数理解析研究所講究録 1447 巻 2005 年 182-186

Hyperbolic Hausdorff dimension is equal to the minimal exponent of conformal measure on Julia set. A simple proof.[1]

by Feliks Przytycki[2]

The fact in the title was proved in [DU] except one point proved later in [P1]. This is a crucial technical fact in the study of dimensions and their continuity for Julia sets. The proofs in these papers use several complicated techniques. Here we give a simple proof.

Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational mapping of degree $d\geq 2$ on the Riemann sphere $\overline{\mathbb{C}}$. We denote by Crit the set of critical points, that is $f'(x)=0$ for $x\in$ Crit. The symbol J stands for the Julia set of f. Absolute values of derivatives and distances are considered with respect to the standard Riemann sphere metric. We consider pressures below for all $t>0$.

Definition 1. *Tree pressure.* For every $z\in\overline{\mathbb{C}}$ define

$$P_{\mathrm{tree}}(z,t):=\limsup_{n\to\infty}\frac{1}{n}\ln\sum_{f^n(x)=z}|(f^n)'(x)|^{-t}.$$

Definition 2. *Hyperbolic pressure.*

$$P_{\mathrm{hyp}}(t):=\sup_X P(f|_X,-t\ln|f'|),$$

where the supremum is taken over all compact f-invariant (that is $f(X)\subset X$) isolated hyperbolic subsets of J.

Isolated (sometimes called *repelling*) means that there is a neighbourhood U of X such that $f^n(x)\in U$ for all $n\geq 0$ implies $x\in X$. *Hyperbolic* means that there is a constant $\lambda_X>1$ such that for all n large enough and all $x\in X$ we have $|(f^n)'(x)|\geq\lambda_X^n$. Sometimes the more adequate term *expanding* is used.

$P(f|_X,-t\ln|f'|)$ denotes the standard topological pressure for the continuous mapping $f|_X:X\to X$ and continuous real-valued potential function $-t\ln|f'|$ on X, see for example [W].

Note that these definitions imply that $P_{\mathrm{hyp}}(t)$ is a continuous monotone decreasing function of t.

Definition 3. *Conformal pressure.* Set $P_{\mathrm{Conf}}(t):=\ln\lambda(t)$, where

$$\lambda(t)=\inf\{\lambda>0:\exists\mu,\text{ a probability measure on } J \text{ with Jacobian } \lambda|f'|^t\}.$$

---

[1] extracted from [PRS] F. Przytycki, J. Rivera-Letelier, S. Smirnov "Equality of pressures for rational functions", to appear in Ergodic Theory and Dynamical Systems

[2] Institut Henri Poincaré, Paris, September 2003 and RIMS, Kyoto, February 2004

We say that $\varphi : J \to \mathbb{R}, \varphi \geq 0$ is the *Jacobian for* $f|_J$ *with respect to* μ if φ is μ-integrable and for every Borel set $E \subset J$ on which f is injective $\mu(f(E)) = \int_E \varphi d\mu$. We write $\varphi = \mathrm{Jac}_\mu(f|_J)$.

We call any probability measure μ on J with Jacobian of the form $\lambda|f'|^t$ a (λ, t)-*conformal measure* and with Jacobian $|f'|^t$ a *conformal measure* with exponent t or t-*conformal measure*.

Proposition 1. For each $t > 0$ the number $P_{\mathrm{Conf}}(t)$ is attained, that is there exists a (λ, t)-conformal measure with $\lambda = P_{\mathrm{Conf}}(t)$.

This Proposition follows from the following

Lemma. If μ_n is a sequence of (λ_n, t)-conformal measures of J for an arbitrary $t > 0$, weakly* convergent to a measure μ and $\lambda_n \to \lambda$ then μ is a (λ, t)-conformal measure.

Proof. Let $E \subset J$ on which f is injective. E can be decomposed into a countable union of critical points and sets E_i pairwise disjoint and such that f is injective on a neighbourhood V of $\mathrm{cl}E_i$. For every ε there exist compact set K and open U such that $K \subset E_i \subset U \subset V$ and $\mu(U) - \mu(K) < \varepsilon$ and $\mu(f(U)) - \mu(f(K)) < \varepsilon$. Consider an arbitrary continuous function $\chi : J \to [0, 1]$ so that χ is 1 on K and 0 on $J \setminus U$. Then there exists $s : 0 < s < 1$ such that for $A = \chi^{-1}([s, 1])$, $\mu(\partial f(A)) = 0$. Then the weak* convergence of μ_n implies $\mu_n(f(A)) \to \mu(f(A))$, as $n \to \infty$. Moreover this weak* convergence and $\lambda_n \to \lambda$ imply $\int \chi \lambda_n |f'|^t d\mu_n \to \int \chi \lambda |f'| d\mu$. Therefore from $\mu_n(f(A)) = \int_A \lambda_n |f'|^t d\mu_n$, letting $\varepsilon \to 0$, we obtain $\mu(f(E_i)) = \int_{E_i} |f'|^t d\mu$.

If $E = \{c\}$ where $c \in \mathrm{Crit} \cap J$ then for every $r > 0$ small enough and for all n, we have $\mu_n(f(B(c, r)) \leq 2(\sup_k \lambda_k)^t (2r)^t$ and since the bound is independent of n we get $\mu(f(c)) = 0$, hence $\mu(f(c)) = \int_c |f'|^t d\mu$, as $f'(c) = 0$.

Definition 4. We call $z \in \overline{\mathbb{C}}$ *safe* if
$z \notin \bigcup_{j=1}^{\infty} f^j(\mathrm{Crit})$ and $\liminf_{n\to\infty} \frac{1}{n} \ln \mathrm{dist}(z, f^n(\mathrm{Crit})) = 0$

Definition 5. We call $z \in \overline{\mathbb{C}}$ *repelling* if there exist Δ and $\lambda = \lambda_z > 1$ such that for all n large enough f^n is univalent on $\mathrm{Comp}_z f^{-n}(B(f^n(z), \Delta))$, where Comp_z means the connected component containing z, and $|(f^n)'(z)| \geq \lambda^n$.

Definition 6. Hyperbolic Hausdorff dimension of Julia set is defined by

$$\mathrm{HD}_{\mathrm{hyp}}(J) = \sup_X \mathrm{HD}(X),$$

where the supremum is taken over all compact f-invariant isolated hyperbolic subsets of J and $\mathrm{HD}(X)$ means the Hausdorff dimension of X.

Proposition 2. The set S of repelling safe points in J is nonempty. Moreover $\mathrm{HD}(S) \geq \mathrm{HD}_{\mathrm{hyp}}(J)$.

Proof. The set NS of non-safe points is of zero Hausdorff dimension. This follows from $NS \subset \bigcup_{j=1}^{\infty} f^j(\mathrm{Crit}) \cup \bigcup_{\xi<1} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B(f^j(\mathrm{Crit}), \xi^j)$, finitness of Crit and from $\sum_n (\xi^n)^t < \infty$ for every $0 < \xi < 1$ and $t > 0$. Therefore the existence of safe repelling points in J follows from the existence of hyperbolic sets $X \subset J$ with $\mathrm{HD}(X) > 0$. Note that every point in a hyperbolic set X is repelling.

Theorem 1. For all $t > 0$, all repelling safe $z \in J$ and all $w \in \overline{\mathbb{C}}$

$$P_{\mathrm{tree}}(z,t) \le P_{\mathrm{hyp}}(t) \le P_{\mathrm{Conf}}(t) \le P_{\mathrm{tree}}(w,t).$$

We will provide a proof later. Now let us state corollaries.

Corollary 1. $P_{\mathrm{hyp}}(t) = P_{\mathrm{Conf}}(t)$ and $\mathrm{HD}_{\mathrm{hyp}}(J) =$ minimal exponent t for a t conformal measure.

Proof. The first equality follows from Theorem 1 and existence of repelling safe points in J. The second from the fact that both quantities are first zeros of $P_{\mathrm{hyp}}(t)$ and $P_{\mathrm{Conf}}(t)$.

We obtain also a simple proof of the following

Corollary 2. $P_{\mathrm{tree}}(z,t)$ does not depend on z for $z \in J$ repelling safe.

Proof of Theorem 1.

1. We prove first that $P_{\mathrm{tree}}(z,t) \le P_{\mathrm{hyp}}(t)$. Fix repelling safe $z = z_0 \in J$ and $\lambda = \lambda_{z_0} > 1$ according to Definition 5. Since z_0 is repelling, we have for $\delta = \Delta/2$, $l = 2\alpha n$ and all n large enough

$$W := \mathrm{Comp}_{z_0} f^{-l} B(f^l(z_0), 2\delta) \subset B(z, \varepsilon\lambda^{-\alpha n}),$$

and f^l is univalent on W. Since z_0 is safe we have

$$B(z_0, \lambda^{-\alpha n}) \cap \bigcup_{j=1}^{2n} f^j(\mathrm{Crit}) = \emptyset$$

for arbitrary constants $\varepsilon, \alpha > 0$.

By the Koebe Distortion Lemma for ε small enough, for every $1 \le j \le 2n$ and $z_j \in f^{-j}(z_0)$ we have

$$\mathrm{Comp}_{z_j} f^{-j} B(z_0, \varepsilon\lambda^{-\alpha n}) \subset B(z_j, \delta).$$

Let $m = m(\delta)$ be be such that $f^m(B(y, \delta/2)) \supset J$ for every $y \in J$. Then, putting $y = f^l(z_0)$, for every $z_n \in f^{-n}(z_0)$ we find $z_n' \in f^{-m}(z_n) \cap f^m(B(y, \delta/2))$. Hence the component W_{z_n} of $f^{-m}(\mathrm{Comp}_{z_n} f^{-n}(B(z_0, \varepsilon\lambda^{-\alpha n}))$ containing z_n' is contained in $\subset B(y, \frac{3}{2}\delta))$ and f^{m+n} is univalent on W_{z_n} (provided $m \le n$).

Therefore f^{m+n+l} is univalent from $W'_{z_n} := \mathrm{Comp}(f^{-(m+n+l)}(B(y,2\delta)) \subset W_{z_n}$ onto $B(y,2\delta)$. The mapping

$$F = f^{m+n+l} : \bigcup_{z_n \in f^{-n}(z_0)} W'_{z_n} \to B(y,2\delta)$$

has no critical points, hence $Z := \bigcap_{k=0}^{\infty} F^{-k}(B(y,2\delta))$ is an isolated expanding F-invariant (Cantor) subset of J.

We obtain for a constant $C > 0$ resulting from distortion and $L = \sup |f'|$,

$$P(F|_Z, -t\ln|F'|) \geq \ln\Big(C \sum_{z_n \in f^{-n}(z_0)} |(f^{m+n+l})'(z'_n)|^{-t}\Big)$$

$$\geq \ln\Big(C \sum_{z_n \in f^{-n}(z_0)} |(f^n)'(z_n)|^{-t} L^{-t(m+l)}\Big). \tag{1.1}$$

Hence on the expanding f-invariant set $Z' := \bigcup_{j=0}^{m+n+l-1} f^j(Z)$ we obtain

$$P(f|_{Z'}, -t\ln|f'|) \geq \frac{1}{m+n+l} P(F, -t\ln|F'|)$$

$$\geq \frac{1}{m+n+l}\Big(\ln C - t(m+l)\ln L + \ln \sum_{z_n \in f^{-n}(z_0)} |(f^n)'(z_n)|^{-t}\Big).$$

Passing with n to ∞ and next letting $\alpha \searrow 0$ we obtain

$$P(f|_{Z'}, -t\ln|f'|) \geq P_{\mathrm{tree}}(z_0, t).$$

2. $P_{\mathrm{hyp}}(t) \leq P_{\mathrm{Conf}}(t)$ is immediate. Let μ be an arbitrary (λ, t)-conformal measure on J. From the *topological exactness* of f on J, which means that for every U an open set intersecting J there exists $N \geq 0$ such that $f^N(U) \supset J$, we get $\int_U \lambda^N |(f^N)'|^t d\mu \geq 1$. Hence $\mu(U) > 0$. Let X be an arbitrary f-invariant non-empty isolated hyperbolic subset of J. Then, for U small enough, $(\exists C)(\forall x_0 \in X)(\forall n \geq 0)(\forall x \in X \cap f^{-n}(x_0))$ f^n maps $U_x = \mathrm{Comp}_x f^{-n}(U)$ onto U univalently with distortion bounded by C. So, for every n,

$$\mu(U) \cdot \sum_{x \in f^{-n}(x_0)\cap X} \lambda^{-n} |(f^n)'(x)|^{-t} \leq C \sum_{x \in f^{-n}(x_0)\cap X} \mu(U_x) \leq C.$$

Hence

$$P(f|_X, -\ln\lambda - t\ln|f'|) \leq 0 \text{ hence } P(f|_X, -t\ln|f'|) \leq \ln\lambda.$$

3. $P_{\mathrm{Conf}}(t) \leq P_{\mathrm{tree}}(w,t)$ follows from Patterson-Sullivan construction. The proof is as follows. Let us assume first that w is such that for any sequence $w_n \in f^{-n}(w)$ we have

$w_n \to J$. This means that w is neither in an attracting periodic orbit, nor in a Siegel disc, nor in a Herman ring. Let $P_{\text{tree}}(w,t) = \lambda$. Then for all $\lambda' > \lambda$

$$\sum_{x\in f^{-n}(w)} (\lambda')^{-n}|(f^n)'(x)|^{-t} \to 0$$

exponentially fast, as $n \to \infty$. We find a sequence of numbers $\phi_n > 0$ such that $\lim_{n\to\infty} \phi_n/\phi_{n+1} \to 1$ and for $A_n := \sum_{x\in f^{-n}(w)} \lambda^{-n}|(f^n)'(x)|^{-t}$ the series $\sum_n \phi_n A_n$ is divergent. For every $\lambda' > \lambda$ consider the measure

$$\mu_{\lambda'} = \sum_n \sum_{x\in f^{-n}(w)} D_x \cdot \varphi_n \cdot (\lambda')^{-n}|(f^n)'(x)|^{-t}/\Sigma_{\lambda'},$$

where D_x is the Dirac delta measure at x and $\Sigma_{\lambda'}$ is the sum of the weights at D_x, so that $\mu_{\lambda'}(J) = 1$. Finally we find a (λ, t)-conformal measure μ as a weak* limit of a convergent subsequence of $\mu_{\lambda'}$ as $\lambda' \searrow \lambda$.

If w is in an attracting periodic orbit which is one of at most two exceptional fixed points (∞ for polynomials, 0 or ∞ for $z \mapsto z^k$, in adequate coordinates) then it is a critical value, so $P_{\text{tree}}(w,t) = \infty$. If w is in a non-exceptional attracting periodic orbit or in a Siegel disc or Herman ring S, take $w' \in f^{-1}(w)$ not in the periodic orbit of w, neither in the periodic orbit of S in the latter cases. Then for w' we have the first case, hence $P_{\text{Conf}}(f) \le P_{\text{tree}}(w',t) \le P_{\text{tree}}(w,t)$. The latter inequality follows from

$$P_{\text{tree}}(w',t) = \limsup_{n\to\infty} \frac{1}{n-1} \sum_{x\in f^{-(n-1)}(w')} |(f^{-(n-1)})'(x)|^{-t} \le$$

$$\le \limsup_{n\to\infty} \frac{1}{n} \sum_{x\in f^{-(n-1)}(w')} |(f^n)'(x)|^{-t} \sup_{z\in\overline{\mathbb{C}}} |f'|^t \le P_{\text{tree}}(w,t).$$

QED

Remark 1. There is a direct simple proof of $P_{\text{tree}}(z,t) \le P_{\text{Conf}}(t)$ for μ-a.e. z, using Borel-Cantelli Lemma, see [P2, Theorem 2.4].

Remark 2. In [P2, Th.3.4] a stronger fact than Corollary 2 has been proved, also by elementary means, namely that $P_{\text{tree}}(z,t)$ does not depend on $z \in \overline{\mathbb{C}}$ except zero Hausdorff dimension set of z's.

References

[DU] M. Denker, M. Urbański. *On Sullivan's conformal measures for rational maps of the Riemann sphere.* Nonlinearity 4 (1991), 365–384.

[P1] F. Przytycki. *Lyapunov characteristic exponents are nonnegative.* Proc. Amer. Math. Soc. 119.1 (1993), 309–317.

[P2] F. Przytycki. *Conical limit sets and Poincaré exponent for iterations of rational functions.* Trans. Amer. Math. Soc. , 351.5 (1999), 2081–2099.

[W] P. Walters. *An Introduction to Ergodic Theory.*, Springer 1982.