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A generalization of the Cremer theorem by the Nevanlinna theoretical argument

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This article is a resume of the paper [11]. We study an irrationally indifferent cycle of points or circles of a rational function, which is either Siegel or Cremer by definition. We give a clear interpretation of some Diophantine quantity associated with an irrationally indifferent cycle as a quantity arising in the Nevanlinna theory. As a consequence, we show that an irrationally indifferent cycle is Cremer if this Nevanlinna-theoretical quantity does not vanish, which generalize the classical Cremer condition.

Let f be a rational function of degree ≥ 2 and $f^k := f^{\circ k}$ for $k \in \mathbb{N}$. $F(f)$ and $J(f)$ denote the Fatou and Julia sets of f respectively. The classification of cyclic Fatou components D is known: The pair (g, D), where g is the first return map of f on D, is an *attractive basin* if $\{g^n\}_{n=1}^{\infty}$ converges to a point in D locally uniformly on D. The *parabolic basin* is similar, but $\{g^n\}_{n=1}^{\infty}$ converges to a point in the boundary of D locally uniformly on D. When g is a proper selfmap of D of degree ≥ 2, (g, D) is one of them. When g is a univalent selfmap of D, D is called a *rotation* domain since (g, D) is conformally conjugate to an irrational rotation on either a disk or an annulus, and called a *Siegel disk* or an *Herman ring* in each of cases. For the details, see [9], [1], [3], [8].

Our main interest is an irrationally indifferent cycle of points or circles.

Definition 1 (irrationally indifferent cycle of points or circles). A point z_0 in $\hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(z_0) = z_0$. The least such p is the *period* of z_0,

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$\{f^n(z_0)\}_{n=1}^p$ is a *cycle* of points, and $\lambda := (f^p)'(z_0)$ is the *multiplier* of it. This cycle of *points* is *irrationally indifferent* if $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$.

A topological circle $S \subset \hat{\mathbb{C}}$ is *periodic* if for some $p \in \mathbb{N}$, $f^p(S) = S$ *and* $f^p|S : S \to S$ is an orientation-preserving homeomorphism. The least such p is the *period* of S, $\{f^n(S)\}_{n=1}^p$ is a cycle of *circles*, and $\lambda := e^{2\pi i\alpha}$ is the *multiplier* of it, where $\alpha \in \mathbb{R}/\mathbb{Z}$ is the *rotation number* (cf. [5]) of a S^1-homeomorphism ϕ which is topologically conjugate to $f^p|S$. This cycle of *circles* is *irrationally indifferent* if α is irrational.

It is known that if irrationally indifferent cycles of points or circles intersect $F(f)$, then they are contained in some rotation domains.

Definition 2 (Siegel and Cremer cycles). An irrationally indifferent cycle of points or circles is a *Siegel cycle* if it is contained in $F(f)$. Otherwise it is a *Cremer cycle.*

The following is an unsolved problem with a *long* history: *Given an irrationally indifferent cycle of points or circles, how can we judge whether it is contained in the Fatou set or not?*

The following answers the problem for points in one direction.

Theorem 1 (Siegel, Brjuno, Rüssmann and Yoccoz). *Every analytic germ* $f(z) = \lambda z + O(z^2)$, $\lambda = e^{2\pi i\alpha}$ $(\alpha \in \mathbb{R} - \mathbb{Q})$, *at the origin is analytically linearizable if* λ *satisfies the Brjuno condition, which is a Diophantine(-type) condition.*

For the precise definition of Brjuno condition, see [2] and [17]. In the reverse direction,

Theorem 2. *Let P be a quadratic polynomial. An irrationally indifferent cycles of points of P is Cremer if its multiplier does not satisfy the Brjuno condition.*

Remark 1. Theorem 2 is proved by Yoccoz ([17]) in the case of period one, and, from this, later generalized by the author ([10]) in the case of arbitrary periods.

Until now, only for quadratic polynomials and only for points, the complete answer of the problem is known. The classical Cremer Theorem is a partial answer for the problem for points in the reverse direction:

Theorem 3 (Cremer[4] (1932)). *Let* f *be a rational function of degree* $d \geq 2$, *and O be an irrationally indifferent cycle of points of period p and of multiplier* λ.

O is Cremer if λ *satisfies*

$$\limsup_{n\to\infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} = \infty. \tag{1}$$

We succeeded in improving the Cremer condition (1) and extending his result to irrationally indifferent cycles of circles.

Main Theorem 1 (Criterion for Cremer [11]). *Let f be a rational function of degree $d \geq 2$, and O be an irrationally indifferent cycle of either points or circles of period p and of multiplier λ.*

O is Cremer if λ satisfies

$$\limsup_{n\to\infty} \frac{1}{d^{pn}} \log \frac{1}{|\lambda^n - 1|} > 0. \tag{2}$$

This improvement is substantial. In fact, For every $x \in [0, \infty]$, it is easy to find $\alpha \in \mathbb{R} - \mathbb{Q}$ such that the left hand side of (2) equals x (cf. [6]).

Remark 2. In the case of polynomials and irrationally indifferent cycle of points, Pierre Tortrat ([16]) showed a similar result to Main Theorem 1 by using a potential theoretical argument. Indeed, his condition, which is slightly technical, coincides with ours.

Main Theorem 1 naturally follows from the interpretation of the left hand side of (1) as a *Nevanlinna-theoretical* quantity, and its *vanishing theorem.*

Let $[p, q]$ be the chordal distance between $p, q \in \hat{\mathbb{C}}$ such that $[0, \infty] = 1$, and σ the spherical area measure on $\hat{\mathbb{C}}$ such that $\sigma(\hat{\mathbb{C}}) = 1$. We write the set of all rational endomorphism of $\hat{\mathbb{C}}$ by Rat, and call a sequence in Rat a *rational sequence.*

The following tools of the Nevanlinna theory for rational sequences was invented by Sodin in [15].

For $f, g \in$ Rat, we define the *pointwise proximity function*:

$$(w(g, f))(z) := \log \frac{1}{[g(z), f(z)]} : \hat{\mathbb{C}} \to [0, \infty],$$

and the *mean proximity*:

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) d\sigma \in [0, \infty).$$

For a rational sequence $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$, we define the *characteristic sequence* $\{T_{\mathcal{F}}(k) := \deg f_k\}_{k=1}^{\infty} \subset \mathbb{N}$.

Using the above tools, we shall define some deficiency of a rational sequence with the increasing characteristic sequence:

Definition 3 (cf. [7]). Let $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ be a rational sequence with the increasing characteristic sequence. For $g \in$ Rat, we define the *Valiron exceptionality*:

$$\mathrm{VE}(g; \mathcal{F}) := \limsup_{k\to\infty} \frac{m(g, f_k)}{T_{\mathcal{F}}(k)} \in [0, \infty].$$

Although the *Valiron exceptionality* is defined by a quite complicate way, we can treat it by the following.

Main Theorem 2 (Fundamental Equality). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$, *and* $g \in \mathrm{Rat}$ *not identically equal to such* $z \in \hat{\mathbb{C}}$ *as* $\# \bigcup_{n \in \mathbb{N}} f^{-n}(z) < \infty$.

Then for every positive continuous function $\phi \not\equiv 0$ *on* $\hat{\mathbb{C}}$,

$$\mathrm{VE}(g; \{f^k\}) = \limsup_{k \to \infty} \frac{\int_{\hat{\mathbb{C}}} \phi \cdot w(g, f^k) d\sigma}{d^k \cdot \int_{\hat{\mathbb{C}}} \phi d\sigma}.$$

By the fundamental equality, we obtain two Main Theorems.

Main Theorem 3 (Vanishing Theorem). *Let* f *be a rational function of degree* ≥ 2 *such that* $F(f) \neq \emptyset$. *Then* $\mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}, \{f^k\}) = 0$.

Problem. Does $\mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}, \{f^k\})$ always vanish even if $J(f) = \hat{\mathbb{C}}$?

Main Theorem 4 (Natural Equality). *Let* O *be an irrationally indifferent cycle of either points or circles of period* p *and of multiplier* λ. *If* O *is Siegel, then*

$$\limsup_{k \to \infty} \frac{1}{d^{pk}} \log \frac{1}{|\lambda^k - 1|} = \mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}; \{f^{pk}\}). \tag{3}$$

Now the proof of Main Theorem 1 is straightforward:

Proof of Main Theorem 1. If O is Siegel, then $F(f) \neq \emptyset$. Hence from the Vanishing Theorem, $\mathrm{VE}(\mathrm{Id}_{\hat{\mathbb{C}}}; \{f^{pk}\}) = 0$, so by the Natural equality, the left hand side of (3) vanishes. □

Remark 3. Our method of studying multipliers of irrationally indifferent cycles unifies the cases both of points and of circles, and in particular, dispenses with such quasiconformal surgeries as in [12] for Siegel cycles of circles. However, when this cycle of circles is contained in an Herman ring, by quasiconformal surgery of it (cf. [12], [13] and [14]), we obtain a rational function $\tilde{f}$ whose degree is less than that of f and which has a Siegel disk with the same rotation number as the original Herman ring of f. Hence by applying Main Theorem 1 to $\tilde{f}$ rather than f, a stronger conclusion than Main Theorem 1 follows.

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