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Non-landing of stretching rays for real cubic polynomials and real biquadratic polynomials

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This is a joint work with Yohei Komori and details will be published in Komori-Nakane [KN].

1 Stretching rays

Let $\mathcal{P}_d$ be the family of monic centered polynomials of degree $d \geq 2$. For $P \in \mathcal{P}_d$, let φ_P be its Böttcher coordinate and $h_P(z) = \log_+ |\varphi_P(z)|$ be the Green function for P. For a complex number $u \in \mathbb{H}_+ = \{u = s+it \in \mathbb{C}, s > 0\}$, put $f_u(z) = z|z|^{u-1}$ and we define a P-invariant almost complex structure σ_u by

$$\sigma_u = \begin{cases} (f_u \circ \varphi_P)^* \sigma_0 \text{ on } U_P, \\ \sigma_0 \text{ on } K(P). \end{cases}$$

Then, by the Measurable Riemann Mapping Theorem, σ_u is integrated by a qc-map F_u and $P_u = F_u \circ P \circ F_u^{-1} \in \mathcal{P}_d$. Thus we define a holomorphic map $W_P : \mathbb{H}_+ \to \mathcal{P}_d$ by $W_P(u) = P_u$. This qc-deformation, what we call *wringing*, is the same as the *Branner-Hubbard motion* in the talk of A. Douady [A]. See also Branner [B] and Branner-Hubbard [BH].

The Böttcher coordinate φ_{P_u} of P_u is equal to $f_u \circ \varphi_P \circ F_u^{-1}$. Since P_u is hybrid equivalent to P, it holds $P_u \equiv P$ for $P \in \mathcal{C}_d$, the *connectedness locus*. For $P \in \mathcal{E}_d$, the *escape locus*, we define the *stretching ray* through P by

$$R(P) = W_P(\mathbb{R}_+) = \{P_s; s \in \mathbb{R}_+\}.$$

In case $d = 2$, stretching rays coincide with the *external rays* for the Mandelbrot set. Because of the conjecture that the Mandelbrot set is locally connected, it seems that all the external rays of the Mandelbrot set land. Here we show the existence of non-landing stretching rays for real cubic polynomials.

2 Stretching rays for real cubic polynomials

We consider the family of real cubic polynomials :

$$P(z) = P_{A,B}(z) = z^3 - 3Az + \sqrt{B}; \quad A, B > 0.$$

We restrict our attention to the first quadrant of the parameter space. Then the connectedness locus $\mathcal{C}_3(\mathbb{R})$ is bounded by two real algebraic curves :

$$\begin{aligned} Per_1(1) &= \{B = 4(A+1/3)^3; 0 \le A \le 1/9\}, \\ Preper_{(1)1} &= \{B = 4A(A-1)^2; 1/9 \le A \le 1\}. \end{aligned}$$

For $Q \in Per_1(1)$, Q has a parabolic fixed point $\beta_Q = \sqrt{A+1/3}$ with multiplier 1.

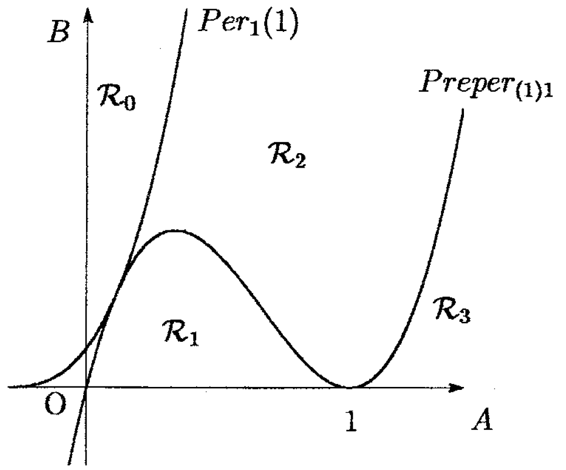

Figure 1: Parameter space for real cubics

We investigate the landing properties of stretching rays in the region $\mathcal{R}_0 : B > 4(A+1/3)^3$, that is, above the parabolic arc $Per_1(1)$. This region is contained in the *shift locus* $\mathcal{S}_3(\mathbb{R})$, i.e. the locus where both critical points $\pm\sqrt{A}$ escape to ∞.

We set $\zeta_P(z) = \dfrac{\log\log\varphi_P(z)}{\log 3}$ and define the *Böttcher vector* $\eta(P)$ for $P \in \mathcal{S}_3(\mathbb{R})$ by

$$\eta(P) = \frac{\log h_P(-\sqrt{A}) - \log h_P(\sqrt{A})}{\log 3} = \zeta_P(P(-\sqrt{A})) - \zeta_P(P(\sqrt{A})).$$

Lemma 2.1. *On the stretching ray $R(P)$ through $P \in \mathcal{S}_3(\mathbb{R})$, $\eta(P_s)$ is invariant.*

Thus each stretching ray in the shift locus $\mathcal{S}_3(\mathbb{R})$ is a level curve $\eta(P) = \eta$ of the Böttcher vector map $P \mapsto \eta(P)$, which we denote by $R(\eta)$. So, we have an explicit description of stretching rays and we can draw their pictures. See Figures 2 and 3.

3 Non-landing stretching rays

For $Q \in Per_1(1)$, the immediate basin $\mathcal{B}_Q$ of the parabolic fixed point β_Q contains both critical points $\pm\sqrt{A}$ and $J(Q) = \partial\mathcal{B}_Q$ is a Jordan curve. Let $\phi_{Q,-}$ and $\phi_{Q,+}$ be the attracting and repelling *Fatou coordinates* respectively normalized appropriately so that they are symmetric with respect to the real axis. They satisfy $\phi_{Q,\pm} \circ Q = T_1 \circ \phi_{Q,\pm}$, where $T_\alpha(w) = w + \alpha$ is a translation by α. We define the *Fatou vector* $\tau(Q)$ of Q by $\tau(Q) = \phi_{Q,-}(-\sqrt{A}) - \phi_{Q,-}(\sqrt{A})$.

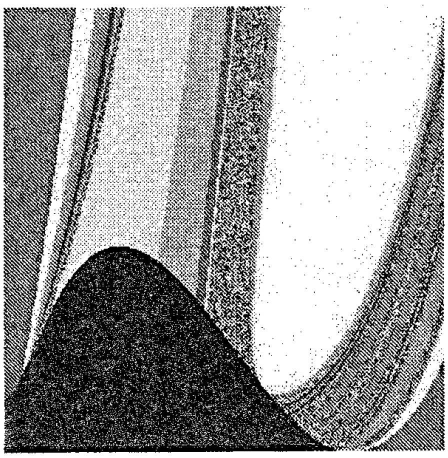

Figure 2: Parameter space for real cubics

Figure 3: Oscillating stretching rays

Lemma 3.1. *The Fatou vector gives a real analytic parametrization of $Per_1(1)$.*

Lemma 3.2. *The stretching ray $R(\eta)$ with $\eta \in \mathbb{Z}$ lands at a map $Q \in Per_1(1)$ with $\tau(Q) = \eta$. Conversely, at a map $Q \in Per_1(1)$ with $\tau(Q) \in \mathbb{Z}$, the stretching ray $R(\eta)$ with $\eta = \tau(Q)$ lands.*

Actually $R(k)$ is a real algebraic curve $P(-\sqrt{A}) - P^{k+1}(\sqrt{A}) = 0$ for $k \geq 1$.

Theorem 3.1. *Suppose η is not integral. Then the stretching ray $R(\eta)$ does not land at any point on $Per_1(1)$. Hence its accumulation set is a non-trivial arc on $Per_1(1)$.*

4 Idea of proof

Let $\mathcal{A}(Q) = \phi_{Q,+}(\Omega_{Q,+} - K(Q))/T_1$ be the annulus in the repelling *Ecalle cylinder* $\phi_{Q,+}(\Omega_{Q,+})/T_1$, bounded by the images of the Julia set $J(Q)$. Note that ζ_Q maps $\Omega_{Q,+} - K(Q)$ conformally onto a right half region Σ in the strip $\{|\mathrm{Im}\,\zeta| < \pi/(2\log 3)\}$ and satisfies $\zeta_Q \circ Q = T_1 \circ \zeta_Q$ there (the same functional equation as the Fatou coordinates). This induces a flat annulus $\mathcal{A}'(Q) = \{\zeta \in \mathbb{C}/\mathbb{Z} \mid |\mathrm{Im}\,\zeta| < \pi/(2\log 3)\}$ of modulus $\pi/\log 3$. Then the quotient map $\psi_Q : \mathcal{A}'(Q) \to \mathcal{A}(Q)$ of the map $\phi_{Q,+} \circ \zeta_Q^{-1} : \Sigma \to \Omega_{Q,+} - K(Q)$ gives a conformal equivalence between the annuli $\mathcal{A}'(Q)$ and $\mathcal{A}(Q)$. See Figure 4.

By virtue of the parabolic implosion analysis, we have the following.

Lemma 4.1. *Suppose $R(\eta)$ lands at $Q \in Per_1(1)$. Then $\psi_Q(\zeta + [\eta]) = \psi_Q(\zeta) + [\tau(Q)]$ holds for any ζ. Especially it follows $\tau(Q) = \eta$.*

Now, we give an idea of the proof of Theorem 3.1. In case $\eta \in \mathbb{R} - \mathbb{Q}$, using the fact that $\{[n\eta]; n \in \mathbb{Z}\}$ is dense in $\mathbb{R}/\mathbb{Z}$, we conclude that $\mathcal{A}(Q)$ must be a flat annulus. Then $J(Q)$ must be a real analytic curve. This contradicts the fact that $J(Q)$ is not differentiable at any point on the inverse orbit of the parabolic fixed point β_Q.

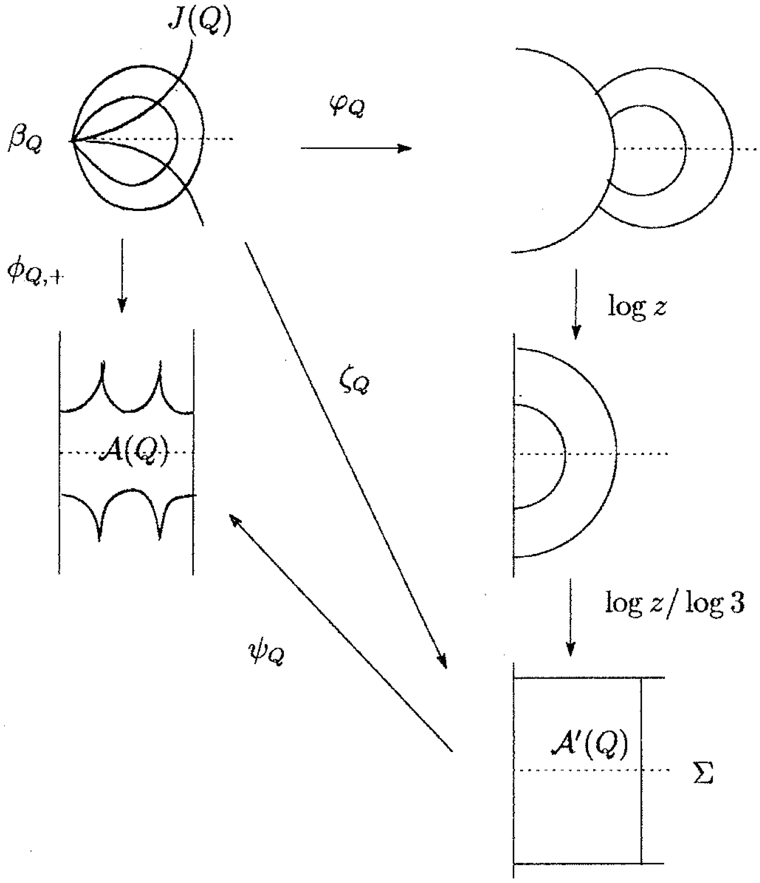

Figure 4: Annuli $\mathcal{A}'(Q)$ and $\mathcal{A}(Q)$

In case $\eta = p/q \in \mathbb{Q} - \mathbb{Z}$, we need a more careful analysis (due to Weixiao Shen). We use the fact that $J(Q)$ is cusp at every point on the inverse orbit of β_Q. On the other hand, the *radial Julia set* $J_{rad}(Q)$ is just the complement in $J(Q)$ of the inverse orbits of β_Q. Lemma 4.1 says that $\phi_{Q,+}(J(Q))$ is invariant under translation $T_{1/q}$, which corresponds to multiplication of the external angles by $3^{1/q} \in \mathbb{R} - \mathbb{Q}$. Let $z_0 \in J(Q)$ be the landing point of the external ray with angle 3^{-n}. Then it lies on the inverse orbit of β_Q and is a cusp. Then $z_1 = \phi_{Q,+}^{-1} \circ T_{1/q} \circ \phi_{Q,+}(z_0) \in J(Q)$ is also a cusp but it has irrational external angle $3^{1/q-n}$, hence $z_1 \in J_{rad}(Q)$. By the Koebe distortion theorem, it follows that, there exists a $k < 1$ so that, for any $r << 1$, $\mathbb{D}(z_1, r) - K(Q)$ contains a disk of radius kr. This contradicts the fact that z_1 is a cusp.

5 Stretching rays for real biquadratic polynomials

In the above proof, we essentially use the fact that $3^{1/q}$ is irrational for any $q \geq 2$. This argument does not work for degree four polynomials. In fact, as was pointed out by Milnor, there exists a landing stretching ray with Böttcher vector $1/2$.

Consider the family of real biquadratic polynomials :

$$P_{a,b}(z) = (z^2 + a)^2 + b, \quad (a, b) \in \mathbb{R}^2.$$

Each $P_{a,b}$ has two critical orbits and we can define Böttcher vectors as above. The connectedness locus of this family is surrounded by three algebraic curves :

- $Per_1^+(1) : -\frac{\sqrt[3]{2}}{4} \leq a \leq \frac{7\sqrt[3]{4}}{16}$,
- $Preper_{(1)1} : b = -a^2 + \sqrt{-2a}, \quad -2 \leq a \leq -\frac{\sqrt[3]{2}}{4}$,
- $Preper_{(2)1} : a = -b^2 + \sqrt{-2b}, \quad -2 \leq b \leq -\frac{\sqrt[3]{2}}{4}$.

Here the curve $Per_1^+(1)$ is one of the two connected components of the parabolic locus :

$$Per_1(1) : a^2b^2 + a^3 + b^3 + \frac{9ab}{8} - \frac{27}{256} = 0,$$

and is characterized by the locus where the β-fixed point (i.e. the maximum real fixed point) is parabolic with multiplier one. This curve is also parametrized by the Fatou vector.

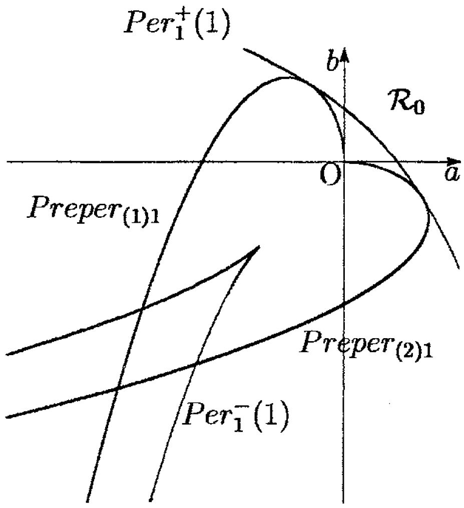

Figure 5: Parameter space for real biquadratics

Another component $Per_1^-(1)$ of the parabolic locus is not related to the boundary of the connectedness locus. See Figure 5.

We consider the stretching rays in the region $\mathcal{R}_0$ above this curve. By the same argument as for cubic polynomials, we can show the following. See Figures 6 and 7.

Theorem 5.1. *The stretching ray with an integral Böttcher vector lands at a map with the same Fatou vector. The stretching ray with Böttcher vector $\eta \notin \mathbb{Z} \cup (\mathbb{Z}+1/2)$ does not land at any point on $Per_1^+(1)$.*

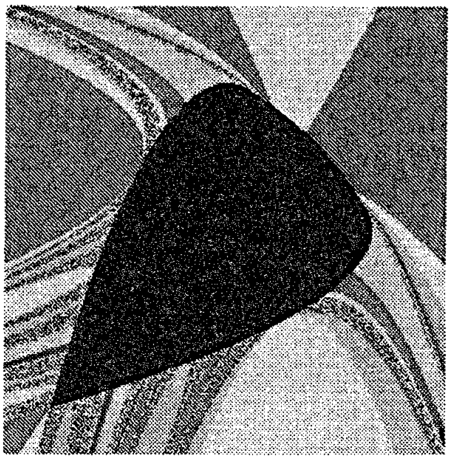

Figure 6: Parameter space for real biquadratics

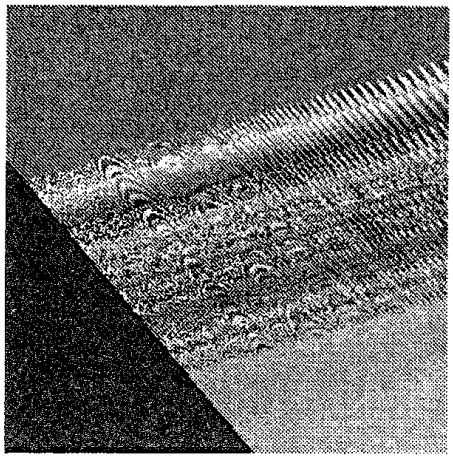

Figure 7: Oscillating stretching rays

The stretching ray with Böttcher vector $1/2$ is the ray $b = a > 1/4$, which lands at $(1/4, 1/4)$. But this seems to be exceptional since $Q_{a,a}$ is the second iterate of the quadratic polynomial $z^2 + a$. The stretching rays $R(\eta)$ with $\eta = n + 1/2$, $n \in \mathbb{Z} - \{0\}$ do not seem to land although we do not have a proof.

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