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Dynamics of postcritically bounded polynomial semigroups and interaction cohomology

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Abstract

We show that for each positive integer n, there exists a finitely generated polynomial semigroup G_n with bounded postcritical set such that the number of connected components of the Julia set of G_n is equal to n. Furthermore, we show that there exists a finitely generated polynomial semigroup G with bounded postcritical set such that the Julia set of G has precisely countably many connected components.

We show that for a finitely generated polynomial semigroup G with bounded postcritical set, the space of connected components of the Julia set is isomorphic to the inverse limit of connected components of the realizations of the nerves of finite coverings $\mathcal{U}$ of the Julia set of G, where each $\mathcal{U}$ consists of backward images of the Julia set of G by finite elements of G. In particular, we give a criterion for the Julia set of G to be connected.

To investige the overlapping of (backward)images of the Julia set (or limit sets), we define a new cohomology theory (interaction cohomology) for a (backward) self-similar set. We give a sufficient condition for H^1 to be infinite rank. As an application, combining it with the Alexander duality, we show that for a finitely generated polynomial semigroup G with bounded postcritical set which has the above condition, the Fatou set of G has infinitely many conntected components. We give an example of semigroup which has the above condition.

1 Preliminaries

Definition 1.1. A **rational semigroup** is a semigroup generated by non-constant rational maps on $\overline{\mathbb{C}}$ with the semigroup operation being the composition of maps([HM]). A **polynomial semigroup** is a semigroup generated by non-constant polynomial maps. Let G be a rational semigroup. We set

1. $F(G) := \{z \in \overline{\mathbb{C}} \mid G$ is normal in a neighborhood of $z\}$(**Fatou set** for G)
2. $J(G) := \overline{\mathbb{C}} \setminus F(G)$ (**Julia set** for G)
3. $E(G) := \{z \in \overline{\mathbb{C}} \mid \sharp \bigcup_{g \in G} g^{-1}(z) < \infty\}$ (the **exceptional set** for G).

We denote by $\langle h_1, h_2, \dots \rangle$ the (rational) semigroup generated by the family $\{h_i\}$. The Julia set (resp.filled-in Julia set) of the semigroup generated by a single map g is denoted by $J(g)$(resp.$K(g)$).

2 Connected components of Julia sets

For a polynomial semigroup G with bounded postcritically bounded in the plane, we investigate the space of all conntected components of the Julia set of G.

Definition 2.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{\text{all critical values of } g\}}.$$

This is called the **postcritical set** for G.

Definition 2.2. Let $\mathcal{G}$ be the set of all polynomial semigroups G with the following properties:

- Each element of G is of degree at least two.
- $P(G) \setminus \{\infty\}$ is bounded in $\mathbb{C}$.

We set $\mathcal{G}_c := \{G \in \mathcal{G} \mid J(G) :$ connected$\}$ and $\mathcal{G}_d := \{G \in \mathcal{G} \mid J(G) :$ disconnected$\}$.

Notation: For a polynomial semigroup G, we denote by $\mathcal{J}$ the set of all connected components J of $J(G)$ with $J \subset \mathbb{C}$.

Question 1. $\sharp\mathcal{J}$?

Proposition 2.3. *For any $n \in \mathbb{N}$ with $n > 1$, there exists a finitely generated polynomial semigroup $G_n = \langle h_1, \dots, h_{2n} \rangle$ in $\mathcal{G}$ satisfying $\sharp\mathcal{J} = n$.*

Proposition 2.4. *There exists a finitely generated polynomial semigroup $G = \langle h_1, h_2, h_3 \rangle$ in $\mathcal{G}$ satisfying that $\mathcal{J} \cong \mathbb{N}$, there exists a superattracting fixed point z_0 of some element of G with $z_0 \in J(G)$, and int $J(G) \neq \emptyset$.*

Definition 2.5. 1. Let X be a metric space. Let $h_j : X \to X$ $(j = 1, \cdots, m)$ be a continuous map. Let $G = \langle h_1, \cdots, h_m \rangle$ be a semigroup generated by $\{h_j\}$. A nonempty compact subset L of X is said to be a **backward self-similar set with respect to** $\{h_1, \cdots, h_m\}$ if

(a) $L = \bigcup_{j=1}^{m} h_j^{-1}(L)$ and

(b) $g^{-1}(z) \neq \emptyset$ for each $z \in L$ and $g \in G$.

For example, if $G = \langle h_1, \cdots, h_m \rangle$ is a finitely generated rational semigroup, then the Julia set $J(G)$ is a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$.

2. We set $\Sigma_m := \{1, \cdots, m\}^{\mathbb{N}}$. For each $x = (x_1, x_2, \cdots,) \in \Sigma_m$, we set $L_x := \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1} L\ (\neq \emptyset)$.

3. For a finite word $w = (w_1, \cdots, w_k) \in \{1 \cdots, m\}^k$, we set $h_w := h_{w_k} \circ \cdots \circ h_{w_1}$ and $\overline{w} = (w_k, w_{k-1}, \cdots, w_1)$. Furthermore, we set $|w| := k$(this is called the word length of w).

4. For any $k \in \mathbb{N}$, let $\mathcal{U}_k = \mathcal{U}_k(L, \{h_1, \cdots, h_m\})$ be the finite covering of L defined as: $\mathcal{U}_k := \{h_{\overline{w}}^{-1}(L) \mid |w| = k\}$. Let $N(\mathcal{U}_k)$ be the nerve: i.e. a simplicial complex whose vertex set is $\{1, \ldots, m\}^k$ and which satisfies that mutually different $w^1, \cdots, w^{r+1} \in \{1, \ldots, m\}^k$ makes an r-simplex if and only if $\bigcap_{j=1}^{r+1} h_{w^j}^{-1}(L) \neq \emptyset$.

   Let $\varphi_k : N(\mathcal{U}_{k+1}) \to N(\mathcal{U}_k)$ be the simplicial map defined as: $(w_1, \cdots, w_{k+1}) \mapsto (w_1, \cdots, w_k)$ for each $(w_1, \cdots, w_{k+1}) \in \{1, \cdots, m\}^{k+1}$. Then $\{\varphi_k : N(\mathcal{U}_{k+1}) \to N(\mathcal{U}_k)\}_k$ is an inverse system of simplicial map.

5. Let $\pi_0(|N(\mathcal{U}_k)|)$ be the set of connected components of the realization $|N(\mathcal{U}_k)|$ of $N(\mathcal{U}_k)$. Let $\{\varphi_k : \pi_0(|N(\mathcal{U}_{k+1})|) \to \pi_0(|N(\mathcal{U}_k)|)\}_k$ be the inverse system induced by $\{\varphi_k\}_k$.

Theorem 2.6. *Let $G = \langle h_1, \cdots, h_m \rangle \in \mathcal{G}$. Then, as regarding the backward self-similar set $J(G)$ with respect to $\{h_1, \cdots, h_m\}$, we have the following.*

1. *There exists a bijection:*

$$\Phi : \varprojlim \pi_0(|N(\mathcal{U}_k)|) \cong \mathcal{J}$$

2. *(**A criterion for the Julia set to be connected**) $J(G)$ is connected if and only if $|N(\mathcal{U}_1)|$ is connected: i.e. for each $i, j \in \{1, \cdots, m\}$ there exists a sequence $(i_t)_{t=1}^{s}$ in $\{1 \cdots, m\}$ such that $i_1 = i$, $i_s = j$ and $h_{i_t}^{-1}(J(G)) \cap h_{i_{t+1}}^{-1}(J(G)) \neq \emptyset$ for each $t = 1, \cdots, s-1$.*

3. *$\sharp\pi_0(|N(\mathcal{U}_k)|) \leq \sharp\pi_0(|N(\mathcal{U}_{k+1})|)$, for each $k \in \mathbb{N}$.*
Furthermore, $\{\sharp\pi_0(|N(\mathcal{U}_k)|)\}_k$ is bounded if and only if $\sharp\mathcal{J} < \infty$. If $\sharp\mathcal{J} < \infty$, then
$$\lim_{k\to\infty} \sharp\pi_0(|N(\mathcal{U}_k)|) = \sharp\mathcal{J}.$$

4. *If $m = 2$ and $G \in \mathcal{G}_d$, then $\mathcal{J} \cong \{1,2\}^{\mathbb{N}}$ (Cantor set).*

5. *If $m = 3$ and $G \in \mathcal{G}_d$, then $\sharp\mathcal{J} \geq \aleph_0$.*

3 Interaction cohomology

We define a new cohomology theory for (backward) iterated function systems.

Definition 3.1. **(Interaction cohomology)** Let X be a metric space. Let $h_j : X \to X$ be a continuous map ($j = 1, \cdots, m$). Let $G = \langle h_1, \cdots, h_m \rangle$. Let L be a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$.

1. We set $\mathrm{Cov}(L, G) :=$
$\{\mathcal{U}$: finite covering of $L \mid \mathcal{U} = \{g_1^{-1}(L), \cdots, g_n^{-1}(L)\},\ g_1, \cdots, g_n \in G,\ n \in \mathbb{N}\}$. This makes an inverse system with respect to refinement.

2. For any module R, we set
$$\check{H}^p(L, G, R) := \varinjlim_{\mathcal{U}\in \mathrm{Cov}(L,G)} H^p(|N(\mathcal{U})|,\ R).$$
This is called the p**-th interaction cohomology for** (L, G) **with coefficient** R. Since $\{\mathcal{U}_k\}$ is cofinal in $\mathrm{Cov}(L, G)$, we have
$$\check{H}^p(L, G, R) \cong \varinjlim_k H^p(|N(\mathcal{U}_k)|,\ R) =: \check{H}^p(L,\ \{h_1, \cdots, h_m\},\ R).$$
($\varinjlim_k H^p(|N(\mathcal{U}_k)|,\ R)$ is the inverse limit of $\{\varphi_k^* : H^p(|N(\mathcal{U}_k)|,\ R) \to H^p(|N(\mathcal{U}_{k+1})|,\ R)\}_k$.) $H^p(|N(\mathcal{U}_k)|,\ R)$ is called **the** p**-th interaction cohomology at** k**-th stage for backward iterated function system** $(L, \{h_1, \cdots, h_m\})$ **with coefficient** R and $\check{H}^p(L,\ \{h_1, \cdots, h_m\},\ R)$ is called **the** p**-th interaction cohomology at** ∞**-stage for backward iterated function system** $(L, \{h_1, \cdots, h_m\})$ **with coefficient** R.

3. There is a natural homomorphism from $\check{H}^p(L, G, R)$ to $\check{H}^p(L, R)$ (the Čech cohomology of L with coefficient R). We denote it by:
$$\Psi : \check{H}^p(L, G, R) \to \check{H}^p(L, R).$$

Definition 3.2. Let X be a metric space. Let $h_j : X \to X$ be a continuous map $(j = 1, \cdots, m)$. Let $G = \langle h_1, \cdots, h_m \rangle$. A non-empty compact subset L of X is said to be a **self-similar set with respect to** $\{h_1, \cdots, h_m\}$ if $L = \cup_{j=1}^{m} h_j(L)$. We define the p-th interaction cohomology for (L, G) etc., by the same way as in Definition 3.1.

Theorem 3.3. *Let $G = \langle h_1, \cdots, h_m \rangle \in \mathcal{G}$. Let R be a field. As regarding the backward self-similar set $J(G)$ with respect to $\{h_1, \cdots, h_m\}$, we have*

1. $\dim \check{H}^0(J(G), G, R) < \infty$ *if and only if* $\sharp \mathcal{J} < \infty$.
2. *If* $\dim \check{H}^0(J(G), G, R) < \infty$, *then* $\dim \check{H}^0(J(G), G, R) = \sharp \mathcal{J}$.

Lemma 3.4. *Let X be a metric space. Let $h_j : X \to X$ be a continuous map $(j = 1, \cdots, m)$. Let $G = \langle h_1, \cdots, h_m \rangle$. Let L be a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$. Let R be a module. If $\bigcap_{g \in G} g^{-1}(L) \neq \emptyset$ (for example, if G is a rational semigroup, $L = J(G)$, and there exists a point $z \in J(G)$ with $h_j(z) = z$ for each $j = 1, \cdots, m$), then $\check{H}^0(L, G, R) = R$ and $\check{H}^p(L, G, R) = 0$ for each $p > 0$.*

Hence there exist a $G \in \mathcal{G}$ with $\check{H}^1(J(G), G, R) = 0$. We show there exists a $G \in \mathcal{G}$ such that $\check{H}^1(J(G), G, R)$ has infinite rank.

Theorem 3.5. *Let $G = \langle h_1, \cdots, h_m \rangle \in \mathcal{G}$. Let R be a field. As regarding the backward self-similar set $J(G)$ with respect to $\{h_1, \cdots, h_m\}$, we assume all of the following:*

1. $|N(\mathcal{U}_1)|$ *is connected* ($\Longleftrightarrow$ $J(G)$ *is connected*).
2. *There exists a number $j \in \{1, \cdots, m\}$ such that* $(h_j^2)^{-1}(J(G)) \cap (\cup_{i: i \neq j} h_i^{-1}(J(G))) = \emptyset$.
3. *There exist mutually distinct elements $j_1 = j$, j_2, $j_3 \in \{1, \cdots, m\}$ such that $\{j_1, j_2\}$, $\{j_2, j_3\}$ and $\{j_1, j_3\}$ are 1-simlexes in $N(\mathcal{U}_1)$.*
4. *For any $r \geq 2$, there exists no r-simplex S in $N(\mathcal{U}_1)$ with $j \in S$.*

Then, we have the following.

1. $\dim_R \check{H}^1(J(G), G, R) = \dim_R \Psi(\check{H}^1(J(G), G, R)) = \infty$.
2. *$F(G)$ has infinitely many connected components.*

Example 3.6. There exists a polynomial semigroup $G = \langle h_1, h_2, h_3, h_4 \rangle$ such that G satisfies the assumption in Theorem 3.5. Hence we have that $\dim_R \check{H}^1(J(G), G, R) = \dim_R \Psi(\check{H}^1(J(G), G, R)) = \infty$ and $F(G)$ has infinitely many connected components. To construct such a G, let $a \in \mathbb{R}$ with $1 < a < 2$ and let $h_1(z) = \frac{1}{a^2} z^3$ and $h_2(z) = z^2$. Let $c_1 := (a^{\frac{2}{3}} - a^{\frac{1}{2}})/2$. Let g_3 be a polynomial such that $J(g_3) = \{|z - c_1| = a^{\frac{2}{3}} - c_1\}$ and let

g_4 be a polynomial such that $J(g_4) = \{|z + c_1| = a^{\frac{2}{3}} - c_1\}$. Take a sufficient large $n \in \mathbb{N}$ and let $h_3 = g_3^n$ and $h_4 = g_4^n$. Let $G = \langle h_1, h_2, h_3, h_4 \rangle$. Then, we can show that $G \in \mathcal{G}$, the set of all 1-simplexes in $N(\mathcal{U}_1)$ is: $\{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$, and there exists no r-simplex S in $N(\mathcal{U}_1)$ for each $r \geq 2$. It is easy to show that G satisfies all of the conditions of the assumption in Theorem 3.5.

Problem 1. (open) Are there any finitely generated $G \in \mathcal{G}$ with $0 < \dim_R \check{H}^1(J(G), G, R) < \infty$?

Definition 3.7.

1. Let L be a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$. For each $(i,j) \in \{1, \cdots, m\}^2$ with $i \neq j$, we set $C_{i,j} := h_i^{-1}(L) \cap h_j^{-1}(L)$. Let $C := \cup_{(i,j):i\neq j} C_{i,j}$. We say that $(L, \{h_1, \cdots, h_m\})$ is **postunbranched** if for any (i,j) with $i \neq j$, there exists a unique $x = x(i,(i,j)) \in \Sigma_m$ such that $h_i(C_{i,j}) \subset L_x$.
2. Similarly, Let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$. We set $L_x = \cap_{j=1}^{\infty} h_{x_1} \cdots h_{x_j}(L)$, for each $x \in \Sigma_m$. For each $(i,j) \in \{1, \cdots, m\}^2$ with $i \neq j$, we set $C_{i,j} := h_i(L) \cap h_j(L)$. Let $C := \cup_{(i,j):i\neq j} C_{i,j}$. We say that $(L, \{h_1, \cdots, h_m\})$ is **postunbranched** if for any (i,j) with $i \neq j$, there exists a unique $x = x(i,(i,j)) \in \Sigma_m$ such that $(h_i|_L)^{-1}(C_{i,j}) \subset L_x$.

Theorem 3.8. *Let L be a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$ (or let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$ such that $h_j : L \to L$ is injective for each $j = 1, \cdots, m$). Let R be a field and let $a_{r,k} = \dim_R H^r(|N(\mathcal{U}_k)|, R)$ for each $r, k \in \mathbb{Z}$ with $r \geq 0, k \geq 1$. Furthermore, let $a_{r,\infty} = \dim_R \check{H}^r(L, G, R)$. Suppose $(L, \{h_1, \cdots h_m\})$ is postunbranched. Then, we have the following.*

1. *For any $r \geq 2$ and $k \geq 1$, $a_{r,k+1} = m a_{r,k+1} + a_{r,1}$ and there exists an exact sequence:*

$$0 \to H^r(|N(\mathcal{U}_1)|, R) \to H^r(|N(\mathcal{U}_{k+1})|, R) \to \bigoplus_{j=1}^{m} H^r(|N(\mathcal{U}_k)|, R) \to 0.$$

2. *If $|N(\mathcal{U}_1)|$ is connected, then we have the following.*
   (a) *$a_{1,k+1} = m a_{1,k} + a_{1,1}$.*
   (b) *If $a_{1,1} = 0$, then $a_{1,\infty} = 0$. If $a_{1,1} \neq 0$, then $a_{1,\infty} = \infty$.*
3. *If $a_{0,\infty}, a_{1,\infty} < \infty$, then $m - a_{0,1} + a_{1,1} = (m-1)(a_{0,\infty} - a_{1,\infty})$.*
4. *If $\frac{m - a_{0,1} + a_{1,1}}{m-1} \in \mathbb{R} \setminus (\mathbb{N} \cup \{0\})$, then at least one of $a_{0,\infty}$ and $a_{1,\infty}$ is equal to ∞.*
5. *$m a_{0,k} - m + a_{0,1} - a_{1,1} \leq a_{0,k+1} \leq m a_{0,k} - m + a_{0,1}$, for each $k \geq 1$.*

6. *If there exists an element* $k_0 \in \mathbb{N}$ *with* $a_{0,k_0} > \frac{1}{m-1}(m - a_{0,1} + a_{1,1})$, *then* $a_{0,k+1} > a_{0,k}$ *for each* $k \geq k_0$.

7. $a_{0,\infty} \in \{x \in \mathbb{N} \mid a_{0,1} \leq x \leq \frac{1}{m-1}(m - a_{0,1} + a_{1,1})\} \cup \{\infty\}$.

Example 3.9. **(Sierpinski gasket)** Let $p_1, p_2, p_3 \in \mathbb{C}$ be three points such that $p_1p_2p_3$ makes a regular triangle. Let $h_i(z) := 2(z - p_i) + p_i$, for each $i = 1, 2, 3$. Let $G = \langle h_1, h_2, h_3 \rangle$. Then, $J(G)$ is equal to the Sierpinski gasket. We see that $(J(G), \{h_1, h_2, h_3\})$ is postunbranched. The set of all 1-simplexes in $N(\mathcal{U}_1)$ is: $\{\{1,2\}, \{1,3\}, \{2,3\}\}$ and there exists no r-simplex in $N(\mathcal{U}_1)$, for each $r \geq 2$. For each field R, we have $\dim_R H^1(|N(\mathcal{U}_1)|, R) = 1$. Hence, by Theorem 3.8, under the notation of Theorem 3.8, we obtain $a_{1,k+1} = 3a_{1,k} + 1$, for each $k \in \mathbb{N}$.

Theorem 3.10. *Let L be a backward self-similar set with respect to $\{h_1, \cdots, h_m\}$ (or let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$ such that $h_j : L \to L$ is injective for each $j = 1, \cdots, m$). Let R be a field. Suppose $\sharp C_{i,j} \leq 1$ for each (i,j) with $i \neq j$. Then, under the notation in Theorem 3.8, we have the following.*

1. *For each $k \in \mathbb{N}$ and each $r \geq 2$, $a_{r,k} = 0$. (If L is a self-similar set, then the injectivity of each h_j is not needed for this statement.)*

2. *For each $k \in \mathbb{N}$, $ma_{1,k} \leq a_{1,k+1}$.*

3. *If $|N(\mathcal{U}_1)|$ is connected and $\check{H}^1(L, G, \mathbb{Z}) \neq 0$, then $a_{1,\infty} = \infty$.*

Proposition 3.11. *Let X be a topological manifold with a distance. Let $h_j : X \to X$ be a continuous map ($j = 1, \cdots, m$, $m \geq 2$). Let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$. Suppose that $h_i : L \to L$ is injective for each $i = 1, 2$ and $\dim_T(C_{i,j}) \leq n$ for each (i,j) with $i \neq j$, where $\dim_T$ denotes the topological dimension. Let R be a field. Then, $\dim_R \check{H}^{n+1}(L, R)$ is either 0 or ∞.*

4 Tools

We give some tools to show the results.

4.1 Fundamental properties of rational semigroups

Lemma 4.1 ([HM],[GR]). *Let G be a rational semigroup.*

1. *For each $g \in G$, we have $g(F(G)) \subset F(G)$ and $g^{-1}(J(G)) \subset J(G)$. Note that we do not have that the equality holds in general.*

2. *If $G = \langle h_1, \cdots, h_m \rangle$ then $J(G)$ is* **backward self-similar***: i.e.*

$$J(G) = h_1^{-1}(J(G)) \cup \cdots \cup h_m^{-1}(J(G)).$$

3. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is a perfect set.*

4. *If $\sharp(J(G)) \geq 3$, then $\sharp E(G) \leq 2$.*

5. *If $z \in \overline{\mathbb{C}} \setminus E(G)$, then $J(G) \subset \overline{\bigcup_{g \in G} g^{-1}(z)}$. In particular if $z \in J(G) \setminus E(G)$, then $\overline{\bigcup_{g \in G} g^{-1}(z)} = J(G)$.*

6. *If $\sharp(J(G)) \geq 3$, then $J(G)$ is the smallest in*
$$\{K \subset \overline{\mathbb{C}} \mid K : \text{compact}, \ \sharp K \geq 3, \ \text{and } g^{-1}(K) \subset K \text{ for each } g \in G.\}$$

Theorem 4.2 ([HM], [GR]). *If $\sharp(J(G)) \geq 3$, then*
$$J(G) = \overline{\{z \in \overline{\mathbb{C}} \mid z \text{ is a repelling fixed point of some } g \in G\}}.$$
In particular, $J(G) = \overline{\bigcup_{g \in G} J(g)}$.

4.2 Fiberwise (Wordwise) dynamics

Notation: Let $G = \langle h_1, \cdots, h_m \rangle$ be a finitely generatd rational semigroup. For a fixed generator system $\{h_1, \cdots, h_m\}$, we set $\Sigma_m = \{1, \cdots, m\}^{\mathbb{N}}$, $\sigma : \Sigma_m \to \Sigma_m$, $\sigma(x_1, x_2, \cdots) := (x_2, x_3, \cdots)$. Moreover, we define a map $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ by: $(x, y), \mapsto (\sigma(x), \ h_{x_1}(y))$, where $x = (x_1, x_2, \cdots)$. This is called **the skew product map associated with the generator system** $\{h_1, \cdots, h_m\}$. Let $\pi : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m$ and $\pi_{\overline{\mathbb{C}}} : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the projections.

Definition 4.3 ([S1],[S3]). Under the above notation,

1. $f_x^n := f^n|_{\pi^{-1}x} : \pi^{-1}x \to \pi^{-1}\sigma^n(x) \subset \Sigma_m \times \overline{\mathbb{C}}$.

2. we denote by $F_x(f)$ the set of points $y \in \pi^{-1}x$ which has a neighborhood U in $\pi^{-1}x$ such that $\{f_x^n : U \to \Sigma_m \times \overline{\mathbb{C}}\}_{n \in \mathbb{N}}$ is normal.

3. $J_x(f) := \pi^{-1}x \setminus F_x(f)$.

4. $\tilde{J}(f) := \overline{\bigcup_{x \in \Sigma_m} J_x(f)}$ in $\Sigma_m \times \overline{\mathbb{C}}$.

5. $\hat{J}_x(f) := \pi^{-1}x \cap \tilde{J}(f)$.

6. $\tilde{F}(f) := (\Sigma_m \times \overline{\mathbb{C}}) \setminus \tilde{J}(f)$.

Lemma 4.4. *1. $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$. $f^{-1}J_{\sigma(x)}(f) = J_x(f)$. $f^{-1}\hat{J}_{\sigma(x)}(f) = \hat{J}_x(f)$. $\hat{J}_x(f) \supset J_x(f)$, note that* **equality does not hold in general.**

2. *$\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$. where $\pi_{\overline{\mathbb{C}}} : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the projection.*

3. $\pi_{\overline{\mathbb{C}}}(\hat{J}_x(f)) = \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$ *for each* $x = (x_1, x_2, \cdots) \in \Sigma_m$.

4. *(*[J], [S1]*)* *(***Lower semicontinuity of** $x \to J_x(f)$*) If* $\deg(h_j) \geq 2$ *for each* $j = 1, \cdots, m$, *then* $J_x(f)$ *is a non-empty perfect set. Furthermore,* $x \to J_x(f)$ *is* **lower semicontinuous.** *That is: for any point* $(x, y) \in \Sigma_m \times \overline{\mathbb{C}}$ *with* $(x, y) \in J_x(f)$ *and any sequence* (x^n) *in* Σ_m *with* $x^n \to x$, *there exists a sequence* (x^n, y^n) *in* $\Sigma_m \times \overline{\mathbb{C}}$ *with* $(x^n, y^n) \in J_{x^n}(f)$ *such that* $(x^n, y^n) \to (x, y)$. *But* $x \mapsto J_x(f)$ *is* **NOT** *continuous with respect to the Hausdorff topology in general.*

5. *If* $\deg(h_j) \geq 2$ *and* h_j *is a polynomial for each* $j = 1, \cdots, m$, *then* $\infty \in F(G)$ *and for each* $x \in \Sigma_m$, *we have* $\infty \in \pi_{\overline{\mathbb{C}}}(F_x(f))$ *and* $J_x(f) = \partial K_x(f)$ *(in* $\pi^{-1}(x)$*), where* $K_x(f) := \{y \in \pi^{-1}x \mid \{\pi_{\overline{\mathbb{C}}}(f_x^n(y))\}_{n\in\mathbb{N}} :$ *bounded in* $\mathbb{C}\}$.

Proposition 4.5. *Let* $G = \langle h_1, \cdots, h_m \rangle \in \mathcal{G}$. *Then, for each* $x \in \Sigma_m$, *the following sets are connected.*

$$(1)\ J_x(f) \qquad (2)\ \hat{J}_x(f) \qquad (3)\ \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G)).$$

Proof. $(1) J_x(f)$: Just the same procedure as the usual dynamics. $(2) \hat{J}_x(f)$: use connectedness of $J_x(f)$ and Lemma 4.4-4. $(3) \bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$: By the result that $\hat{J}_x(f)$ is connected and Lemma 4.4-3. □

To show Theorem 2.6, we need the fact $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$(Lemma 4.1.2) and $\bigcap_{j=1}^{\infty} h_{x_1}^{-1} \cdots h_{x_j}^{-1}(J(G))$ is connected for each $x \in \Sigma_m$(Proposition 4.5).

4.3 Dynamics of postcritically bounded polynomial semigroups

Theorem 4.6 ([S6]). *Let* $G \in \mathcal{G}$. *Then for any conntected component* J *of* $J(G)$ *and any element* $g \in G$, $g^{-1}(J)$ *is conntected.*

4.4 Fundamental properties of interaction cohomology

Lemma 4.7. *Let* X *be a metric space. Let* $h_j : X \to X$ *be a continuous map* $(j = 1, \cdots, m)$. *Let* $G = \langle h_1, \cdots, h_m \rangle$. *Let* L *be a backward self-similar set with respect to* $\{h_1, \cdots, h_m\}$. *Let* R *be a module. Then, we have the following:*

1. *For each* $k \in \mathbb{N}$, $\varphi_k^* : H^0(|N(\mathcal{U}_k)|, R) \to H^0(|N(\mathcal{U}_{k+1})|, R)$ *is injective. In particular, for each* $k \in \mathbb{N}$, *the projection map* $\psi_k : H^0(|N(\mathcal{U}_k)|, R) \to \check{H}^0(L, G, R)$ *is injective.*

2. $\Psi : \check{H}^0(L, G, R) \to \check{H}^0(L, R)$ *is injective.*

3. *If* $|N(\mathcal{U}_k)|$ *is conntected, then for each* k, $\varphi_k^* : H^1(|N(\mathcal{U}_k)|, R) \to H^1(|N(\mathcal{U}_{k+1})|, R)$ *is injective. In particular, the projection map* $\psi_k : H^1(|N(\mathcal{U}_k)|, R) \to \check{H}^1(L, G, R)$ *is injective.*

4. *If* $g^{-1}(L)$ *is connected for each* $g \in G$, *then we have that* $\Psi : \check{H}^1(L, G, R) \to \check{H}^1(L, R)$ *is injective.*

5. *If an inverse map* $h_j^{-1} : L \to L$ *can be defined and it is a contraction for each* $j = 1, \cdots m$, *then* $\Psi : \check{H}^p(L, G, R) \to \check{H}^p(L, R)$ *is bijective for each* $p \geq 0$.

5 Proofs of results

We give proofs of some results.

Proof of Proposition 2.3: Let $n \in \mathbb{N}$ with $n > 1$. Let $\epsilon > 0$ be a sufficiently small number. For each $j = 1, \cdots, n$, let g_j be a polynomial such that $J(g_j) = \{|z| = j\}$. Similarly, for each $j = 1, \cdots, n$, let $g_{\epsilon,j}$ be a polynomial such that $J(g_{\epsilon,j}) = \{|z - \epsilon| = j\}$. Take a sufficiently large $l \in \mathbb{N}$. Let $G_n = \langle g_1^l, \cdots, g_n^l, g_{\epsilon,1}^l, \cdots, g_{\epsilon,n}^l \rangle$. Then, we can show $G_n \in \mathcal{G}$. By using Theorem 4.6, we can show $\sharp \mathcal{J} = n$. □

Proof of Proposition 2.4: Let $g_1(z)$ be the second iterate of $z \mapsto z^2 - 1$. Let g_2 be a polynomial such that $J(g_2) = \{|z| = 1\}$ with $g_2(-1) = (-1)$. Then $g_1(i) = 3 \in \mathbb{C} \setminus K(g_1)$. Take a sufficiently large $m_1 \in \mathbb{N}$ and let $a := g_1^{m_1}(i)$. Let g_3 be a polynomial such that $J(g_3) = \{|z| = a\}$. Take sufficiently large m_2 and m_3 in $\mathbb{N}$ and let $G := \langle g_1^{m_1}, g_2^{m_2}, g_3^{m_3} \rangle$. Then, we can show $G \in \mathcal{G}$. By using some results in [S6], we can show $\mathcal{J} \cong \mathbb{N}$. □

Theorem 5.1. *Let* L *be a backward self-similar set with respect to* $\{h_1, \cdots, h_m\}$ *such that* L_x *is connected for each* $x \in \Sigma_m$. *Let* $\mathcal{L}$ *be the set of all conntected components of* L. *Then, we have the following.*

1. *There exists a bijection:*

$$\Phi : \varprojlim \pi_0(|N(\mathcal{U}_k)|) \cong \mathcal{L}.$$

   (Φ *is defined as follows: let* $B = (B_k)_k \in \varprojlim \pi_0(|N(\mathcal{U}_k)|)$ *where* $B_k \in \pi_0(|N(\mathcal{U}_k)|)$ *and* $\varphi_k(B_{k+1}) = B_k$ *for each* k. *Take a point* $x \in \Sigma_m$ *such that* $(x_k, \cdots, x_1) \in B_k$ *for each* k. *Take an element* $C \in \mathcal{L}$ *such that* $L_x \subset C$. *Let* $\Psi(B) = C$.*)*

2. L *is connected if and only if* $|N(\mathcal{U}_1)|$ *is connected: i.e. for each* $i, j \in \{1, \cdots, m\}$ *there exists a sequence* $(i_t)_{t=1}^s$ *in* $\{1 \cdots, m\}$ *such that* $i_1 = i$, $i_s = j$ *and* $h_{i_t}^{-1}(L) \cap h_{i_{t+1}}^{-1}(L) \neq \emptyset$ *for each* $t = 1, \cdots, s-1$.

3. $\sharp\pi_0(|N(\mathcal{U}_k)|) \leq \sharp\pi_0(|N(\mathcal{U}_{k+1})|)$, *for each* $k \in \mathbb{N}$.
Furthermore, $\{\sharp\pi_0(|N(\mathcal{U}_k)|)\}_k$ *is bounded if and only if* $\sharp\mathcal{L} < \infty$. *If* $\sharp\mathcal{L} < \infty$, *then*
$$\lim_{k\to\infty} \sharp\pi_0(|N(\mathcal{U}_k)|) = \sharp\mathcal{L}.$$

4. *If* $m = 2$ *and* L *is disconnected, then* $\mathcal{L} \cong \{1,2\}^{\mathbb{N}}$ *(Cantor set).*

5. *If* $m = 3$ *and* L *is disconnected, then* $\sharp\mathcal{L} \geq \aleph_0$.

6. $\dim \check{H}^0(L,G,R) < \infty$ *if and only if* $\sharp\mathcal{L} < \infty$.

7. *If* $\dim \check{H}^0(L,G,R) < \infty$, *then* $\dim \check{H}^0(L,G,R) = \sharp\mathcal{L}$.

Remark 1. Let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$ such that L_x is connected for each $x \in \Sigma_m$. Then, we obtain a result similar to Theorem 5.1.

Proof of Theorem 2.6 and Theorem 3.3: By Theorem 5.1 and Proposition 4.5. □

Theorem 5.2. *Let* L *be a backward self-similar set with respect to* $\{h_1, \cdots, h_m\}$. *Let* $G = \langle h_1, \cdots, h_m \rangle$. *Let* R *be a field. We assume all of the following:*

1. $|N(\mathcal{U}_1)|$ *is connected.*

2. *There exists a number* $j \in \{1, \cdots, m\}$ *such that* $(h_j^2)^{-1}(L) \cap (\cup_{i:i\neq j} h_i^{-1}(L)) = \emptyset$.

3. *There exist mutually distinct elements* $j_1 = j$, j_2, $j_3 \in \{1, \cdots, m\}$ *such that* $\{j_1, j_2\}$, $\{j_2, j_3\}$ *and* $\{j_1, j_3\}$ *are* 1*-simlexes in* $N(\mathcal{U}_1)$.

4. *For any* $r \geq 2$ *there exists no* r*-simplex* S *in* $N(\mathcal{U}_1)$ *with* $j \in S$.

Then, $\dim_R \check{H}^1(L,G,R) = \infty$.

Remark 2. Let L be a self-similar set with respect to $\{h_1, \cdots, h_m\}$. Then, we obtain a result similar to Theorem 5.2.

Proof of Theorem 3.5: By Theorem 5.2, Proposition 4.5, Theorem 4.6, and Lemma 4.7-4, we obtain $\dim_R \check{H}^1(J(G),G,R) = \dim_R \Psi(\check{H}^1(J(G),G,R) = \infty$. Hence, $\dim_R \check{H}^1(J(G),R) = \infty$. By the Alexander duality (see [Sp]), we have $\check{H}^1(J(G),R) \cong \tilde{H}_0(\overline{\mathbb{C}} \setminus J(G),\ R)$, where $\tilde{H}_0$ denotes the 0-th reduced homology. Hence, $F(G)$ has infinitely many connnected components. □

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