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Relations of formal diffeomorphisms

Isao Nakai (中居　功), Kana Yanai(柳井　佳奈) *

Abstract

Germs of holomorphic diffeomorphisms of $\mathbb{C}, 0$ are formally conjugated to time-1 maps of some holomorphic vector fields on $\mathbb{C}$. Thus a word of germs of holomorphic diffeomorphisms is a composite of some time-1 maps of formal vector fields. We give a formula for the Taylor coefficients of the time-1 transport maps of formal ordinary differential equations. We apply the formula to the differential equations corresponding to words of diffeomorphisms. We investigate the various results on the existence of relations of formal diffeomorphisms. The complete account of these results will appear in a forthcoming paper[22].

1 Diff$(\mathbb{C}, 0)$ and associated First Order ODE

Let Diff$(\mathbb{C}, 0)$ denote the group of germs of holomorphic diffeomorphisms of $\mathbb{C}$ fixing $0 \in \mathbb{C}$, and $\overline{\mathrm{Diff}}(\mathbb{C}, 0)$ the group of formal diffeomorphisms without constant term. The classification problem of subgroups of Diff$(\mathbb{C}, 0)$ arises naturally in the study of foliations as well as differential equations. A *relation of length* l of n elements in Diff$(\mathbb{C}, 0)$ is a word

$$W(f_1, \ldots, f_n) = f_{i_1}^{(\pm 1)} \circ \cdots \circ f_{i_l}^{(\pm 1)} = 1,$$

where W is not a priori 1 and $f^{(m)}$ denotes the m-hold iteration of f. A subgroup G is *free* if there exists no relation of elements of G. A word, or a set of words, of n diffeomorphisms $W(f_1, \ldots, f_n)$ is *holomorphically* (*respectively formally*) *conjugated* to a $W(g_1, \ldots, g_n)$ (the same word with substituted letters) if there exists a holomorphic (resp. formal) diffeomorphism $\phi \in$ Diff$(\mathbb{C}, 0)$ such that $f_i = \phi^{(-1)} \circ g_i \circ \phi$ for $i = 1, \ldots, n$. If two germs f, g are tangent to identity (i.e. the linear terms $= 1$) and *commutative* : $f \circ g = g \circ f$, then f, g are simultaneously formally conjugated to a time-s and a time-t maps of an equal holomorphic vector field χ for some $s, t \in \mathbb{C}$. It is also known that a holomorphic vector field is holomorphically conjugated to

*Isao Nakai: お茶の水女子大学・理学部・数学教室、京都大学・数理解析研究所、Department of Mathematics, Facluty of Science, Ochanomizu University, and Rsearch Institute of Mathematical Science, Kyoto University. *E-mail address*:nakai@math.ocha.ac.jp

Kana Yanai: お茶の水女子大学・人間文化研究科、Graduate School of Humanities and Sciences, Ochanomizu University. *E-mail address*:g0140625@edu.cc.ocha.ac.jp

either a linear vector field or a $z^{p+1}/(1-mz^p)\ \partial z$ for some $m \in \mathbb{C}$ and positive integer p invariant under the linear action of $\mathbb{Z}_p$ (m being the *residue* of χ.) Thus the commutativity relation is formally embedded to either the linear subgroup $\mathbb{C}^*$ or the commutative subgroup $\mathbb{C} \times \mathbb{Z}_p \subset \mathrm{Diff}(\mathbb{C}, 0)$ generated by the complex flow of the χ and the linear action. When the residue $m = 0$, the linear conjugate of χ by an arbitrary $\lambda \in \mathbb{C}^*$ remains in the form of its constant multiple. Therefore the linear conjugate action of $\mathbb{C}^*$ and the complex one parameter group of χ generate a semidirect product $\mathbb{C} \times \mathbb{C}^*$, which is nothing but the affine transformation group $\mathrm{Aff}(\mathbb{C})$. This group contains also the various other relarions. We call the relations formally conjugated to those relations *elementary relations.*

In the geometric study of a codimension 1 foliation $\mathcal{F}$, the question whether there exist non elementary relations or not in $\mathrm{Diff}(\mathbb{C}, 0)$ has been at crucial issue. When $\mathcal{F}$ is deformed leaving a leaf L stable, relations in the holonomy group $\mathrm{Hol}(L)$ of L may be violated or some extra relations may appear for some values of the deformation parameter. While, there exist only countably many words. Thus it follows if there exists no relation stable under deformation, the holonomy group is free for a generic parameter. Ilyashenko and Pyartli [18] showed that for a generic rational differential equation

$$dy/dx = P(x,y)/Q(x,y)$$

on $\mathbb{C}P^2$, the holonomy group $\mathrm{Hol}(L_\infty)$ of L_∞ is free by showing that the set of k-jets of diffeomorphisms with a relation is smaller than the set of k-jets of the generators of holomnomy when k tends to ∞. This implies that no relation in $\mathrm{Hol}(L_\infty)$ is stable under deformation, thus $\mathrm{Hol}(L_\infty)$ is free for a generic differential equation. This argument relies on the dimension estimate and basically on the countableness of the set of relations, and does not tell any concrete free groups or non free groups with non elementary relations. Our argument can tell a precise asymptotics of the codimension of the set of k-jets of diffeomorphisms with a relation.

Cerveau [4] showed the diffeomorphisms $f(z) = z/1+z$ and $g(z) = z/\sqrt{1+z^2}$ play no relations using the result by Cohen [8] that asserts $z+1$ and z^2 generate a free group. On the other hand Ecalle [12] constructed many relations in the formal group $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ of the various types such as

$$\{f,\{f,\{f,g\}\}\}^{(p)} \circ \{g,\{g,f\}\}^{(q)} = 1\ ,$$

$\{f,g\}$ being the commutator $f^{(-1)} \circ g^{(-1)} \circ f \circ g$, by solving f, g in formal power series. In the paper [12] it is stated that some of those f, g are not convergent but summable in a certain manner, and also predicted that non convergence is a mathematical law.

In §8,9 of this paper we calculate the Taylor coefficients of words of diffeomorphisms in terms of their *phase diagram* (*Feynman diagram*, see the next section for the definition). As an immediate consequence we show that, if initial jets of lower order of diffeomorphisms satisfy a certain algebraic condition on the various moments of a Feynman diagram, we can choose their subsequent infinite jets properly so that the relation associated to the diagram holds. Our method is in the same vein as the classical linearlization of $(\mathbb{C},0)$-diffeomorphisms due to Poincaré and Siegel. The difference in contrast in our case is that the denominators behave tamely in a polynomial growth while the numerators might tend to infinity rapidly.

By simple observation we see that a relation $W(f_1,f_2)=1$ implies a series of algebraic relations $P_1=1, P_k=0, k=2,3,\dots$ of Taylor coefficients P_k of the z^k terms of $W(f_1(z),f_2(z))$, which are polynomials of Taylor coefficients of f_1,f_2 of order equal or lower than i. So clearly if these coefficients of f,g are all algebraically independent, it would follow some special conditions on the word W. In §7 it is shown that P_k are presented in terms of integration of 1-forms on the Feynman diagram of γ. So the above conditions imply that the winding number of $\gamma=0$ at each point.

It would be worth to note that a generic finite subset of a non solvable Lie group generate a free subgroup [15]. The group of truncated diffeomorphisms $\mathrm{Diff}^k(\mathbb{C},0)$ of degree k is solvable for all k, while $\mathrm{Diff}(\mathbb{C},0)$ is non solvable. We define a subgroup $G^k\subset \mathrm{Diff}^k(\mathbb{C},0)$ is *stably free* if any subgroup $G\subset\mathrm{Diff}(\mathbb{C},0)$ with k-jet G^k is free. The existence or non existence of stably free subgroup is not clear [23].

To end this section we give some words on the calculation of Taylor coefficients of a word $W(f_1,\dots,f_n)$. Assume $f_i=\exp a_i\partial_z$ with a holomorphic vector field $a_i\partial_z$ and consider the piecewise holomorphic non linear ordinary differential equation

$$\frac{dz}{dt}=\lambda_t(z)=\quad \pm a_i(z)\quad \text{if}\quad t\in[\,i-1,i)\ ,\quad \text{for } z\in\mathbb{C}, 0\le t\le l$$

where the right hand side is determined corresponding to the i-th letter and its sign in the word $W(f_1,\dots,f_k)$ from the right hand side. It is easily seen the word W is the *time-l transport map h_l* (or *the product integral*[9]) of the equation. To solve the equation we employ the classic method due to Picard, Volterra, Dyson, Chen and Chacon and Chacon, Fomenko. The logarithm of h_n as a diffeomorphism is a formal vector field of one valuable such that its time-1 map is h_n. Such a vector field is uniquely determined by analytic continuation of the branch $\log\mathrm{id}=0$, and called the *Lie integral* (of the left hand side of the above differential equation).

Now we suppose the right hand side as a piecewise smooth function of t valued in the Lie algebra of formal vector fields $\hat{\chi}(\mathbb{C},0)$. The *Feynman diagram* γ and its *dual diagram* γ^* associated to the above W are the integral curves in $\hat{\chi}(\mathbb{C},0)$ with the initial point 0 and the velocities $X_t=\lambda_t\partial_z$ and $X_{l-t}=\lambda_{l-t}\partial_z$ respectively. One of our results is that for a closed γ the Taylor coefficient L_k of z^k in the Taylor expansion of Lie integral is expressed in terms of integration on of 1-forms ω_k on γ^* for some small $k=2,3,\dots$:

$$\log h_n=\Big(\int_{\gamma^*}\omega_2\,z^2+\int_{\gamma^*}\omega_3\,z^3+\cdots\Big)\,\partial_z,$$

in other words,

$$h_n=\exp\Big(\int_{\gamma^*}\omega_2\,z^2+\int_{\gamma^*}\omega_3\,z^3+\cdots\Big)\,\partial_z\ .$$

And also in the case where the velocity vectors of γ and γ^* have all trivial linear parts, in other words, γ,γ^* are confined in the subspace of formal diffeomorphisms without linear terms, all the coefficients are "holomorphic functions" of γ. Although, the coefficints of higher order terms may not be written in integration of 1-forms on γ, since even in the case where $W = W(f_1, f_2)$ and the winding number of γ^* is 0 everywhere on the plane spanned by $a_1\partial_z, a_2\partial_z$, the relation $W = 1$ does not hold in general for highly non commutative f_1, f_2. We investigate some applications of the formula in the later sections of the note.

2 Some examples

Before we define Feynman diagram, let us consider the real n dimensional sub space L_n in the Lie algebra $\hat{\chi}(\mathbb{R})$ spanned by n formal vector fields $a_i\partial_z, i = 1, 2, \ldots, n$. We suppose L_n is the n dimensional vector space $\mathbb{R}^n$ whose coordinate is $(x_1, x_2, \ldots, x_n)$: x_i axes correspond respectively to $a_i\partial_z$ direction, $i = 1, 2, \ldots, n$. From now on, we identify L_n with $\mathbb{R}^n$.

Let H_i be a segment of length 1 in x_i direction, $i = 1, 2, \ldots, n$, in $\mathbb{R}^n$. Feynman diagram $\gamma = \gamma(t)$, $0 \le t \le l$, is then defined by a composite of some of these segments, say,

$$\gamma = {H_1}^{k_1} \circ {H_2}^{k_2} \circ \cdots \circ {H_n}^{k_n} \circ \cdots \circ {H_1}^{k_{np-(n-1)}} \circ {H_2}^{k_{np-(n-2)}} \circ \cdots \circ {H_n}^{k_{np}}$$

with $\gamma(0) = 0$ and $|k_1| + |k_2| + \cdots + |k_{np}| = l$, where ${H_i}^{k_m}$ denotes the k_m-times composites of H_i. And let us define the dual diagram $\gamma^* = \gamma^*(t)$, $0 \le t \le l$, by the transpose

$$\gamma^* = {H_n}^{k_{np}} \circ {H_{n-1}}^{k_{np-1}} \circ \cdots \circ {H_1}^{k_{np-(n-1)}} \circ \cdots \circ {H_n}^{k_n} \circ {H_{n-1}}^{k_{n-1}} \circ \cdots \circ {H_1}^{k_1}$$

with $\gamma^*(0) = 0$ for γ. We get then clearly $(\gamma^*)^* = \gamma$ for a diagram γ.

Figure 1 shows examples of closed Feynman diagram and its dual diagram in the real (x_1, x_2)-plane $\mathbb{R}^2$. Let H and V be segments of length 1 in x_1-direction and x_2-direction respectively, then $\gamma_i, i = 1, 2, 3$, in Figure 1 are as follows.

$$\begin{aligned}
\gamma_1 &= H \circ V^2 \circ H^{-1} \circ V^{-2} \circ H^{-2} \circ V \circ H^2 \circ V^{-1}, \\
\gamma_2 &= H \circ V \circ H^{-2} \circ V^{-1} \circ H^{-1} \circ V \circ H^2 \circ V^{-1} \circ H^{-1} \circ V \circ H^2 \circ V \circ H^{-1} \circ V^{-2}, \\
\gamma_3 &= H \circ V \circ H^{-1} \circ V^{-1} \circ V \circ H \circ V \circ H^{-1} \circ V^{-2} \\
&\qquad \circ H^{-1} \circ V \circ H \circ V^{-1} \circ H^{-2} \circ V \circ H \circ V^{-1} \circ H^{-1} \circ H^2.
\end{aligned}$$

And $\gamma_i{}^*, i = 1, 2, 3$, are as follows.

$$\begin{aligned}
\gamma_1{}^* &= V^{-1} \circ H^2 \circ V \circ H^{-2} \circ V^{-2} \circ H^{-1} \circ V^2 \circ H, \\
\gamma_2{}^* &= V^{-2} \circ H^{-1} \circ V \circ H^2 \circ V \circ H^{-1} \circ V^{-1} \circ H^2 \circ V \circ H^{-1} \circ V^{-1} \circ H^{-2} \circ V \circ H, \\
\gamma_3{}^* &= H^2 \circ H^{-1} \circ V^{-1} \circ H \circ V \circ H^{-2} \circ V^{-1} \circ H \circ V \\
&\qquad \circ H^{-1} \circ V^{-2} \circ H^{-1} \circ V \circ H \circ V \circ V^{-1} \circ H^{-1} \circ V \circ H.
\end{aligned}$$

$\gamma_i{}^*$ is obtained by rotating γ_i 180 degrees with respect to the origin and reversing the orientation of it for $i = 1, 2, 3$. For the domain D enclosed by a closed diagram

γ in $\mathbb{R}^2$, i.e. $\gamma = \partial D$, let denote the domain enclosed by the dual diagram γ^* by D^*, i.e. $\gamma^* = \partial D^*$. We get $(D^*)^* = D$ then, for $(\gamma^*)^* = \gamma$.

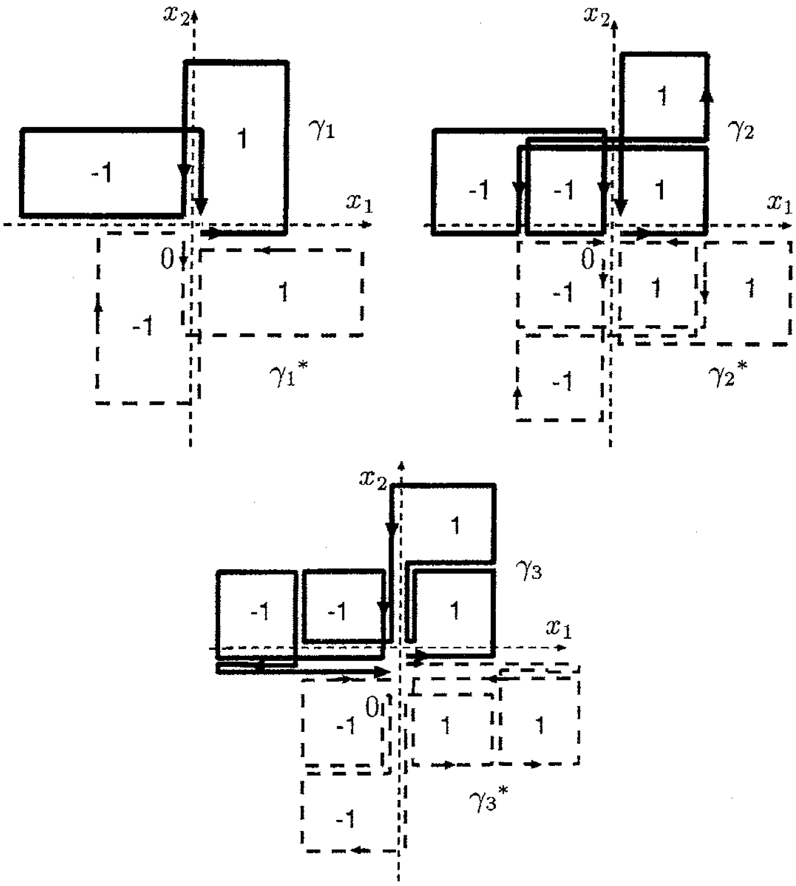

Figure 1: closed Feynman diagram and its dual diagram

The integer written in the domain enclosed by Feynman diagram or its dual diagram in Figure 1 is its winding number. We denote the winding number of a closed diagram γ in $\mathbb{R}^2$ at a point (x_1, x_2) by $\rho(\gamma)(x_1, x_2)$. We get then $\rho(\gamma)(x_1, x_2) = -\rho(\gamma^*)(-x_1, -x_2)$ for a closed diagram γ in $\mathbb{R}^2$.

Now, let us show some examples of relations of time-1 maps $f = \exp a_1\partial_z$ of $a_1\partial_z$ and $g = \exp a_2\partial_z$ of $a_2\partial_z$ in $\overline{\mathrm{Diff}}(\mathbb{C}, 0)$ for two formal vector fields

$$\begin{aligned} a_1\partial_z &= (a_{11}z + a_{12}z^2 + a_{13}z^3 + \cdots)\partial_z, \\ a_2\partial_z &= (a_{21}z + a_{22}z^2 + a_{23}z^3 + \cdots)\partial_z \end{aligned}$$

$\in \hat{\chi}(\mathbb{C}, 0)$ on $\mathbb{C}$. We see that there exist relations of f and g in $\overline{\mathrm{Diff}}(\mathbb{C}, 0)$ with

$(a_{11}, a_{21}) = (0,0)$, $(a_{12}, a_{22}) = (-1,3)$ and $(a_{13}, a_{23}) = (\frac{63}{8}, \frac{107}{8})$, as follows

$$\begin{aligned} W_{\gamma_1}(f,g) &= \{g, f^{(-2)}\} \circ \{g^{(2)}, f\} = 1, \\ W_{\gamma_2}(f,g) &= g^{(-1)} \circ \{g,f\} \circ \{f^{(-1)}, g^{(-1)}\} \circ f \circ \{f^{(-1)}, g^{(-1)}\} \circ f^{(-2)} \circ g \circ f = 1, \\ W_{\gamma_3}(f,g) &= f^{(2)} \circ \{g,f\}^{(-1)} \circ f^{(-1)} \circ \{g,f\}^{(-1)} \\ &\qquad \circ f^{(-1)} \circ g^{(-1)} \circ \{g,f\} \circ g \circ \{g,f\} = 1 \end{aligned}$$

by Theorem 8.2, since the *Area* of $\gamma_i{}^* = \iint_{D_i{}^*} \rho \; dx_1 \wedge dx_2$ is 0 and the *moment* of $\gamma_i{}^* = (\iint_{D_i{}^*} \rho \; x_1 dx_1 \wedge dx_2, \iint_{D_i{}^*} \rho \; x_2 dx_1 \wedge dx_2)$ is $(3,1) \neq (0,0)$ and $\iint_{D_i{}^*} \rho \; (a_{12}x_1 + a_{22}x_2)^2 dx_1 \wedge dx_2 = 16 \neq 0$ for the vector $(a_{12}, a_{22}) = (-1,3)$ orthogonal to the *moment* of $\gamma_i{}^*$ for $i = 1,2,3$. $A_3 = (a_{13}, a_{23})$ is determined by the solution of the simultaneous linear equations in the proof of Theorem 8.2. It is noted that the pairs (f,g) of formal diffeomorphism satisfying the above three relations can be different from each other since (a_{1k}, a_{2k}), $k \geq 4$, can be arbitrary.

Although we get the relations in the form of $W_\gamma(f,g) = 1$ for a closed Feynman diagram γ in the above by computing the *Area* and the *moment* of γ^*, and $\iint_{D^*} \rho \; K_2{}^2 dx_1 \wedge dx_2 = \iint_{D^*} \rho \; (a_{12}x_1 + a_{22}x_2)^2 dx_1 \wedge dx_2$ for a vector (a_{12}, a_{22}) orthogonal to the *moment* of γ^*, we consider relations in the form of $W_{\gamma^*}(f,g) = 1$ for a dual closed diagram γ^* in §6-9 of the note for simplicity of notations of Lie integral.

It seems that there exist many different relations in $\overline{\mathrm{Diff}}(\mathbb{C},0)$. We will treat the existence of relations of many diffeomorphisms in $\overline{\mathrm{Diff}}(\mathbb{C},0)$ in a forthcoming paper [22]. We investigate the existence of relations of two formal diffeomorphisms in this note.

3 Transport formula by Picard iteration and Dyson exponential

Let us consider the linear differential equation

$$\frac{du(t)}{dt} = K(t)\; u(t), \qquad u \in \mathbb{C}^\nu,\; t \in \mathbb{R},$$

where $K(t)$ is a $n \times n$ matrix valued analytic function of t. For small t, the solution can be uniquely presented as

$$u(t) = \exp\; Z(t)\; u(0)\;.$$

The $Z(t)$ is called the *Lie element* or *Lie integral* of $K(t)$ and denoted

$$Z(t) = L \int_0^t K(t) dt\;.$$

Now let us follow the begining of the paper of Dyson [11]. In the manner of numerical method of differential equations, the matrix $\exp Z(t)$, $0 < t$, can be approximated by a composite (or product)

$$\exp(K(\xi_N)\Delta_N) \cdot \;\cdots\; \cdot \exp(K(\xi_2)\Delta_2) \cdot \exp(K(\xi_1)\Delta_1) \tag{1}$$

with a division $\Delta : 0 = t_0 < t_1 < \cdots < t_N = t$ of the interval $[0, t]$, $\Delta_i = t_i - t_{i-1}$ and $t_{i-1} \leq \xi_i \leq t_i$. And as $\max \Delta_i$ tends to 0, this is convergent to $\exp Z(t)$. The limit is called the product integral [9]. One may substitute the above finite composite by

$$(1 + \int_{t_{N-1}}^{t} K(t)dt) \ (1 + \int_{t_{N-2}}^{t_{N-1}} K(t)dt) \cdots (1 + \int_{0}^{t_1} K(t)dt) .$$

Simply by expanding the product we see this is equal, modulo $\max \Delta_i$, to

$$1 + \int_0^t K(t)dt + \int_{0 \leq t_1 \leq t_2 \leq t} K(t_2)K(t_1)\, dt_1 dt_2 + \int_{0 \leq t_1 \leq t_2 \leq \leq t_3 \leq t} K(t_3)K(t_2)K(t_1)\, dt_1 dt_2 dt_3 + \cdots$$

It is an elementary exercise to verify the first and the second products are convergent to the third formula. This presentation is called Volterra Expansion or Dyson exponential, and in Chen's notation, presented as

$$T_\gamma(\omega) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots \tag{2}$$

where $\omega = K(t)dt$ and γ denotes the path from 0 to t. It is easily seen by the above argument that

$$T_{\gamma_1 \circ \gamma_2}(\omega) = T_{\gamma_2}(\omega)\ T_{\gamma_1}(\omega) \tag{3}$$

where $\gamma_1 \circ \gamma_2$ stands for the composite of γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$. This is nothing but the formula due to Chen [7, 16].

4 Lie integral for linear differential equations

To expand the composite in (1), we may apply the so-called BACH formula (Campbell-Baker-Hausdorff-Dynkin formula) for $\nu \times \nu$ matrices X, Y as follows.

$$\log\ (\exp X \exp Y) = X + Y + \frac{1}{2}[\, X, Y] + \frac{1}{12}[\, X, [\, X, Y]\,] + \frac{1}{12}[\, Y, [\, Y, X]\,] + \cdots$$

This formula has been generalized in the following manner. Let $X_1, \ldots, X_N$ be noncommuting indeterminates and let $\exp Z = \exp X_1 \exp X_2 \cdots \exp X_N$. Then

$$\begin{aligned} Z &= \log\ (\exp X_1 \exp X_2 \cdots \exp X_N) \\ &= X_1 + X_2 + \cdots + X_N + \frac{1}{2}\sum_{i<j}[\, X_i, X_j] + \cdots \end{aligned}$$

The generalized BACH formula is

Theorem 4.1 (Strichartz[25]).

$$\begin{aligned} Z = \sum_{m=1}^{\infty} \sum_{p_{jk}} \Big((-1)^{m-1} \,/\, m\ \big(\sum_{j=1}^{N}\sum_{k=1}^{N} p_{jk}\big)\ \Pi_{j=1}^{m}\Pi_{k=1}^{N}(p_{jk}!) \Big) \\ \times (\mathrm{ad}X_N)^{p_{mN}} \cdots (\mathrm{ad}X_1)^{p_{m1}} \quad \cdots \quad (\mathrm{ad}X_N)^{p_{1N}} \cdots (\mathrm{ad}X_1)^{p_{11}} \end{aligned}$$

where $(\mathrm{ad}\ X)(Y) = [\, X, Y] = XY - YX$, $(\mathrm{ad\ X}) = X$ *and the sum over* p_{jk} *is taken over all* $p_{jk} > 0$ *such that* $\sum_{k=1}^{N} p_{jk} > 0$ *for* $j = 1, 2, \ldots, m$.

The next theorem explains the convergence.

Theorem 4.2 (Chacon, Fomenko[3]). *Let H_n denote the homogeneous part of Z in $X_1, \dots, X_N$ of degree n:*

$$Z = H_1 + H_2 + H_3 + \cdots \tag{4}$$

Then H_n satisfies the recursion relation

$$\begin{aligned}(n+1)\ H_{n+1} = \ & \frac{1}{n!}T^{(n)}(0) \ + \ \sum_{r=1}^{n} \frac{1}{(n-r)!}\left(\frac{1}{2}[\ H_r, T^{(n-r)}(0)]\right. \\ & + \left.\sum_{p\geq 1, 2p\leq r} k_{2p} \sum_{m_i>0, m_1+\cdots+m_{2p}=r} [\ H_{m_1}, [\ \cdots, [\ H_{m_{2p}}, T^{(n-r)}(0)]\ \cdots]\ ]\right)\end{aligned}$$

$(2p)!k_{2p} = B_{2p}$being Bernoulli's number, and

$$\begin{aligned}T^{(k)}(0) = \ & \mathrm{ad}_{-X_N}^{\ k}(X_{N-1}) \\ & + \sum_{i=2}^{N-1}\sum_{\alpha_1=0}^{k}\sum_{\alpha_2=0}^{\alpha_1}\cdots\sum_{\alpha_{N-i}=0}^{\alpha_{N-i-1}} C_{\alpha_1}^{k} C_{\alpha_2}^{\alpha_1}\cdots C_{\alpha_{\alpha_{N-i}}}^{\alpha_{N-i-1}} \\ & \times \mathrm{ad}_{-X_N}^{\ k-\alpha_1}\mathrm{ad}_{-X_{N-1}}^{\ \alpha_1-\alpha_2}\cdots\mathrm{ad}_{-X_{i+1}}^{\ \alpha_{N-i-1}-\alpha_{N-i}}\mathrm{ad}_{-X_i}^{\ \alpha_{N-i}}(X_{i-1}) + X_N^{(k)}\end{aligned}$$

where

$$X_N^{(k)} = \frac{d^k}{dt^k}X_N = \begin{cases} 0 & k \geq 1 \\ X_N & k = 0. \end{cases}$$

Moreovere, there exisists $\delta > 0$ such that if $||X|| = \sum_{i=1}^{N}|X_i| < \delta$, the series (4) converges absolutely uniformly in N.

By taking the limit as $N \to \infty$ of the above formulas, Strichartz and Chacon, Fomenko obtained the formulas for the Lie integral. Here we employ the following formula by Chacon and Fomenko.

Theorem 4.3 (Chacon, Fomenko[3]). *Assume $K(t), 0 \leq t \leq 1$, is Riemann integrable. Then Taylor expansion of $L\int_0^t \lambda K(s)ds$ in λ at $\lambda = 0$*

$$L\int_0^t \lambda K(s)ds = \lambda H_1[t] + \lambda^2 H_2[t] + \cdots$$

is convergent. Here $H_1[t] = \int_0^t K(s)ds$, and $H_n[t]$ is uniquely defined by the recursion formula

$$(n+1)H_{n+1} = T_n + \sum_{r=1}^{n}\left(\frac{1}{2}[H_r, T_{n-r}] + \sum_{\substack{p\geq 1\\ 2p\leq r}} k_{2p} \sum_{\substack{m_i>0\\ m_1+\cdots+m_{2p}=r}} [H_{m_1}, [\dots, [H_{m_{2p}}, T_{n-r}]\cdots]]\right),$$

for $n \geq 1$, and

$$T_k = \int\cdots\int_{0\leq u_{k+1}\leq\cdots\leq u_1\leq t} du_1\cdots du_{k+1}[\cdots[[K(u_1), K(u_2)], \dots, K(u_k)], K(u_{k+1})] \quad (k \geq 1).$$

This formula is nothing but the logarithm of the transport map $T_\gamma(\omega)$ in (3) §3.

5 Solving non linear ordinary differential equations by Lie integral

Consider the ordinary differential equation

$$\frac{dz}{dt} = f(t,z) = f_t(z) \tag{5}$$

where f_t is a family of germs of holomorphic functions on $\mathbb{C}$ at 0 with $f_t(0) = 0$. Let $h_t(z_0)$ denote the solution with the initial value $h_0(z_0) = z_0$ at $t = 0$. Clearly h_t is a holomorphic diffeomorphism in z_0. We call h_t a *time-t transport map* . Differentiating the equation

$$h_t^* \; g = g \circ h_t$$

we obtain

$$\frac{d}{dt} h_t^* \; g = \frac{d}{dt}(g \circ h_t) = X_t \; g \; (h_t) = h_t^* \; X_t \; g,$$

$X_t = f_t \partial_z$, from which

$$\frac{d}{dt} h_t^* = h_t^* \; X_t$$

and

$$\frac{d}{dt} h_t^{*-1} = -X_t \; h_t^{*-1} \; , \tag{6}$$

which is a linear differential equation in the space of linear operator on the germs of holomorphic functions. In order to reduce the equation to a finite dimensional space and assure the existence of solution, we consider truncated those operators to the space of degree n polynomials, which we denote with the suffix $[n]$. Applying truncation to the both sides of the equation (6), we obtain the linear ordinary equation

$$\frac{d}{dt}\{(h_t{}^*)^{[n]}\}^{-1} = -X_t^{[n]}\{(h_t{}^*)^{[n]}\}^{-1}.$$

By Thorem 4.3 the Lie integral of $-X_t^{[n]}$ is well defined for small t. By definition

$$(h_t{}^*)^{-1[n]} = \exp \; L\int_0^t -X_t^{[n]} dt$$

and by the definition of the transpose map in §3

$$(h_t{}^*)^{[n]} = \exp \; L\int_t^0 -X_t^{[n]} dt. \tag{7}$$

This is seen also by the relation

$$L\int_0^t -X_t^{[n]} dt + L\int_t^0 -X_t^{[n]} dt = 0 \; ,$$

which follows from Chen's formula (3) in §3 and also directly from the formula in Theorem 4.3. The equation (7) holds for all n, however, the domain of definition for

t may shrink to 0 as n tends to infinity. It is easily seen by the formula in Theorem 4.3 that if f_t dose not have linear term, the Lie integral exists for all t, hence the Lie integral of $-X_t$ also exists for all t. Although the Lie integral of $-X_t$ may not be well defined in general case, the relation (7) holds as long as the Lie integral

$$L\int_t^0 -X_t^{[n]}dt$$

is analytically prolonged.

6 Relations in the Lie algebra $\hat{\chi}(\mathbb{C})$

First we consider relations of two holomorphic vector fields $a\partial_z, b\partial_z$. So let $\omega = a\partial_z dx + b\partial_z dy$ be a closed holomorphic 1-form on the real xy-plane valued in $\chi(\mathbb{C})$, and consider the equation

$$\nabla: \qquad \partial_z dz = \omega = a\partial_z dx + b\partial_z dy. \tag{8}$$

The $-\omega$ may be regarded as a connection form of a $\chi(\mathbb{C})$-valued connection on the trivial $(\mathbb{C}, 0)$-bundle over the xy-plane, which is defined by the (horizontal) plane field

$$H: \ dz = a\ dx + b\ dy\ .$$

On a path $\gamma(t) = (x(t), y(t))$ this restricts to an ordinary differential equaiton

$$\frac{dz}{dt} = a\frac{dx}{dt} + b\frac{dy}{dt}. \tag{9}$$

The time-t transport map (h_t in the previous section) of the equation corresponds to the parallel transport of the connection along γ. When γ is a segment of length 1 in x-direction H, (9) becomes $dz/dt = a$, and the time-1 transport map $h_1 = f$ is the time-1 map of $a\ \partial_z$. When γ is a segment of length 1 in y-direction V, (9) becomes $dz/dt = b$, and the time-1 transport map $h_1 = g$ is the time-1 map of $b\ \partial_z$. More in general let γ be a composite of some of these segments of length k, say,

$$\gamma = H^{n_1} \circ V^{n_2} \circ H^{n_3} \circ V^{n_4} \circ \cdots \circ H^{n_{2p-1}} \circ V^{n_{2p}}$$

with $|n_1| + \cdots + |n_{2p}| = k$, where H^m, V^m denote the m-composites of H, V respectively. Then the time-k transport map h_k is a word of f, g of length k

$$g^{(n_{2p})} \circ f^{(n_{2p-1})} \circ g^{(n_{2p-2})} \circ f^{(n_{2p-3})} \circ \cdots \circ g^{(n_2)} \circ f^{(n_1)}.$$

On the other hand we have the formula

$$h_t^* = \exp L\int_t^0 -\left(a\frac{dx}{dt} + b\frac{dy}{dt}\right)dt.$$

In the later sections we shall compute, instead, the Lie integral

$$L_\gamma = L\int_0^t \left(a\frac{dx}{dt} + b\frac{dy}{dt}\right)dt$$

for simplicity of notations, of which the exponential is the transpose

$$W_{\gamma^*}(f,g) = f^{(n_1)} \circ g^{(n_2)} \circ f^{(n_3)} \circ g^{(n_4)} \circ \cdots \circ f^{(n_{2p-1})} \circ g^{(n_{2p})}$$

If the Lie integral vanishes, the relation $W_{\gamma^*}(f,g) = 1$ holds. Clearly this relation holds whenever γ is closed and f,g commute, in other words, a,b are linearly dependent.

Next let us consider relations of many diffeomorphisms. Recall that all germs of diffeomorphisms in $\mathrm{Diff}(\mathbb{C},0)$ are formal conjugate with a time-1 map of holomorphic flow. In other words, all germs of holomorphic diffeomorphisms are time-1 maps of some formal vector fields: time-t map is not neccessarily convergent for a general t. So we may seek relations of time-1 maps of formal vetor fields. To a word of those time-1 maps there corresponds a piecewise linear Feynman diagram γ in the space of formal vector fields $\hat{\chi}(\mathbb{C},0)$ without constant terms: To a time-1 map $\exp a\partial_z$ of $a\partial_z \in \hat{\chi}(\mathbb{C},0)$ corresponds a segment which is a parallel translation of the vector $a\partial_z \in \hat{\chi}(\mathbb{C},0)$. By the view point of differential geometry we are led to a generalization the notion of word of diffeomorphisms, i.e., piecewise smooth curves in $\hat{\chi}(\mathbb{C})$.

Now consider the tautological $\hat{\chi}(\mathbb{C})$-valued 1-form on $\hat{\chi}(\mathbb{C})$

$$\nabla : \qquad \partial_z \, dz = \omega \; .$$

This defines a $\hat{\chi}(\mathbb{C})$-valued connection on the trivial $(\mathbb{C},0)$-bundle over $\hat{\chi}(\mathbb{C})$. The holonomy map of the bundle along the piecewise linear path γ is nothing but the word of time-1 maps of formal vector fields corresponding to each segment of the path. All results on the Lie integral remain valid for piecewise smooth loop γ in the Lie algebra.

7 Taylor coefficients of Lie integral of $\omega = a_1\partial_z dx_1 + a_2\partial_z dx_2$

Let us calculate some coefficients of Taylor expansion of Lie integral $L\int_\gamma \omega$ for formal vector fields without linear term

$$a_1\partial_z = \left(a_{12}z^2 + a_{13}z^3 + \cdots\right)\partial_z,$$
$$a_2\partial_z = \left(a_{22}z^2 + a_{23}z^3 + \cdots\right)\partial_z.$$

Let

$$A_i = \begin{bmatrix} a_{1i} \\ a_{2i} \end{bmatrix} \quad , \qquad \text{and} \qquad K_i = a_{1i}x_1 + a_{2i}x_2 \; .$$

Assume γ is closed. Then $H_1 = T_0 = \int_\gamma \omega = 0$. Let $K(t) = a_1\partial_z \frac{dx_1}{dt} + a_2\partial_z \frac{dx_2}{dt}$. Then

$$
\begin{aligned}
T_k &= \int\limits_{0\le u_{k+1}\le\cdots\le u_1\le 1}\!\!\cdots\!\int du_1\cdots du_{k+1}[\cdots[[K(u_1),K(u_2)],\ldots,K(u_k)],K(u_{k+1})] \\
&= -\sum_{1\le i_1,i_2,\cdots i_{k-1}\le 2}[\cdots[[a_1\partial_z,a_2\partial_z],a_{i_1}\partial_z],\cdots,a_{i_{k-1}}\partial_z] \\
&\qquad\times \int\limits_{0\le u_{k+1}\le\cdots\le u_1\le 1}\!\!\cdots\!\int du_1\cdots du_{k+1}\frac{dx_{i_{k-1}}}{du_{k+1}}\cdots\frac{dx_{i_1}}{du_3}\left[\frac{dx_1}{du_2},\frac{dx_2}{du_1}\right] \\
&= -\sum_{1\le i_1,i_2,\cdots i_{k-1}\le 2}[\cdots[[a_1\partial_z,a_2\partial_z],a_{i_1}\partial_z],\cdots,a_{i_{k-1}}\partial_z]\int_\gamma dx_{i_{k-1}}\cdots dx_{i_2}dx_{i_1}[dx_1,dx_2].
\end{aligned}
$$

First assume $\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix} \neq 0$. Then Taylor expansion of H_2, H_3 and H_4 are respectively

$$
\begin{aligned}
H_2 &= \frac{1}{2}T_1 \\
&= -\frac{1}{2}[a_1\partial_z,a_2\partial_z]\int_\gamma [dx_1,dx_2] \\
&= -\left\{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix} z^4 + 2\begin{vmatrix} a_{12} & a_{14} \\ a_{12} & a_{14}\end{vmatrix} z^5 + \left(3\begin{vmatrix} a_{12} & a_{15} \\ a_{22} & a_{25}\end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24}\end{vmatrix}\right)z^6 + \cdots\right\} \\
&\qquad\times \iint_D \rho\; dx_1\wedge dx_2\;\partial_z,
\end{aligned}
$$

Since $T_0 = 0$,

$$
\begin{aligned}
H_3 &= \frac{1}{3}T_2 + \frac{1}{6}[T_0,H_2] = \frac{1}{3}T_2 \\
&= -\frac{1}{3}\left\{[[a_1\partial_z,a_2\partial_z],a_1\partial_z]\int_\gamma dx_1[dx_1,dx_2] + [[a_1\partial_z,a_2\partial_z],a_2\partial_z]\int_\gamma dx_2[dx_1,dx_2]\right\} \\
&= \Bigg[-\frac{2}{3}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix}\left(a_{12}\int_\gamma dx_1[dx_1,dx_2] + a_{22}\int_\gamma dx_2[dx_1,dx_2]\right)z^5 \\
&+ \Bigg\{-2\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24}\end{vmatrix}\left(a_{12}\int_\gamma dx_1[dx_1,dx_2] + a_{22}\int_\gamma dx_2[dx_1,dx_2]\right) \\
&- \frac{1}{3}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix}\left(a_{13}\int_\gamma dx_1[dx_1,dx_2] + a_{23}\int_\gamma dx_2[dx_1,dx_2]\right)\Bigg\}z^6 + \cdots\Bigg]\partial_z \\
&= \Bigg\{-2\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix}\iint_D \rho\; K_2 dx_1\wedge dx_2 z^5 \\
&+ \left(-6\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24}\end{vmatrix}\iint_D \rho\; K_2 dx_1\wedge dx_2 - \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{vmatrix}\iint_D \rho\; K_3 dx_1\wedge dx_2\right)z^6 + \cdots\Bigg\}\partial_z,
\end{aligned}
$$

Since $T_0 = H_1 = 0$,

$$\begin{aligned} H_4 &= \frac{1}{4}T_3 + \frac{1}{8}[H_1, T_2] + \frac{4}{k_2}[H_1,[H_1,T_0]] + \frac{1}{8}[H_3, T_0] = \frac{1}{4}T_3 \\ &= -\frac{1}{4}\sum_{1\le i_1,i_2\le 2} [[[a_1\partial_z, a_2\partial_z], a_{i_1}\partial_z], a_{i_2}\partial_z] \times \int_\gamma dx_{i_1}dx_{i_2}[dx_1, dx_2] \\ &= -\frac{1}{4}\left\{ \left(6 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \sum_{1\le i_1,i_2\le 2} a_{i_1}a_{i_2} \int_\gamma dx_{i_1}dx_{i_2}[dx_1,dx_2] \right) z^6 + \cdots \right\}\partial_z \\ &= \left\{ \left(-3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \iint_D \rho\ {K_2}^2 dx_1 \wedge dx_2 \right) z^6 + \cdots \right\}\partial_z, \end{aligned}$$

Thus Lie integral $L\int_\gamma \omega = H_1 + H_2 + H_3 + \cdots$ is in the form

$$\begin{array}{rcr} H_1 &=& 0 \\ H_2 &=& (*z^4 + *z^5 + *z^6 + *z^7 + \cdots)\ \partial_z \\ H_3 &=& (*z^5 + *z^6 + *z^7 + \cdots)\ \partial_z \\ H_4 &=& (*z^6 + *z^7 + \cdots)\ \partial_z \\ +) \qquad & \vdots & \\ \hline L\int_\gamma \omega &=& (*z^4 + *z^5 + *z^6 + *z^7 + \cdots)\ \partial_z \end{array}$$

From above calculation we see that the Taylor coefficients of $L\int_\gamma \omega$ are as follows.

The coefficient of $z^4 = -\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \iint_D \rho\ dx_1 \wedge dx_2 = -\iint_D \rho\ dK_2 \wedge dK_3$.

The coefficient of $z^5 = 2\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \iint_D \rho\ K_2 dx_1 \wedge dx_2 = 2\iint_D \rho\ K_2 dK_2 \wedge dK_3$,

$$\bmod \iint_D \rho\ dx_1 \wedge dx_2.$$

The coefficient of $z^6 = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \iint_D \rho\ (K_3 - 3{K_2}^2) dx_1 \wedge dx_2$

$$= \iint_D \rho\ (K_3 - 3K_2^2) dK_2 \wedge dK_3,$$

$$\bmod \iint_D \rho\ dx_1 \wedge dx_2, \iint_D \rho\ K_2 dx_1 \wedge dx_2.$$

The coefficient of $z^7 = 4\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \iint_D \rho\ ({K_2}^3 - K_2K_3) dx_1 \wedge dx_2$

$$= 4\iint_D \rho\ (K_2^3 - K_2K_3) dK_2 \wedge dK_3,$$

$$\bmod \iint_D \rho\ dx_1 \wedge dx_2, \iint_D \rho\ K_2 dx_1 \wedge dx_2,$$

$$\iint_D \rho\ (K_3 - 3{K_2}^2) dx_1 \wedge dx_2 .$$

In general, the coefficient of $z^k (k \geq 8)$ is

$$\iint_D \rho \left((k-5)^2 K_3 - \frac{(k-5)(k-4)(k-3)}{2} K_2^2\right) dK_2 \wedge dK_{k-3}$$
$$- (k-7)\rho \ K_{k-3} dK_2 \wedge dK_3$$
$$\mod \iint_D \rho \ dx_1 \wedge dx_2, \iint_D \rho \ K_2 dx_1 \wedge dx_2, R_k(A_2, A_3, \ldots, A_{k-4})$$
$$= \iint_D \rho \left\{ -\frac{1}{6}(k-5)(k-6)(k-7)K_3 + \frac{1}{a_{22}}(k-7) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} x_1 \right\} dK_2 \wedge dK_{k-3}$$
$$\mod \iint_D \rho \ (K_3 - 3K_2^2) dx_1 \wedge dx_2$$

8 Relations in two formal diffeomorphisms with trivial linear terms

Here we consider time-1 maps of two vector fields

$$a_1 \partial_z = (a_{11} z + a_{12} z^2 + a_{13} z^3 + \cdots) \ \partial_z$$
$$a_2 \partial_z = (a_{21} z + a_{22} z^2 + a_{23} z^3 + \cdots) \ \partial_z,$$

and their relations such that the diagram γ and its dual γ^* are closed. From now on we write $x_1 = x$ and $x_2 = y$ for simplicity. All those diagrams are confined in the 2-plane in the Lie algebra $\hat{\chi}(\mathbb{R})$ spanned by $a_1 \partial_z$ and $a_2 \partial_z$. Thus we suppose the plane is the xy-plane: x, y axes correspond respectively to $a_1 \partial_z$ and $a_2 \partial_z$ directions, and we draw the diagram γ in the xy-plane.

By the results in the previous section, the condition

$$z^4\text{-term} = 0, \ z^5\text{-term} = 0, \ z^6\text{-term} = 0, \ z^7\text{-term} = 0$$

is equivalent to the following condition

$$\left\{ \begin{array}{ll} \iint_D \rho \ dK_2 \ \wedge dK_3 = 0, & \text{(i)} \\ \iint_D \rho \ K_2 \ dK_2 \wedge dK_3 = 0, & \text{(ii)} \\ \iint_D \rho \ (K_3 - 3K_2^2) \ dK_2 \wedge dK_3 = 0, & \text{(iii)} \\ \iint_D \rho \ (K_2^3 - K_2 K_3) \ dK_2 \wedge dK_3 = 0 & \text{(iv)} \end{array} \right.$$

The *Area* of $\gamma \subset \mathbb{R}^2$ is

$$Area(\gamma) = \iint_D \rho \ dx \wedge dy \ .$$

The *moment* of $\gamma \subset \mathbb{R}^2$ is

$$G(\gamma) = (\iint_D \rho \ x \ dx \wedge dy, \iint_D \rho \ y \ dx \wedge dy) \ .$$

The above second condition is equivalent that the vector A_2 be orthogonal to the moment $G(\gamma)$.

Theorem 8.1. *Let γ be a closed Feynman diagram with* $\mathrm{Area}(\gamma) = 0$ *and* $G(\gamma) \neq 0$. *Assume* $A_1 = 0, A_2 \neq 0, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \neq 0$ *and the 3-jets of* a_1, a_2 *satisfy the above 4 conditions. Then the equation* $L \int_\gamma \omega = 0$ *admits formal solutions* a_1, a_2. *The 4-th order term of* a_1, a_2 *can be arbitrary. If the y-moment* $\iint_D \rho\ y\ dx \wedge dy$ *is not 0, then the Taylor coefficients of* a_1 *of order* ≥ 5 *can be arbitrary, and if the x-moment* $\iint_D \rho\ x\ dx \wedge dy$ *is not 0, then the Taylor coefficients of* a_2 *of order* ≥ 5 *can be arbitrary.*

Proof. Under the above 4 conditions, the z^k term has the coefficient

$$C_k = \iint_D \rho \left\{ -\frac{1}{6}(k-5)(k-6)(k-7)K_3 + \frac{1}{a_{22}}(k-7) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} x_1 \right\} dK_2 \wedge dK_{k-3}$$
$$+ R_k\ (A_2, \dots, A_{k-4})$$

for $k \geq 8$. The first integration part is the inner product

$$((k-5)(k-6)a_{13}, (k-5)(k-6)a_{23}) - (\frac{6}{a_{22}} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, 0)) \bullet G(\gamma) \times \iint_D \rho\ dK_2 \wedge dK_{k-3},$$

where $\bullet$ stands for the inner product of the plane vectors. One can solve the linear equation $C_k = 0$ in terms of K_{k-3}, if the inner product is not zero. Since A_2 is orthogonal to $G(\gamma)$ by the second condition, we may replace $G(\gamma)$ with $(-a_{22}, a_{12})$. Then the inner product becomes

$$((k-5)(k-6)+6) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

which is not 0 for all $k \geq 8$. The second statment is easily seen as A_2 is orthogonal to $G(\gamma)$ and we may add $\lambda K_2, \lambda \in \mathbb{C}$ to K_{k-3} without changing $\iint_D \rho\ dK_2 \wedge dK_{k-3}$ for $k = 8, 9, \dots$. □

Theorem 8.2. *Let* $\gamma \subset \mathbb{R}^2$ *be a closed Feynman diagram. Assume* $\mathrm{Area}(\gamma) = 0$ *and* $G(\gamma) \neq 0$. *Let* $A_2 = (a_{12}, a_{22}) \neq 0$ *be orthogonal to* $G(\gamma)$, *and assume*

$$\iint_D \rho\ {K_2}^2\ dx \wedge dy \neq 0.$$

Then the relation $W_{\gamma^*}(f, g) = 1$ *admits formal non commuting solutions* f, g *such that* $f'(0) = g'(0) = 1, (f''(0), g''(0)) = A_2$. *And the 4-jet of* f, g *can be arbitrary. If the y-moment* $\iint_D \rho\ y\ dx \wedge dy$ *is not 0, then the Taylor coeficients of* f. *of order* ≥ 5 *can be arbitrary, and if the x-moment* $\iint_D \rho\ x\ dx \wedge dy$ *is not 0, then the Taylor coeficients of* g *of order* ≥ 5 *can be arbitrary.*

Proof. Assume $a_{11}, a_{21} = 0$. By the results in the previous section, the holonomy map along the diagram γ^* has a Taylor expansion starting with the z^4-term.

The second condition tells the vector $A_2 = (a_{12}, a_{22})$ is orthogonal to the moment. So we may suppose

$$K_2 = a_{12}x + a_{22}y = -\iint_D \rho\ y\ dx \wedge dy\ x + \iint_D \rho\ x\ dx \wedge dy\ y$$

Conditions (iii), (iv) are equivalent to the linear equation

$$\begin{bmatrix} \iint_D \rho\, x\, dx \wedge dy & \iint_D \rho\, y\, dx \wedge dy \\ \iint_D \rho\, K_2 x\, dx \wedge dy & \iint_D \rho\, K_2 y\, dx \wedge dy \end{bmatrix} A_3 = \begin{bmatrix} 3 \iint_D \rho\, K_2^2\, dx \wedge dy \\ \iint_D \rho\, K_2^3\, dx \wedge dy \end{bmatrix}.$$

The determinant of the above 2×2-matrix is $\iint_D \rho\, K_2^2\, dx \wedge dy$, which is not 0 by the assumption. Thus the above linear equation has a solution $A_3 = (a_{13}, a_{23})$. By the assumption $\iint_D \rho\, {K_2}^2\, dx \wedge dy \neq 0$ and Condition (iii), we see A_3 is not parallel to A_2 hence f, g do not commute. The solution A_3 depends on A_2: for λA_2 the solution is $\lambda^2 A_3$. The second part of the theorem follows from the same argument as in the proof of the previous theorem.

□

9 Relations in two formal diffeomorphisms with non trivial linear terms

Next let us consider the case $A_1 \neq 0$. From now on we assume γ is a composite of horizontal or vertical segments of length 1 in the plane. Similarly to §7, we obtain the followings.
The coefficient of z^2 in the Taylor expansion of $L \int_\gamma \omega = \sum_{s=1}^{\infty} H_s$ is

$$L_2 = \int_\gamma e^{-K_1} dK_2.$$

Thus the coefficient of z^3 in the Taylor expansion of $L \int_\gamma \omega$ is

$$L_3 = \int_\gamma e^{-2K_1} dK_3.$$

In general, the coefficient of $z^k (k \geq 4)$ in the Taylor expansion of $L \int_\gamma \omega$ is of the form

$$L_k = \int_\gamma e^{-(k-1)K_1} dK_k + R_k(A_1, A_2, \ldots, A_{k-1}).$$

In the case of non zero linear terms, the nature of the remainder term R_k is completely unkown. By the convergence theorem by Chacon and Fomenko in §4, each coefficient is well defined and analytic for sufficiently small γ. But the infimum of the *radii* of convergence might be 0.

The first term

$$\hat{L}_k = \int_\gamma e^{-(k-1)K_1} dK_k = -(k-1) \iint_D \rho\, e^{-(k-1)K_1}\, dK_1 \wedge dK_k$$

is of the form

$$-(k-1) \sum_{(i,j) \in \mathbb{Z}^2} \rho(i,j)\, e^{-(k-1)K_1(i,j)} \begin{vmatrix} a_{11} & a_{1k} \\ a_{21} & a_{2k} \end{vmatrix} \int_0^1 dx \int_0^1 dy\, e^{-(k-1)K_1(x,y)}$$

where $\rho\ (i,j)$ denotes the winding number of γ on the domain $\{i < x < i+1, j < y < j+1\}$. The summation part is a polynomial in $e^{-(k-1)a_{11}}, e^{-(k-1)a_{21}}$. Let the hights of γ in x and y directions be X and Y respectively. Then the degree of L_2 in $e^{-a_{11}}$ is $X-1$ and the degree in $e^{-a_{21}}$ is $Y-1$. Thus the equation

$$L_2 = 0 \quad , \quad L_3 = 0$$

has at most $2(X+Y-2)^2$-solutions counting multiplicity by Bézout theorem if $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \neq 0$ and L_2, L_3 do not have a common factor. The following theorem is a simple corollary of the above argument.

Theorem 9.1. *Let $f_i = \exp a_i \partial_z$ for $i = 1,2$ and assume $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ or $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \neq 0$, and $e^{-a_{11}}, e^{-a_{21}}$ admit no algebraic relation with integer coefficients, then f_1, f_2 play no relation with closed γ^* such that the winding number $\rho(\gamma^*)$ is not identically 0.*

On the other hand, if

$$\sum_{(i,j)\in\mathbb{Z}^2} \rho(i,j)\ e^{-(k-1)K_1(i,j)} \neq 0\ ,\ k = 3,4,\ldots$$

holds for one of the solutions of $\hat{L}_2 = \hat{L}_3 = 0$, then we can solve the equation

$$\int_\gamma e^{-(k-1)K_1} dK_k + R_k(A_1, \ldots, A_{k-1}) = 0$$

successively for $k = 4,5,\ldots$ and obtain a relation, provided the remainder terms are all well defined.

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