is either an *attracting basin*, a *parabolic basin*, a *Siegel disk* or a *Baker domain.* In particular, if $p = 1$, U is called *an invariant component.*

Here we briefly explain the history of wandering domains. For more details, see [**R**]. It was I. N. Baker who proved the existence of wandering domains for the first time. In 1963 he proved the following:

Theorem 1.1 (Baker, 1963 [**Ba1**, p.206 Statement (A), p.210 Theorem 1]). *There is an entire function $g(z)$ given by the canonical product*

$$g(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right)$$

such that $g(z)$ has at least one multiply connected Fatou component, where $C > 0$ is a constant and r_n is defined by some recursive formula and satisfies $1 < r_1 < r_2 < \cdots$.

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More precisely, let A_n *be the annulus*

$$A_n : r_n^2 < |z| < r_{n+1}^{\frac{1}{2}},$$

then there is an integer $N > 0$ *such that for all* $n > N$ *the mapping* $z \to g(z)$ *maps* A_n *into* A_{n+1} *and* $g^n(z) \to \infty$ *uniformly in* A_n. *For each* $n > N$, A_n *belongs to a multiply connected component* G_n *of* F_g.

At this moment, he did not assert that the above Fatou component was a wandering domain, because there was a possibility that G_n were equal for any $n > N$ and hence it was an invariant component. That is, it might be a *Baker domain*, on which, by definition, every point goes to ∞ under the iterate of g. But about ten years later, he proved the following result.

Theorem 1.2 (Baker, 1976 (Received 1 November 1974) [**Ba2**, p.174, Theorem]). *For* $n > N$ *the components* G_n *of* F_g *described above are all different and each is a wandering domain of* g.

More generally he proved

Theorem 1.3 (Baker, 1975 (Received 26 May 1975) [**Ba3**, p.278, Theorem 1]). *If* f *is transcendental and entire, then* F_f *has no unbounded multiply connected component. That is, any unbounded Fatou component is simply connected*

Thus the first example of a wandering domain was multiply connected. On the other hand, the example of simply connected wandering domains are known by M. Herman (see [**Ba4**, p.567, Example 5.1]). In this paper we consider the connectivity of a Fatou component, which is defined as follows:

Definition. For a domain D of $\mathbb{C}$, the *connectivity* $\mathrm{conn}(D)$ is defined to be the number of connected components of $\widehat{\mathbb{C}} \setminus D$, which may be ∞.

Note that $\mathrm{conn}(D) = 1$ if and only if D is simply connected, and $\mathrm{conn}(D) = 2$ if and only if D is doubly connected and conformally equivalent to a round annulus

$$\{z \mid 0 \leq r_1 < |z| < r_2 \leq \infty\}.$$

By density of periodic points in the Julia set and Theorem 1.3, it is easily shown that if a Fatou component D is multiply connected, then it must be a wandering domain and $f^n|_D \to \infty$ $(n \to \infty)$. In 1985, Baker constructed an example of a transcendental entire function g with a wandering domain D of infinite connectivity.

Theorem 1.4 (Baker, 1985 [**Ba5**, p.164, Theorem 2]). *There is an entire function* $g(z)$ *given by the canonical product*

$$g(z) = C^2 \prod_{j=1}^{\infty} \left(1 + \frac{z}{r_j}\right)^2, \quad 1 < r_1 < r_2 < \cdots, \; C > 0$$

such that $g(z)$ *has a wandering domain with infinite connectivity.*

So the following is a natural question to ask:

Question: Is there a wandering domain D with finite connectivity, or more precisely, with $\mathrm{conn}(D) = p$, $p \in \mathbb{N}$?

This question was raised by Baker in [**Ba5**] and is also explicitly stated as "Question 7" in [**Ber**, p.167]. Main purpose of this paper is to construct such an example. Incidentally, the connectivity of the wandering domain discussed in Theorem 1.1 and 1.2 is still unknown.

In this paper we first show the following:

Theorem A. *For a wandering domain D of a transcendental entire function f, the connectivity* $\operatorname{conn}(f^n(D))$ *is constant for large n and it is either 1, 2 or ∞. If it is 1, then* $\operatorname{conn}(D) = 1$. *If it is 2, then $f : f^n(D) \to f^{n+1}(D)$ is a covering of annuli for every sufficiently large n.*

According to this theorem, we make the following:

Definition. We define the *eventual connectivity* of a wandering domain D to be $\operatorname{conn}(f^n(D))$ for sufficiently large n.

Main result of this paper is as follows:

Theorem B. *There exists a transcendental entire function f with a wandering domain D such that $f^n(D)$ are doubly connected for all $n \geq 0$, i.e. the eventual connectivity of D is 2. Moreover f has no asymptotic values and all critical values are mapped to 0 which is a repelling fixed point.*

Theorem B gives a negative answer to the following open problem raised by W. Bergweiler.

Problem (Bergweiler, 1994 [**YWLC**, p.354]): Let f be an entire transcendental function. Suppose that $f^n|_U \to \infty$ as $n \to \infty$ for some connected component U of the Fatou set of f. Does there exist $\zeta \in \operatorname{sing}(f^{-1})$ such that $f^n(\zeta) \to \infty$? If not, does there exist at least $\zeta \in \operatorname{sing}(f^{-1})$ such that $f^n(\zeta)$ is unbounded?

Main technique to construct this kind of examples is the quasiconformal surgery. By using the same technique and some additional arguments, we can also show the following:

Theorem C. *For every $p \in \mathbb{N}$ with $p \geq 3$ there exists a transcendental entire function f with a wandering domain D with* $\operatorname{conn}(D) = p$ *and* $\operatorname{conn}(f^n(D)) = 2$ *for every $n \geq 1$.*

Theorem D. *There exists a transcendental entire function f with a wandering domain D such that the eventual connectivity of D is 2. Moreover f has no asymptotic values and all critical values are mapped to 0 which is an attracting fixed point.*

Theorem E. *There exists a transcendental entire function f such that the set $\overline{f(\operatorname{sing}(f^{-1}))}$ is equal to the whole plane $\mathbb{C}$ but f has a wandering domain, hence $J_f \neq \mathbb{C}$.*

Theorem F. *There exists a transcendental entire function f with infinitely many grand orbits of doubly connected wandering domains. That is, there exist doubly connected wandering domains D_i ($i \in \mathbb{N}$) such that if $i \neq j$, then $f^m(D_i) \cap f^n(D_j) = \emptyset$ for any $m, n \in \mathbb{N}$. Furthermore, this f can be constructed so that the Lebesgue measure of the Julia set J_f is positive.*

Theorem D answers the following question also by W. Bergweiler:

Question ("Question 10" in [**Ber**, p.170]): Can a meromorphic function f have wandering domains if all (or all but finitely many) points of $\operatorname{sing}(f^{-1})$ are contained in preperiodic domains?

Incidentally, Baker constructed an example of an entire function with infinitely many grand orbits of simply connected wandering domains in [**Ba4**, p.567, Theorem 5.2]. Also Baker, Kotus and Lü considered the similar problem of existence of multiply connected Fatou components for transcendental meromorphic functions with at least one pole ([**BKL1**], [**BKL2**]).

In §2, we prove Theorem A and §3 is devoted to the explanation of the quasiconformal surgery, which is a main tool for the proof of the main Theorem B. We give a proof of Theorem B in §4.

2. Proof of Theorem A

We need some lemmas.

Lemma 2.1 (Baker, 1984 [**Ba4**, p.565, Theorem 3.1]). *Let D be a multiply connected wandering domain of an entire function f and $\gamma \subset D$ is a nontrivial curve in D. Then $f^n \to \infty$ $(n \to \infty)$ in D and for every sufficiently large n the winding number of $f^n(\gamma)$ with respect to the origin is positive.*

Lemma 2.2 (cf. Baker, 1984 [**Ba4**, p.565, Corollary]). *If f has an asymptotic value, then every Fatou component of F_f is simply connected.*

Proof of Theorem A. By Lemma 2.1 and Lemma 2.2, we may assume that f has no asymptotic values and D is bounded. Then $f : D \to f(D)$ is a branched covering. If $\text{conn}(D) = \infty$, then $\text{conn}(f^n(D)) = \infty$ for every $n \in \mathbb{N}$. So we assume that $\text{conn}(D) < \infty$. By Riemann-Hurwitz Theorem, we have

$$2 - \text{conn}(D) = (\deg f|_D)(2 - \text{conn}(f(D))) - {}^{\#}\{\text{critical points in } D\}. \qquad (2.1)$$

Then it easily follows that $\text{conn}(D) \geq \text{conn}(f(D))$ and hence $\text{conn}(f^n(D))$ is constant for large n. Let us denote it by p. Suppose that $3 \leq p < \infty$, then by replacing D with $f^n(D)$ in (2.1) it follows that $\deg f|_{f^n(D)} = 1$ and hence $f : f^n(D) \to f^{n+1}(D)$ is conformal. By the Argument Principle, f is also 1 to 1 on the bounded components of $\mathbb{C} \setminus f^n(D)$. Then from Lemma 2.1, f must be 1 to 1 on whole $\mathbb{C}$, which is a contradiction, since f is transcendental. Therefore if p is finite, then $p = 1$ or 2. If $p = 1$, then it is easy to see that $\text{conn}(D) = 1$. If $p = 2$, then from (2.1) we have ${}^{\#}\{\text{critical points in } f^n(D)\} = 0$ and hence the result follows. □

3. Surgery and conformal structure

In this section, we recall the definition of quasiconformal map and explain the quasiconformal surgery (Theorem 3.1).

Definition 3.1. An orientation preserving homeomorphism $\varphi : D \to D'$ between two domains D and D' is called a *quasiconformal map* if it is absolutely continuous on lines on any rectangle $R = \{z = x + iy \mid a \leq x \leq b,\ c \leq y \leq d\} \subset D$, that is,

(i) $\varphi(x+iy)$ is absolutely continuous as a function of $x \in [a, b]$ for almost every y, and $\varphi(x + iy)$ is absolutely continuous as a function of $y \in [c, d]$ for almost every x

and moreover,

(ii) $|\mu_\varphi(z)| \le k < 1$ a.e. $z \in D$,

where $\mu_\varphi = \varphi_{\bar{z}}/\varphi_z$ and k is some constant with $0 \le k < 1$. If $k = 0$, then φ is conformal. If $k \ne 0$, we set $K = (1+k)/(1-k)$ and call φ a *K-quasiconformal* (*K-qc* for short) *map*. The constant

$$K_\varphi = \inf\{K \mid \varphi \text{ is } K\text{-qc}\}$$

is called the *maximal dilatation of* φ. A map $g : D \to D'$ is called a *K-quasiregular map* if g can be written as $g = f \circ \varphi$ with a K-quasiconformal map φ and an analytic map f.

For properties of quasiconformal maps, see [**A1**].

In order to construct an entire function with doubly-connected wandering domains, we first construct a quasiregular map g with the similar properties as what we really want to construct by gluing suitable polynomials together by using interpolation. Then we choose a suitable quasiconformal map φ so that $\varphi \circ g \circ \varphi^{-1}$ is a desired entire function. We call this procedure the *quasiconformal surgery*. More precisely, we can formulate this procedure as follows, which is slightly different from the one discussed in [**Sh**]:

Theorem 3.1 (quasiconformal surgery). *Let g be a quasiregular mapping from $\mathbb{C}$ to $\mathbb{C}$. Suppose that there are (disjoint) measurable sets $E_j \subset \mathbb{C}$ $(j = 1, 2, \ldots)$ satisfying:*

(i) *For almost every $z \in \mathbb{C}$, the g-orbit of z passes E_j at most once for every j;*

(ii) *g is K_j-quasiregular on E_j;*

(iii) $K_\infty = \prod_{j=1}^{\infty} K_j < \infty$;

(iv) *g is holomorphic a.e. outside $\bigcup_{j=1}^{\infty} E_j$ (i.e. $\frac{\partial g}{\partial \bar{z}} = 0$ a.e. on $\mathbb{C} \setminus \bigcup_{j=1}^{\infty} E_j$).*

Then there exists a K_∞-quasiconformal map φ such that $f = \varphi \circ g \circ \varphi^{-1}$ is an entire function.

Proof. A measurable conformal structure is the measurable conformal equivalence of measurable Riemannian metrics, and can be represented by the metric of the form

$$ds = |dz + \mu(z) d\bar{z}|,$$

where $\mu(z)$ is a $\mathbb{C}$-valued measurable function with

$$||\mu||_\infty = \operatorname{ess.sup} |\mu(z)| < 1.$$

The distance between two measurable conformal structures $\sigma = [|dz + \mu(z)d\bar{z}|]$ and $\sigma' = [|dz + \mu'(z)d\bar{z}|]$ is defined by

$$d(\sigma, \sigma') = \operatorname{ess.sup} d_{\mathbb{D}}(\mu(z), \mu'(z)),$$

where $d_{\mathbb{D}}$ denotes the Poincaré distance on the unit disk $\mathbb{D}$. A quasiregular map defines the pull-back $g^*(\sigma)$ of the measurable conformal structure σ, and the pull-back preserves the above distance. Let $\sigma_0 = [|dz|]$ denote the standard conformal structure. If g is K-quasiregular, then we have $d(g^*(\sigma_0), \sigma_0) \le \log K$.

Now define the conformal structures

$$\sigma_n(z) = (g^n)^*(\sigma_0(g^n(z))),$$

which are defined almost everywhere. The pointwise distance (when defined) satisfies

$$d_{\mathbb{D}}(\sigma_{n+1}(z), \sigma_n(z)) = d_{\mathbb{D}}(g^*(\sigma_0(g^{n+1}(z))), \sigma_0(g^n(z))) \le \log K_m$$

if $g^n(z)$ is in some E_m and it is 0 otherwise.

By the hypotheses (i) and (iii), $\{\sigma_n(z)\}_{n=0}^{\infty}$ is defined and a Cauchy sequence for almost all z. Therefore the pointwise limit $\sigma(z) = \lim_{n\to\infty} \sigma_n(z)$ exists a.e. and satisfies

$$d(\sigma, \sigma_0) \leq \sum_{j=1}^{\infty} \log K_j = \log K_\infty.$$

Then σ can be written as

$$\sigma(z) = [|dz + \mu(z)d\bar{z}|]$$

with

$$|\mu(z)| \leq \frac{K_\infty - 1}{K_\infty + 1} \quad \text{a.e.}$$

By Measurable Riemann Mapping Theorem ([**A1**, p.98, Theorem 3]), there exists a K_∞-quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\frac{\partial\varphi}{\partial\bar{z}}/\frac{\partial\varphi}{\partial z} = \mu(z)$ a.e., in other words $\varphi^*(\sigma_0) = \sigma$. Then $f = \varphi \circ g \circ \varphi^{-1}$ is quasiregular and satisfies $f^*(\sigma_0) = \sigma_0$. This implies that f is locally conformal except at critical points, hence it is analytic. □

Remark. Theorem 3.1 also follows from the idea by Sullivan ([**Su**, p.750, Theorem 9]).

4. Construction (Proof of Theorem B)

Part I : Construction of a model map f_0.

Definition 4.1. For a closed concentric annulus A with center 0, we use a notation

$$A = A(r_1, r_2) = \{z \mid r_1 \leq |z| \leq r_2\}, \quad (0 < r_1 < r_2)$$

and

$$\partial_{\text{inner}} A = \{z \mid |z| = r_1\}, \quad \partial_{\text{outer}} A = \{z \mid |z| = r_2\},$$

which denote the *inner boundary* and the *outer boundary* of A, respectively. We define the *modulus* of A by

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

The *core curve* Core(A) is the unique closed geodesic of A and given by

$$\operatorname{Core}(A) = \{z \mid |z| = \sqrt{r_1 r_2}\}.$$

We first construct a model map f_0 which roughly describes the dynamics of what we really want to construct. Let $k_n \in \mathbb{N}$ be given integers with $k_0 \leq k_1 \leq \cdots \leq k_n \leq \cdots$. In what follows we choose suitable $R_n \in \mathbb{R}$ with $0 = R_0 < R_1 < R_2 < \cdots$ and set

$$A_n = A(R_n, R_{n+1}) \quad (n \geq 0).$$

(Note that here we abuse the notation — A_0 is a disk, not an annulus). Then we want to construct a map $f_0 : \mathbb{C} \to \mathbb{C}$ with the following dynamical properties:

$$f_0(z) = a_0 z^{k_0}, \quad z \in A_0 \setminus \partial_{\text{outer}} A_0$$

such that $f_0(A_0) = A_0 \cup A_1$ and

$$f_0(z) = a_n z^{k_n}, \quad z \in A_n \setminus \partial_{\text{outer}} A_n$$

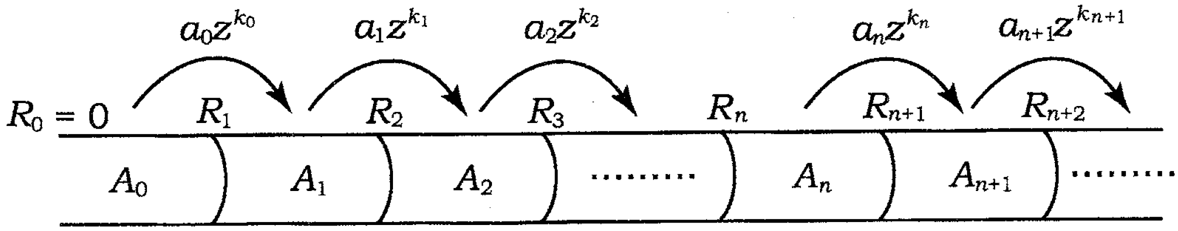

FIGURE 1. The model map f_0. Note that this is only a schematic picture and in reality, $\mathrm{mod}\,(A_n)$ rapidly increases as n tends to ∞. The same is also true for the following figures.

such that $f_0 : A_n \to A_{n+1}$ is a covering map of degree k_n. (See Figure 1, where we describe the annuli A_n as subsets of an infinite cylinder, instead of round annuli in the complex plane.)

For this purpose, we have to choose appropriate $a_n \in \mathbb{C}^*$ and $R_n > 0$. So first we take a_0 and R_1 so that $R_2 = |a_0|R_1^{k_0} > R_1$ holds and $M_1 = \exp(2\pi \,\mathrm{mod}\,(A_1))$ is large enough to be able to apply Proposition 4.1 in the next part. (Actually we have to choose so that $\mathrm{mod}\,(A_1) > m_0$, where m_0 is the constant in Proposition 4.1.) Once the constants a_0, R_1 and k_n ($n \in \mathbb{N}$) are chosen, then the constants $a_n \in \mathbb{C}^*$ ($n \geq 1$) and $R_n > 0$ ($n \geq 2$) are determined inductively as follows: Define $M_n > 0$ by

$$M_1 = \exp(2\pi \,\mathrm{mod}\,(A_1)), \quad M_{n+1} = M_n^{k_n} \ (n \geq 1)$$

and set

$$R_{n+1} = M_n R_n \quad (n \geq 1).$$

Also take $a_n \in \mathbb{C}^*$ with the condition

$$R_{n+1} = |a_n| R_n^{k_n}.$$

Note that only $|a_n|$ is determined by the condition above and we can choose $\arg a_n$ freely. Then it is easy to see that

$$\lim_{|z| \nearrow R_n} |f_0(z)| = \lim_{|z| \searrow R_n} |f_0(z)|,$$

because

$$|f_0(z)| = |a_{n-1} z^{k_{n-1}}| \to |a_{n-1}| R_n^{k_{n-1}} \quad (|z| \nearrow R_n)$$

and

$$|a_{n-1}| R_n^{k_{n-1}} = |a_{n-1}| R_{n-1}^{k_{n-1}} M_{n-1}^{k_{n-1}} = R_n M_n = R_{n+1},$$

on the other hand we have

$$|f_0(z)| = |a_n z^{k_n}| \to |a_n| R_n^{k_n} = R_{n+1} \quad (|z| \searrow R_n).$$

Hence f_0 itself is discontinuous on $|z| = R_n$ but the map $|f_0| : \mathbb{C} \to \mathbb{R}$ is continuous. According to Lemma 2.1, in general, f^n goes to ∞ on a multiply connected wandering domain D and $f^n(D)$ is mapped to an "outer" region by f. So our f_0 indeed satisfies this situation.

Part II : Construction of a quasiregular map f_1 from the model map f_0.

Now we modify the map f_0 to construct a new quasiregular map f_1 and then perform the quasiconformal surgery to obtain the desired map f. First we put $k_n = n + 1$. For each n we replace f_0 with some different polynomial around some annulus containing the circle $|z| = R_n$ and glue this polynomial and the original map f_0 together by interpolation. More precisely we prepare the following proposition.

Proposition 4.1. (1) *Let A, A' and $\widehat{A}$, $\widehat{A}'$ be two pairs of concentric round annuli with center* 0 *which satisfy*

$$\partial_{\text{outer}} A = \partial_{\text{inner}} A' = \{z \mid |z| = R\}, \quad \partial_{\text{outer}} \widehat{A} = \partial_{\text{inner}} \widehat{A}' = \{z \mid |z| = \widehat{R}\},$$

and

$$\operatorname{mod}(\widehat{A}) = k \cdot \operatorname{mod}(A), \quad \operatorname{mod}(\widehat{A}') = (k+1) \cdot \operatorname{mod}(A').$$

Let $F_A : A \to \widehat{A}$, $F_A(z) = c_A z^k$, $(k \geq 2)$ be a covering map of degree k which maps A onto $\widehat{A}$. Similarly let $F_{A'} : A' \to \widehat{A}'$, $F_{A'}(z) = c_{A'} z^{k+1}$, $(k \geq 2)$ be a covering map of degree $k+1$ which maps A' onto $\widehat{A}'$. For the annulus A, take annuli $B^\sharp(A)$, $E^\sharp(A)$, $E^\flat(A)$ and $B^\flat(A)$ as in Figure 2 such that

$$\operatorname{mod}(B^\sharp(A)) = \operatorname{mod}(E^\sharp(A)) = \operatorname{mod}(E^\flat(A)) = \operatorname{mod}(B^\flat(A)) = \sqrt{\operatorname{mod}(A)} \tag{4.1}$$

and define

$$A^- = A \setminus (B^\sharp(A) \cup E^\sharp(A) \cup E^\flat(A) \cup B^\flat(A)).$$

Take similar annuli for each A', $\widehat{A}$ and $\widehat{A}'$. Then there exists a constant $m_0 > 0$ such that if $\operatorname{mod}(A) > m_0$ *and* $\operatorname{mod}(A') > m_0$*, then there exists a quasiregular map*

$$g : A^- \cup E^\flat(A) \cup B^\flat(A) \cup B^\sharp(A') \cup E^\sharp(A') \cup A'^- \to \mathbb{C}$$

which satisfies the following conditions (I) $\sim$ (III):

(I-a) $g = F_A$ *on* A^- *and* $g = F_{A'}$ *on* A'^-.

(I-b) g *is holomorphic on* int $B = \operatorname{int}(B^\flat(A) \cup B^\sharp(A'))$ *with a unique critical point* $\zeta \in B^\flat(A)$. *Also g satisfies $g(\zeta) = \widehat{R}$ and $g(R) = 0$.*

(I-c) g *is K-quasiregular on* int $E = \operatorname{int}(E^\flat(A) \cup E^\sharp(A'))$ *and the maximal dilatation K_g satisfies*

$$\begin{aligned} K = K_g &\leq \max\left(1 + \frac{2}{\sqrt{k \cdot \operatorname{mod}(A)}},\ 1 + \frac{2}{\sqrt{(k+1) \cdot \operatorname{mod}(A')}}\right) \\ &= \max\left(1 + \frac{2}{\sqrt{\operatorname{mod}(\widehat{A})}},\ 1 + \frac{2}{\sqrt{\operatorname{mod}(\widehat{A}')}}\right). \end{aligned} \tag{4.2}$$

(II-a) $g(\operatorname{Core}(A^-)) = \operatorname{Core}(\widehat{A}^-)$. *Similarly,* $g(\operatorname{Core}(A'^-)) = \operatorname{Core}(\widehat{A}'^-)$.

(II-b) $g(A^-) \subset \widehat{A}^-$ *and this inclusion is essential. That is, $g(A^-)$ is an annulus in $\widehat{A}^-$ and its core curve is not 0-homotopic in $\widehat{A}^-$. Similarly, $g(A'^-) \subset \widehat{A}'^-$ essentially.*

(III-a) $g(E^\sharp(A')) \subset E^\sharp(\widehat{A}') \cup \widehat{A}'^-$ *essentially.*

(III-b) $g(E^\flat(A)) \subset \widehat{A}^- \cup E^\flat(\widehat{A})$ *essentially.*

(2) *In the case of $k = 1$, the same conclusion holds if we replace the condition* (4.1) *for the annulus A with*

$$\operatorname{mod}(E^\flat(A)) = \operatorname{mod}(B^\flat(A)) = \operatorname{mod}(B^\sharp(A)) = \operatorname{mod}(E^\sharp(A)) = 2\sqrt{\operatorname{mod}(A)}. \tag{4.3}$$

(*Note that we need not change the conditions* (4.1) *for the annuli A', $\widehat{A}$ and $\widehat{A}'$.*)

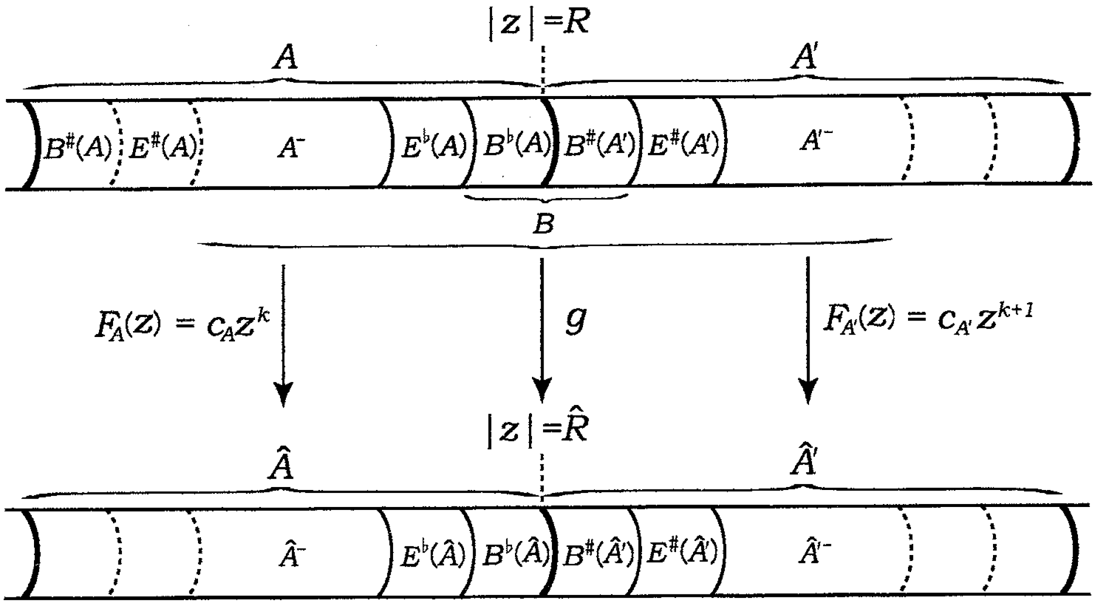

FIGURE 2. Interpolation between the two maps F_A and $F_{A'}$. We glue these two maps together in a neighborhood of the circle $\{z \mid |z| = R\}$.

Remark. Note that $g(B)$ covers not only a neighborhood of $\{|z| = \widehat{R}\}$ but also both $\widehat{A}$ and the bounded component of $\mathbb{C} \setminus \widehat{A}$.

Now we apply Proposition 4.1 (1) to each pair of annuli $(A, A') = (A_{n-1}, A_n)$ and maps $F_A(z) = a_{n-1}z^n$, $F_{A'}(z) = a_n z^{n+1}$ $(n = 2, 3, \cdots)$ to obtain a new map $g(z) = g_n(z)$. In this case, of course, $\widehat{A} = A_n$ and $\widehat{A}' = A_{n+1}$. We use the following notations:

$$B_n^\sharp = B^\sharp(A_n), \quad E_n^\sharp = E^\sharp(A_n), \quad E_{n+1}^\flat = E^\flat(A_n), \quad B_{n+1}^\flat = B^\flat(A_n).$$

Also we define

$$B_n = B_n^\flat \cup B_n^\sharp = B^\flat(A_{n-1}) \cup B^\sharp(A_n).$$

See Figure 3. Note that these notations are somehow different from what we have defined in Proposition 4.1. Here we use "$\sharp$" and "$\flat$" with respect to the circle $\{z \mid |z| = R_n\}$ so, for example, the annuli $E_n^\flat$, B_n and $E_n^\sharp$ are located in this order as in Figure 3.

For $n = 1$, we consider the pair $(A_0^\diamondsuit, A_1)$ rather than (A_0, A_1). More precisely, we take $A_0^\diamondsuit$ to be a preimage of A_1 by the map $a_0 z$. Then we have $\operatorname{mod}(A_0^\diamondsuit) = \operatorname{mod}(A_1)$. Define subannuli $B_0^\sharp$, $E_0^\sharp$, A_0^-, $E_1^\flat$ and $B_1^\flat$ such that

$$A_0^\diamondsuit = B_0^\sharp \cup E_0^\sharp \cup A_0^- \cup E_1^\flat \cup B_1^\flat,$$
$$\operatorname{mod}(E_1^\flat) = \operatorname{mod}(B_1^\flat) = \operatorname{mod}(B_0^\sharp) = \operatorname{mod}(E_0^\sharp) = 2\sqrt{\operatorname{mod}(A_0^\diamondsuit)} \ (= 2\sqrt{\operatorname{mod}(A_1)}).$$

Then we apply Proposition 4.1 (2) instead of (1) to the pair $(A_0^\diamondsuit, A_1)$ to construct $g_1(z)$.

From the condition (I-b), it follows that the critical point ζ_n of g_n satisfies $g_n(\zeta_n) = R_{n+1}$ and $g_{n+1}(R_{n+1}) = 0$. Also g_n satisfies an estimate on its maximal dilatation which is obtained from (4.2) in Proposition 4.1. Since we take a_0 so that $R_2 = |a_0|R_1 > R_1$, $z = 0$ is a repelling fixed point of f_0.

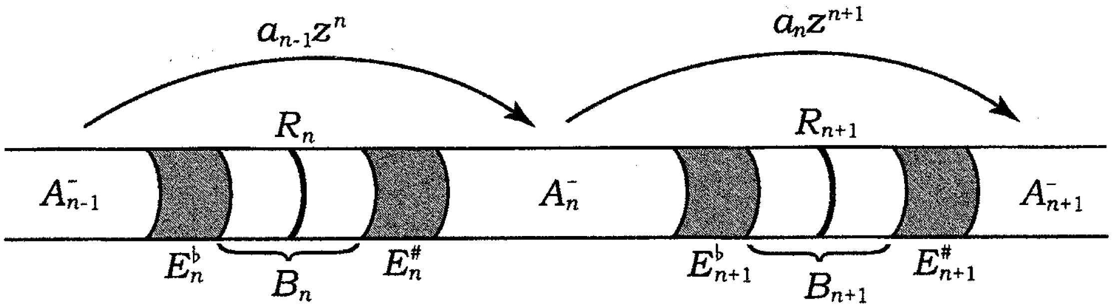

FIGURE 3. Construction of f_1 from f_0 by interpolation.

Then define a new map f_1 by

$$f_1(z) = \begin{cases} f_0(z) & z \in A_0 \setminus (E^\flat_1 \cup B^\flat_1) \\ f_0(z) & z \in A^-_n & n = 1, 2, \cdots \\ g_n(z) & z \in E^\flat_n \cup B_n \cup E^\sharp_n & n = 1, 2, \cdots \end{cases} .$$

Part III : Application of the quasiconformal surgery to f_1.

The new map f_1 is a quasiregular map with the desired dynamical properties. Hence we can apply the quasiconformal surgery (Theorem 3.1) to obtain a transcendental entire function f with the desired properties. More precisely, the following holds:

Proposition 4.2. *The new map f_1 satisfies the following conditions* (I) $\sim$ (IV):

(I-a) $f_1(z) = a_n z^{n+1}$ *on* A^-_n.

(I-b) f_1 *is holomorphic on* B_n.

(I-c) f_1 *is* K_n*-quasiregular on* $E_n = E^\flat_n \cup E^\sharp_n$ *with*

$$K_n \leq 1 + \frac{2}{\sqrt{n! \cdot \mathrm{mod}\,(A_1)}}.$$

(I-d) f_1 *has a critical point* $\zeta_n \in B^\flat_n$ *which satisfies* $f_1(\zeta_n) = R_{n+1}$ *and* $f_1^2(\zeta_n) = 0$ $(n = 1, 2, \cdots)$. $\{\zeta_n\}_{n=1}^\infty$ *is the set of all critical points of* f_1.

(II-a) $f_1(\mathrm{Core}\,(A^-_n)) = \mathrm{Core}\,(A^-_{n+1})$.

(II-b) $f_1(A^-_n) \subset A^-_{n+1}$ *and this inclusion is essential.*

(III-a) $f_1(E^\sharp_n) \subset E^\sharp_{n+1} \cup A^-_{n+1}$ *essentially.*

(III-b) $f_1(E^\flat_n) \subset A^-_n \cup E^\flat_{n+1}$ *essentially.*

(IV) $f_1(B_n) \subset \bigcup_{j=0}^{n+1} A_j$.

Hence there exists a quasiconformal mapping φ such that $f = \varphi \circ f_1 \circ \varphi^{-1}$ is holomorphic and entire.

Proof. All the conditions (I) $\sim$ (III) are obtained by applying Proposition 4.1 to each pair of annuli $(A, A') = (A_{n-1}, A_n)$ and maps $F_A(z) = a_{n-1}z^n$, $F_{A'}(z) = a_n z^{n+1}$ $(n = 1, 2, \cdots)$. Note that

$$K_n \leq 1 + \frac{2}{\sqrt{\mathrm{mod}\,(A_n)}} = 1 + \frac{2}{\sqrt{n! \cdot \mathrm{mod}\,(A_1)}}.$$

Condition (IV) holds from the construction. Then (II-b), (III-a) and (III-b) show that for any $z \in \mathbb{C}$ the f_1-orbit of z passes E_n at most once for every n. Also from (I-c), f_1 is K_n-quasiregular on E_n with

$$K_\infty = \prod_{n=1}^{\infty} K_n \leq \prod_{n=1}^{\infty} \left(1 + \frac{2}{\sqrt{n! \cdot \operatorname{mod}(A_1)}}\right) < \infty.$$

Finally f_1 is holomorphic outside $\bigcup_{n=1}^{\infty} E_n$ by (I-a) and (I-b). Therefore we can apply Theorem 3.1 to the map f_1 and hence there exists a K_∞-quasiconformal map φ such that $f = \varphi \circ g \circ \varphi^{-1}$ is a transcendental entire function. □

Part IV : The map f has the desired properties.

Let $\widetilde{A}_n = \varphi(A_n)$, $\widetilde{B}_n = \varphi(B_n), \cdots$ etc. Then f satisfies exactly the same conditions for $\widetilde{A}_n$, $\widetilde{B}_n$ etc in Proposition 4.2 as f_1 satisfies for A_n, B_n, etc.

Lemma 4.3. *The annuli $\widetilde{A}_n^-$ $(n = 1, 2, \cdots)$ are contained in the Fatou set F_f.*

Proof. By the construction, we have $f(\widetilde{A}_n^-) \subset \widetilde{A}_{n+1}^-$ and the iterates tend to ∞ uniformly on $\widetilde{A}_n^-$, hence $\widetilde{A}_n^-$ is contained in F_f. □

Let us denote by D_n the Fatou component containing $\widetilde{A}_n^-$ $(n \geq 1)$.

Lemma 4.4. $D_n \neq D_{n+1}$.

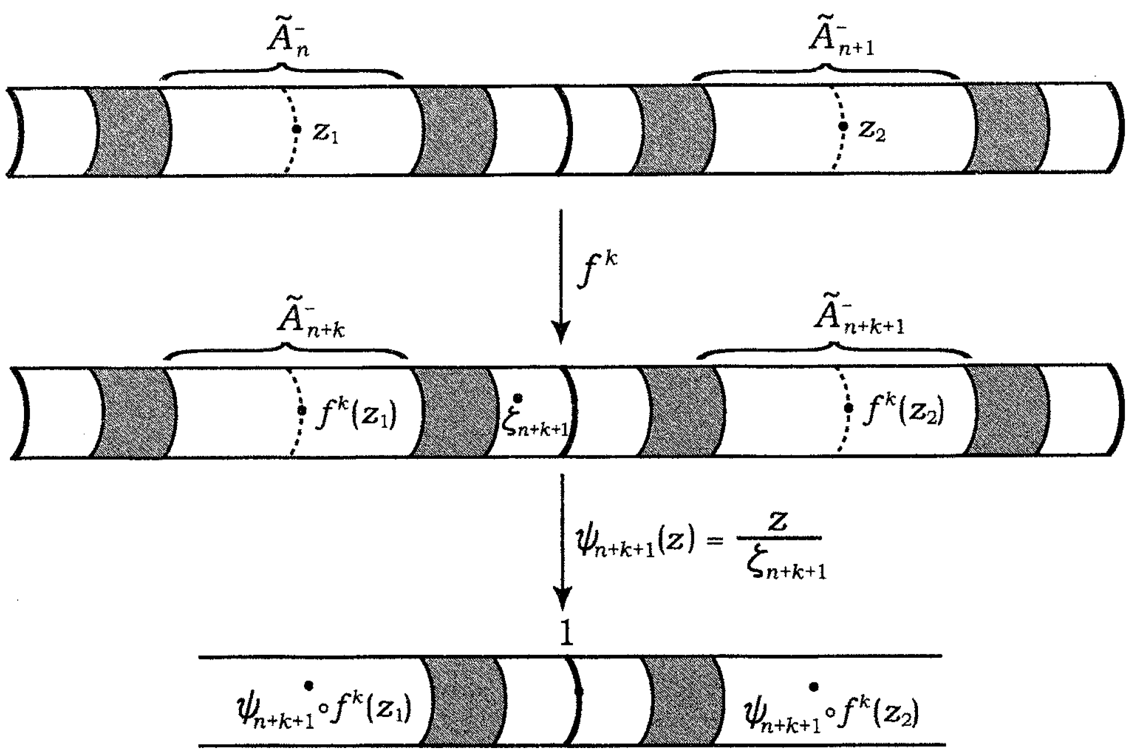

FIGURE 4

Proof. Suppose $\widetilde{A}_n^-$ and $\widetilde{A}_{n+1}^-$ belong to the same Fatou component $D = D_n = D_{n+1}$. Take $z_1 \in \mathrm{Core}(\widetilde{A}_n^-)$ and $z_2 \in \mathrm{Core}(\widetilde{A}_{n+1}^-)$. See Figure 4. Then $f^k(z_1) \in \mathrm{Core}(\widetilde{A}_{n+k}^-)$ and $f^k(z_2) \in \mathrm{Core}(\widetilde{A}_{n+k+1}^-)$ from Proposition 4.2 (II-a). By the construction $0 \notin D$, since 0 is a repelling fixed point. Also for $m \geq 1$, the critical point ζ_m of f satisfies $\zeta_m \in B_m \setminus D$, since $f^2(\zeta_m) = 0$. Let $\psi_m(z) = z/\zeta_m$ then

$$\psi_{n+k+1} \circ f^k(D) \subset \Omega \equiv \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}.$$

Therefore

$$d_\Omega(\psi_{n+k+1} \circ f^k(z_1), \psi_{n+k+1} \circ f^k(z_2)) \leq d_D(z_1, z_2),$$

where d_Ω and d_D are the Poincaré distances of Ω and D, respectively. By the construction we have

$$\psi_{n+k+1} \circ f^k(z_1) \to 0 \quad (k \to \infty).$$

In fact, $\{0, f^k(z_1)\}$ and $\{\zeta_{n+k+1}, \infty\}$ are separated by an annulus which is the outer half of $\widetilde{A}_{n+k}^- \setminus \mathrm{Core}(\widetilde{A}_{n+k}^-)$, and its modulus tends to ∞ as $k \to \infty$. Similarly $\psi_{n+k+1} \circ f^k(z_2) \to \infty$ holds. Hence it follows that

$$d_\Omega(\psi_{n+k+1} \circ f^k(z_1), \psi_{n+k+1} \circ f^k(z_2)) \to \infty.$$

This contradicts with the previous statement. □

Remark. This Lemma also follows immediately from the general result Theorem 1.3 by Baker. His proof of Theorem 1.3 is based on the construction of the hyperbolic metric and so the main idea of our proof of Lemma 4.4 is very similar to his.

Proposition 4.5. *The Fatou component D_n containing $\widetilde{A}_n^-$ can be written as*

$$D_n = \bigcup_{k=0}^{\infty} \widetilde{A}_{n,k}^-, \tag{4.4}$$

where $\widetilde{A}_{n,k}^-$ is the component of $f^{-k}(\widetilde{A}_{n+k}^-)$ containing $\widetilde{A}_n^-$. Moreover if all D_n do not contain critical points, then they are doubly connected, i.e. the eventual connectivity of D_n is 2.

Note that (4.4) is an increasing union, since $f(\widetilde{A}_{n+k}^-) \subset \widetilde{A}_{n+k+1}^-$. In order to prove Proposition 4.5, we need some lemmas.

Lemma 4.6. *Let $a, b > 0$ and $A = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}\, z < a\}/\sim$, where $z \sim z + nbi$ $(n \in \mathbb{Z})$. Suppose that φ is a quasiconformal mapping from A onto another annulus A'. Denote $\mu = \frac{\partial \varphi}{\partial \bar{z}} / \frac{\partial \varphi}{\partial z}$. (In other words, A' can be considered as an annulus A with the conformal structure $|dz + \mu(z)\, d\bar{z}|$.) Then the moduli of A and A' satisfy*

$$\frac{\iint_A 1\, dx\, dy}{\iint_A K_\mu(z)\, dx\, dy} \leq \frac{\mathrm{mod}\,(A')}{\mathrm{mod}\,(A)} \leq \frac{\iint_A K_\mu(z)\, dx\, dy}{\iint_A 1\, dx\, dy},$$

where $K_\mu(z) = \dfrac{1 + |\mu(z)|}{1 - |\mu(z)|}$.

In particular, if $K_\mu(z) = 1$ outside a measurable set $X \subset A$ and $K_\mu(z) \leq K$ on X, then

$$\frac{\mathrm{mod}\,(A')}{\mathrm{mod}\,(A)} \leq K \frac{|X|}{|A|} + \left(1 - \frac{|X|}{|A|}\right),$$

where $|X|$ (resp. $|A|$) denotes the Lebesgue measure of X (resp. A).

The first half is called Grötzsch inequality and the second half is an easy consequence. See [**GL**] §1.4, the proof of Proposition 3. (The proof was for a rectangle but it can be easily adapted for annuli of the above form.)

In the construction in Part III, we had

$$\frac{|A_n^+|}{|A_n^-|} \to 1 \quad (n \to \infty),$$

where

$$A_n^+ = B_n^\flat \cup A_n \cup B_{n+1}^\sharp$$

and $|\cdot|$ denotes the Lebesgue measure in the cylinder model $\mathbb{C}/2\pi i\mathbb{Z}$. Since φ is conformal on A_n^-, it follows from Lemma 4.6 that

$$\frac{\mathrm{mod}\,(\widetilde{A}_n^+)}{\mathrm{mod}\,(A_n^+)} \to 1 \quad (n \to \infty).$$

Combining with $\mathrm{mod}\,(\widetilde{A}_n^-) = \mathrm{mod}\,(A_n^-)$ and $\mathrm{mod}\,(A_n^+)/\mathrm{mod}\,(A_n^-) \to 1$ $(n \to \infty)$, we have:

Corollary 4.7.

$$\frac{\mathrm{mod}\,(\widetilde{A}_n^+)}{\mathrm{mod}\,(\widetilde{A}_n^-)} \to 1 \quad (n \to \infty).$$

Lemma 4.8. *For $m > 0$ and $L > 0$, there exists an $\varepsilon = \varepsilon(m, L) > 0$ such that if A_1 is an essential subannulus of an annulus A_2 with $m \le \mathrm{mod}\,(A_1) \le \infty$ and $\mathrm{mod}\,(A_2)/\mathrm{mod}\,(A_1) < 1 + \varepsilon$, then any point $z \in A_2$ with $d_{A_2}(z, \mathrm{Core}\,(A_1)) \le L$ belongs to A_1.*

Proof. Fix constants $m > 0$ and $L > 0$. Suppose that A_1 is an essential subannulus of an annulus A_2 with $m \le \mathrm{mod}\,(A_1) \le \infty$ and that there exists a point $z_0 \in A_2 \setminus A_1$ with $d_{A_2}(z_0, \mathrm{Core}\,(A_1)) \le L$. We want to show that $\mathrm{mod}\,(A_2)/\mathrm{mod}\,(A_1)$ cannot be arbitrarily close to 1.

Choose $z_1 \in \mathrm{Core}\,(A_1)$ such that $d_{A_2}(z_0, z_1) = d_{A_2}(z_0, \mathrm{Core}\,(A_1))$. There exist universal covering maps $\pi_j : \mathbb{D} \to A_j$ with $\pi_j(0) = z_1$ $(j = 1, 2)$. Since A_1 is essential in A_2, there exists a lift $\psi : \mathbb{D} \to \mathbb{D}$ of the inclusion map $\iota : A_1 \hookrightarrow A_2$ such that $\pi_2 \circ \psi = \iota \circ \pi_1 = \pi_1$ and $\psi(0) = 0$. There exists a point $\zeta_1 \in \mathbb{D}$ such that the segment $[0, \zeta_1]$ maps onto $\mathrm{Core}\,(A_1)$ by π_1, $d_{\mathbb{D}}(0, \zeta_1) = \mathrm{length}_{A_1}(\mathrm{Core}\,(A_1))$ and $\pi_1(\zeta_1) = z_1$. Let $\zeta_2 = \psi(\zeta_1)$, then $\pi_2(\zeta_2) = z_1$ and $|\zeta_2| \le |\zeta_1|$. There is also a point $\zeta_0 \in \mathbb{D}$ such that $\pi_2(\zeta_0) = z_0$, $d_{\mathbb{D}}(0, \zeta_0) \le L$ and $\zeta_0 \notin \mathrm{Image}\ \pi_2 \circ \psi$.

It is well known [**M**, p.12] that

$$\mathrm{length}_{A_j}(\mathrm{Core}\,(A_j)) = \frac{\pi}{\mathrm{mod}\,(A_j)} \qquad (j = 1, 2).$$

It follows from the Schwarz-Pick Theorem ([**A2**, p.3 Theorem 1-1]) and the definition of geodesics that

$$\begin{aligned}\frac{\pi}{\mathrm{mod}\,(A_2)} &= \mathrm{length}_{A_2}(\mathrm{Core}\,(A_2)) \le d_{\mathbb{D}}(0, \zeta_2) \le \mathrm{length}_{A_2}(\mathrm{Core}\,(A_1)) \\ &\le \mathrm{length}_{A_1}(\mathrm{Core}\,(A_1)) = d_{\mathbb{D}}(0, \zeta_1) = \frac{\pi}{\mathrm{mod}\,(A_1)}.\end{aligned}$$

Hence we have
$$\frac{\operatorname{mod}(A_1)}{\operatorname{mod}(A_2)} \le \frac{d_{\mathbb{D}}(0,\zeta_2)}{d_{\mathbb{D}}(0,\zeta_1)} \le 1 \quad \text{and} \quad d_{\mathbb{D}}(0,\zeta_1) \le \frac{\pi}{m}.$$

Define $\psi_0(0) = \psi'(0)$ and $\psi_0(z) = \psi(z)/z$ ($0 \ne z \in \mathbb{D}$). The Schwarz Lemma applied to ψ implies $|\psi_0(z)| < 1$ since ψ is not surjective. We have
$$|\psi_0(\zeta_1)| = \frac{|\zeta_2|}{|\zeta_1|} \ge \frac{d_{\mathbb{D}}(0,\zeta_2)}{d_{\mathbb{D}}(0,\zeta_1)} \ge \frac{\operatorname{mod}(A_1)}{\operatorname{mod}(A_2)},$$
where the left inequality follows from the fact that the coefficient $\frac{2}{1-|z|^2}$ of the Poincaré metric in $\mathbb{D}$ is increasing in $[0,1)$. Since
$$d_{\mathbb{D}}(\psi_0(0), \psi_0(\zeta_1)) \le d_{\mathbb{D}}(0,\zeta_1) \le \frac{\pi}{m},$$
there exists a function $\delta(\varepsilon, m) > 0$ such that if $\operatorname{mod}(A_2)/\operatorname{mod}(A_1) < 1+\varepsilon$ then $|\psi'(0)| = |\psi_0(0)| > 1 - \delta(\varepsilon, m)$ and $\delta(\varepsilon, m) \to 0$ as $\varepsilon \to 0$.

Now decompose ψ as $\psi = \psi_3 \circ \psi_2 \circ \psi_1$, where
$$\psi_3(z) = \frac{z+\zeta_0}{1+\overline{\zeta_0}z} : \mathbb{D} \to \mathbb{D}$$
is a Möbius transformation sending $-\zeta_0$, 0 to 0, ζ_0,
$$\psi_2 : \mathbb{D}^* \equiv \mathbb{D} - \{0\} \hookrightarrow \mathbb{D}$$
is the inclusion and $\psi_1 : \mathbb{D} \to \mathbb{D}^*$ is a holomorphic map sending 0 to $-\zeta_0$ and its image avoids 0. By the Schwarz-Pick Theorem, we have
$$\begin{aligned} |\psi'(0)| &= ||\psi'(0)||_{\mathbb{D},\mathbb{D}} \\ &= ||\psi_3'(-\zeta_0)||_{\mathbb{D},\mathbb{D}} \cdot ||\psi_2'(-\zeta_0)||_{\mathbb{D}^*,\mathbb{D}} \cdot ||\psi_1'(0)||_{\mathbb{D},\mathbb{D}^*} \\ &\le ||\psi_2'(-\zeta_0)||_{\mathbb{D}^*,\mathbb{D}}, \end{aligned}$$
where $||\cdot||_{X,Y}$ denotes the norm of the derivative with respect to the Poincaré metric of the domain X and that of the range Y. Since the Poincaré metric of $\mathbb{D}^*$ is $\frac{|dz|}{|z|\log(1/|z|)}$, we can write down explicitly as
$$||\psi_2'(-\zeta_0)||_{\mathbb{D}^*,\mathbb{D}} = \frac{2|\zeta_0|\log(1/|\zeta_0|)}{1-|\zeta_0|^2} = \frac{t}{\sinh t} \quad \text{with} \quad t = \log(1/|\zeta_0|).$$
Hence there exists $\lambda(L) < 1$ such that if $d(0,\zeta_0) \le L$ then $|\psi'(0)| \le ||\psi_2'(-\zeta_0)||_{\mathbb{D}^*,\mathbb{D}} \le \lambda(L)$.

Finally, choose $\varepsilon > 0$ so that $1 - \delta(\varepsilon, m) > \lambda(L)$. If $\operatorname{mod}(A_2)/\operatorname{mod}(A_1) < 1+\varepsilon$, we have a contradiction, therefore we have thus proved the lemma. □

Proof of Proposition 4.5. The connected component of $f^{-k}(\widetilde{A}^-_{n+k})$ containing $\widetilde{A}^-_n$ must be contained in D_n. Hence the right hand side is contained in the left hand side.

In order to show the converse, take any point $z_0 \in D_n$. Join z_0 with $\operatorname{Core}(\widetilde{A}^-_n)$ by a smooth curve γ in D_n. See Figure 5. Let $L = \operatorname{length}_{D_n}(\gamma)$. Note that $f^k(\operatorname{Core}(\widetilde{A}^-_n)) = \operatorname{Core}(\widetilde{A}^-_{n+k})$ by Proposition 4.2 (II)(i) and that $D_{n+k} \subset \widetilde{A}^+_{n+k}$ by Lemma 4.4. Then by the Schwarz-Pick Theorem again, for $z_0 \in \gamma$, we have
$$d_{\widetilde{A}^+_{n+k}}(f^k(z_0), \operatorname{Core}(\widetilde{A}^-_{n+k})) \le d_{D_{n+k}}(f^k(z_0), \operatorname{Core}(\widetilde{A}^-_{n+k})) \le L, \quad (k \ge 0).$$
Since obviously $\operatorname{mod}(\widetilde{A}^-_{n+k}) \to \infty$ ($k \to \infty$), we can apply Lemma 4.8 with $A_1 = \widetilde{A}^-_{n+k}$ and $A_2 = \widetilde{A}^+_{n+k}$ together with Corollary 4.7 and conclude that there exists $k_0 \ge 0$ such

that $f^k(\gamma) \subset \widetilde{A}_{n+k}^-$ for $k \geq k_0$. This implies that for large k, γ (and hence z_0) is contained in $\widetilde{A}_{n,k}^-$. Thus D_n is contained in $\bigcup_{k=0}^{\infty} \widetilde{A}_{n,k}^-$.

Moreover, if all D_n do not contain critical points, then $\widetilde{A}_{n,k}^-$ is doubly connected. Since $\widetilde{A}_{n,k}^- \subset \widetilde{A}_{n,k+1}^-$ essentially, D_n is also doubly connected as an increasing union of annuli. □

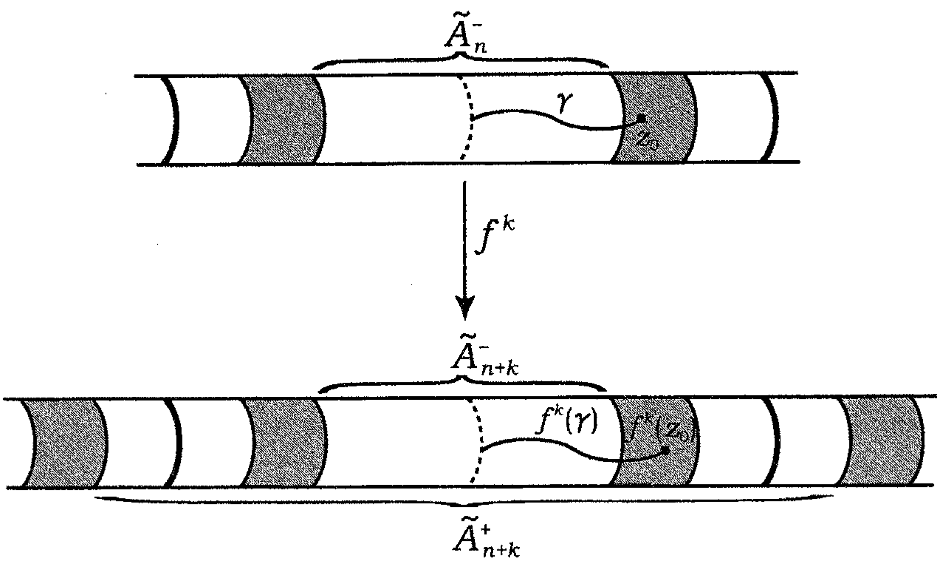

FIGURE 5

By the construction, all the critical points of f are mapped to 0 by f^2. Since 0 is a repelling fixed point, which is in J_f, all the critical points are in J_f and hence all D_n do not contain critical points. Therefore D_n is doubly connected for every n from Proposition 4.5. This completes the proof of Theorem B. □

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