Hyperbolic Hausdorff dimension is equal to the minimal exponent of conformal measure on Julia set. A simple proof.\footnote{extracted from [PRS] F. Przytycki, J. Rivera-Letelier, S. Smirnov "Equality of pressures for rational functions", to appear in Ergodic Theory and Dynamical Systems}

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The fact in the title was proved in [DU] except one point proved later in [P1]. This is a crucial technical fact in the study of dimensions and their continuity for Julia sets. The proofs in these papers use several complicated techniques. Here we give a simple proof.

Let $f : \overline{\mathcal{O}} \rightarrow \overline{\mathcal{O}}$ be a rational mapping of degree $d \geq 2$ on the Riemann sphere $\overline{\mathcal{O}}$. We denote by $\text{Crit}$ the set of critical points, that is $f'(x) = 0$ for $x \in \text{Crit}$. The symbol $J$ stands for the Julia set of $f$. Absolute values of derivatives and distances are considered with respect to the standard Riemann sphere metric. We consider pressures below for all $t > 0$.

**Definition 1.** Tree pressure. For every $z \in \overline{\mathcal{O}}$ define

$$P_{\text{tree}}(z, t) := \lim_{n \to \infty} \frac{1}{n} \sum_{f^n(z) = z} |(f^n)'(x)|^{-t}.$$ 

**Definition 2.** Hyperbolic pressure.

$$P_{\text{hyp}}(t) := \sup_{X} P(f | x, -t \ln |f'|),$$

where the supremum is taken over all compact $f$-invariant (that is $f(X) \subset X$) isolated hyperbolic subsets of $J$.

Isolated (sometimes called repelling) means that there is a neighbourhood $U$ of $X$ such that $f^n(x) \in U$ for all $n \geq 0$ implies $x \in X$. Hyperbolic means that there is a constant $\lambda_X > 1$ such that for all $n$ large enough and all $x \in X$ we have $|(f^n)'(x)| \geq \lambda_X^n$. Sometimes the more adequate term expanding is used.

$P(f | x, -t \ln |f'|)$ denotes the standard topological pressure for the continuous mapping $f|_X : X \rightarrow X$ and continuous real-valued potential function $-t \ln |f'|$ on $X$, see for example [W].

Note that these definitions imply that $P_{\text{hyp}}(t)$ is a continuous monotone decreasing function of $t$.

**Definition 3.** Conformal pressure. Set $P_{\text{Conf}}(t) := \ln \lambda(t)$, where

$$\lambda(t) = \inf \{ \lambda > 0 : \exists \mu, \text{ a probability measure on } J \text{ with Jacobian } \lambda |f'|^t \}. $$
We say that $\varphi : J \to \mathbb{R}, \varphi \geq 0$ is the Jacobian for $f|_J$ with respect to $\mu$ if $\varphi$ is $\mu$-integrable and for every Borel set $E \subset J$ on which $f$ is injective $\mu(f(E)) = \int_E \varphi \, d\mu$. We write $\varphi = \text{Jac}_\mu(f|_J)$.

We call any probability measure $\mu$ on $J$ with Jacobian of the form $\lambda |f'|^t$ a $(\lambda, t)$-conformal measure and with Jacobian $|f'|^t$ a conformal measure with exponent $t$ or $t$-conformal measure.

**Proposition 1.** For each $t > 0$ the number $P_{\text{Conf}}(t)$ is attained, that is there exists a $(\lambda, t)$-conformal measure with $\lambda = P_{\text{Conf}}(t)$.

This Proposition follows from the following

**Lemma.** If $\mu_n$ is a sequence of $(\lambda_n, t)$-conformal measures of $J$ for an arbitrary $t > 0$, weakly* convergent to a measure $\mu$ and $\lambda_n \to \lambda$ then $\mu$ is a $(\lambda, t)$-conformal measure.

**Proof.** Let $E \subset J$ on which $f$ is injective. $E$ can be decomposed into a countable union of critical points and sets $E_i$ pairwise disjoint and such that $f$ is injective on a neighbourhood $V$ of $cE_i$. For every $\varepsilon$ there exist compact set $K$ and open $U$ such that $K \subset E_i \subset U \subset V$ and $\mu(K) < \varepsilon$ and $\mu(f(U)) - \mu(f(K)) < \varepsilon$. Consider an arbitrary continuous function $\chi : J \to [0, 1]$ so that $\chi$ is 1 on $K$ and 0 on $J \setminus U$. Then there exists $s : 0 < s < 1$ such that for $A = \chi^{-1}([s, 1])$, $\mu(\partial f(A)) = 0$. Then the weak* convergence of $\mu_n$ implies $\mu_n(f(A)) \to \mu(f(A))$, as $n \to \infty$. Moreover this weak* convergence and $\lambda_n \to \lambda$ imply $\int \chi \lambda_n |f'|^t \, d\mu_n \to \int \chi \lambda |f'|^t \, d\mu$. Therefore from $\mu_n(f(A)) = \int_A \lambda_n |f'|^t \, d\mu_n$, letting $\varepsilon \to 0$, we obtain $\mu(f(A)) = \int_A \lambda |f'|^t \, d\mu$.

If $E = \{c\}$ where $c \in \text{Crit} \cap J$ then for every $r > 0$ small enough and for all $n$, we have $\mu_n(f(B(c, r)) \leq 2(\sup z \lambda)^t (2r)^t$ and since the bound is independent of $n$ we get $\mu(f(c)) = 0$, hence $\mu(f(c)) = \int \lambda |f'|^t \, d\mu$, as $f'(c) = 0$.

**Definition 4.** We call $z \in \overline{E}$ safe if $z \notin \bigcup_{j=1}^{\infty} f^j(\text{Crit})$ and $\lim \inf_{n \to \infty} \frac{1}{n} \ln \text{dist}(z, f^n(\text{Crit})) = 0$.

**Definition 5.** We call $z \in \overline{E}$ repelling if there exist $\Delta$ and $\lambda = \lambda_z > 1$ such that for all $n$ large enough $f^n$ is univalent on $\text{Comp}_z f^{-n}(B(f^n(z), \Delta))$, where $\text{Comp}_z$ means the connected component containing $z$, and $|(f^n)'(z)| \geq \lambda^n$.

**Definition 6.** Hyperbolic Hausdorff dimension of Julia set is defined by

$$\text{HD}_{\text{hyp}}(J) = \sup_X \text{HD}(X),$$

where the supremum is taken over all compact $f$-invariant isolated hyperbolic subsets of $J$ and $\text{HD}(X)$ means the Hausdorff dimension of $X$.

**Proposition 2.** The set $S$ of repelling safe points in $J$ is nonempty. Moreover $\text{HD}(S) \geq \text{HD}_{\text{hyp}}(J)$. 

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Proof. The set $NS$ of non-safe points is of zero Hausdorff dimension. This follows from $NS \subset \bigcup_{j=1}^{\infty} f^j(Crit) \cup \bigcup_{\xi<1} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B(f^j(Crit), \xi^j)$, finiteness of Crit and from $\sum_{n}(\xi^n)^t < \infty$ for every $0 < \xi < 1$ and $t > 0$. Therefore the existence of safe repelling points in $J$ follows from the existence of hyperbolic sets $X \subset J$ with $HD(X) > 0$. Note that every point in a hyperbolic set $X$ is repelling.

Theorem 1. For all $t > 0$, all repelling safe $z \in J$ and all $w \in \mathcal{F}$

$$P_{\text{tree}}(z, t) \leq P_{\text{hyp}}(t) \leq P_{\text{Conf}}(t) \leq P_{\text{tree}}(w, t).$$

We will provide a proof later. Now let us state corollaries.

Corollary 1. $P_{\text{hyp}}(t) = P_{\text{Conf}}(t)$ and $HD_{\text{hyp}}(J) =$ minimal exponent $t$ for a $t$ conformal measure.

Proof. The first equality follows from Theorem 1 and existence of repelling safe points in $J$. The second from the fact that both quantities are first zeros of $P_{\text{hyp}}(t)$ and $P_{\text{Conf}}(t)$.

We obtain also a simple proof of the following

Corollary 2. $P_{\text{tree}}(z, t)$ does not depend on $z$ for $z \in J$ repelling safe.

Proof of Theorem 1.

1. We prove first that $P_{\text{tree}}(z, t) \leq P_{\text{hyp}}(t)$. Fix repelling safe $z = z_0 \in J$ and $\lambda = \lambda_{z_0} > 1$ according to Definition 5. Since $z_0$ is repelling, we have for $\delta = \Delta/2$, $l = 2\alpha n$ and all $n$ large enough

$$W := \text{Comp}_{z_0} f^{-l} B(f^l(z_0), 2\delta) \subset B(z, \epsilon \lambda^{-\alpha n}),$$

and $f^l$ is univalent on $W$. Since $z_0$ is safe we have

$$B(z_0, \epsilon \lambda^{-\alpha n}) \cap \bigcup_{j=1}^{2n} f^j(Crit) = \emptyset$$

for arbitrary constants $\epsilon, \alpha > 0$.

By the Koebe Distortion Lemma for $\epsilon$ small enough, for every $1 \leq j \leq 2n$ and $z_j \in f^{-j}(z_0)$ we have

$$\text{Comp}_{z_j} f^{-j} B(z_0, \epsilon \lambda^{-\alpha n}) \subset B(z_j, \delta)$$

Let $m = m(\delta)$ be be such that $f^m(B(y, \delta/2)) \supset J$ for every $y \in J$. Then, putting $y = f^l(z_0)$, for every $z_n \in f^{-n}(z_0)$ we find $z'_n \in f^{-m}(z_n) \cap f^m(B(y, \delta/2))$. Hence the component $W_{z_n}$ of $f^{-m} (\text{Comp}_{z_n} f^{-n} (B(z_0, \epsilon \lambda^{-\alpha n}))$ containing $z'_n$ is contained in $\subset B(y, \frac{3}{2}\delta)$ and $f^{m+n}$ is univalent on $W_{z_n}$ (provided $m \leq n$).
Therefore $f^{m+n+l}$ is univalent from $W_{z_{n}}' := \text{Comp}(f^{-(m+n+l)}(B(y, 2\delta)) \subset W_{z_{n}}$ onto $B(y, 2\delta)$. The mapping

$$F = f^{m+n+l} : \bigcup_{z_{n} \in f^{-n}(z_{0})} W_{z_{n}}' \rightarrow B(y, 2\delta)$$

has no critical points, hence $Z := \bigcap_{k=0}^{\infty} F^{-k}(B(y, 2\delta))$ is an isolated expanding $F$-invariant (Cantor) subset of $J$.

We obtain for a constant $C > 0$ resulting from distortion and $L = \sup |f'|$

$$P(F|z, -t \ln |F'|) \geq \ln\left(C \sum_{z_{n} \in f^{-n}(z_{0})}|(f^{m+n+l})'(z_{n}')|^{-t}\right) \geq \ln\left(C \sum_{z_{n} \in f^{-n}(z_{0})}|(f^{n})'(z_{n})|^{-t}L^{-(m+l)}\right).$$

Hence on the expanding $f$-invariant set $Z' := \bigcup_{j=0}^{m+n+l-1} f^{j}(Z)$ we obtain

$$P(f|z', -t \ln |f'|) \geq \frac{1}{m+n+l}P(F, -t \ln |F'|) \geq \frac{1}{m+n+l}(\ln C - t(m+l) \ln L + \ln \sum_{z_{n} \in f^{-n}(x_{0})}|(f^{n})'(z_{n})|^{-t}).$$

Passing with $n$ to $\infty$ and next letting $\alpha \searrow 0$ we obtain

$$P(f|_{Z'}, -t \ln |f'|) \geq P_{\text{tree}}(z_{0}, t).$$

2. $P_{\text{hyp}}(t) \leq P_{\text{Conf}}(t)$ is immediate. Let $\mu$ be an arbitrary $(\lambda, t)$-conformal measure on $J$. From the topological exactness of $f$ on $J$, which means that for every $U$ an open set intersecting $J$ there exists $N \geq 0$ such that $f^{N}(U) \supset J$, we get $\int_{J} \lambda^{n}|(f^{N})'|^{t}d\mu \geq 1$. Hence $\mu(U) > 0$. Let $X$ be an arbitrary $f$-invariant non-empty isolated hyperbolic subset of $J$. Then, for $U$ small enough, $(\exists C)(\forall x_{0} \in X)(\forall n \geq 0)(\forall x \in X \cap f^{-n}(x_{0}))$ $f^{n}$ maps $U_{x} = \text{Comp}_{x}f^{-n}(U)$ onto $U$ univalently with distortion bounded by $C$. So, for every $n$,

$$\mu(U) \cdot \sum_{x_{0} \in f^{-n}(x_{0}) \cap X} \lambda^{-n}|(f^{n})'(x)|^{-t} \leq C \sum_{x_{0} \in f^{-n}(x_{0}) \cap X} \mu(U_{x}) \leq C.$$

Hence

$$P(f|x, -\ln \lambda - t \ln |f'|) \leq 0 \text{ hence } P(f|x, -t \ln |f'|) \leq \ln \lambda.$$

3. $P_{\text{Conf}}(t) \leq P_{\text{tree}}(w, t)$ follows from Patterson-Sullivan construction. The proof is as follows. Let us assume first that $w$ is such that for any sequence $w_{n} \in f^{-n}(w)$ we have
$w_n \to J$. This means that $w$ is neither in an attracting periodic orbit, nor in a Siegel disc, nor in a Herman ring. Let $P_{\text{tree}}(w, t) = \lambda$. Then for all $\lambda' > \lambda$

$$
\sum_{x \in f^{-n}(w)} (\lambda')^{-n}|(f^n)'(x)|^{-t} \to 0
$$

exponentially fast, as $n \to \infty$. We find a sequence of numbers

$\phi_n > 0$ such that $\lim_{n \to \infty} \phi_n/\phi_{n+1} \to 1$ and for $A_n := \sum_{x \in f^{-n}(w)} \lambda^{-n}|(f^n)'(x)|^{-t}$ the series $\sum_n \phi_n A_n$ is divergent. For every $\lambda' > \lambda$ consider the measure

$$
\mu_{\lambda'} = \sum_{n} \sum_{x \in f^{-n}(w)} D_x \cdot \varphi_n \cdot (\lambda')^{-n}|(f^n)'(x)|^{-t}/\Sigma_{\lambda'},
$$

where $D_x$ is the Dirac delta measure at $x$ and $\Sigma_{\lambda'}$ is the sum of the weights at $D_x$, so that $\mu_{\lambda'}(J) = 1$. Finally we find a $(\lambda, t)$-conformal measure $\mu$ as a weak* limit of a convergent subsequence of $\mu_{\lambda'}$ as $\lambda' \searrow \lambda$.

If $w$ is in an attracting periodic orbit which is one of at most two exceptional fixed points (for polynomials, 0 or $\infty$ for $z \mapsto z^k$, in adequate coordinates) then it is a critical value, so $P_{\text{tree}}(w, t) = \infty$. If $w$ is in a non-exceptional attracting periodic orbit or in a Siegel disc or Herman ring $S$, take $w' \in f^{-1}(w)$ not in the periodic orbit of $w$, neither in the periodic orbit of $S$ in the latter cases. Then for $w'$ we have the first case, hence $P_{\text{Conf}}(f) \leq P_{\text{tree}}(w', t) \leq P_{\text{tree}}(w, t)$. The latter inequality follows from

$$
P_{\text{tree}}(w', t) = \limsup_{n \to \infty} \frac{1}{n-1} \sum_{x \in f^{-(n-1)}(w')} |(f^{-(n-1)})'(x)|^{-t} \leq 
$$

$$
\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{x \in f^{-(n-1)}(w')} |(f^n)'(x)|^{-t} \sup_{x \in \overline{S}} |f'|^t \leq P_{\text{tree}}(w, t).
$$

QED

Remark 1. There is a direct simple proof of $P_{\text{tree}}(z, t) \leq P_{\text{Conf}}(t)$ for $\mu$-a.e. $z$, using Borel-Cantelli Lemma, see [P2, Theorem 2.4].

Remark 2. In [P2, Th.3.4] a stronger fact than Corollary 2 has been proved, also by elementary means, namely that $P_{\text{tree}}(z, t)$ does not depend on $z \in \overline{S}$ except zero Hausdorff dimension set of $z$'s.

References


