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INTERSECTIONS, RESIDUE THEOREMS ON SINGULAR SURFACES AND APPLICATIONS

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This is a summary of the joint work [BS1] with F. Bracci.

0. Motivation

In the celebrated paper [CS] published in 1982, C. Camacho and P. Sad proved that, for a holomorphic vector field v on a neighborhood of the origin 0 in $\mathbb{C}^2$ with isolated singularity, there always exists a separatrix (complex analytic integral curve through 0) for v. Main ingredients of the proof are (1) the results of Poincaré et al. on generic vector fields, (2) reduction of singularities by Seidenberg et al. and (3) the Camacho-Sad index theorem:

Theorem [CS]. *Let S be a complex surface, C a compact non-singular curve in S and $\mathcal{F}$ a one-dimensional foliation on S leaving C invariant. Let $p_1, \dots, p_r$ denote the singularities of $\mathcal{F}$ on C.*
(i) *For each p_i, we may associate a complex number $\mathrm{Ind}_C(\mathcal{F}, p_i)$, called the index.*
(ii) *We have*

$$\sum_{i=1}^{r} \mathrm{Ind}_C(\mathcal{F}, p_i) = C \cdot C,$$

the self-intersection number of C.

Generalizations of this theorem are done in [L1] and [Su1] for singular invariant curves in surfaces, in [G] and [L2] for codimension one foliations and in [LS] for general case.

Then in 1988, Camacho went on to prove the existence of separatrices for vector fields on a surface with an isolated singularity whose resolution graph is a tree ([C]), using (1) resoluton of surface singularities and reduction of singularities of vector fields, (2) Camacho-Sad index theorem and (3) a lemma on the resolution graphs.

An analogous problem in discrete dynamics is to investigate if there exist "parabolic curves" for holomorphic self-maps. In one-dimensional case, this is known as the Leau-Fatou flower theorem. In two-dimensional case, M. Abate proved in 2001 that for a holomorphic self-map of $(\mathbb{C}^2, 0)$ tangent to the identity, there always exists a parabolic curve for f.

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Here a parabolic curve for f means a continuous map $\varphi : \Delta \to \mathbb{C}^2$ of the unit disk $\Delta = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ such that $0 \in \varphi(\partial\Delta)$, $f(\varphi(\Delta)) \subset \varphi(\Delta)$ and that for any $z \in \Delta$, $\lim_{k\to\infty} \underbrace{f \circ \cdots \circ f}_{k}(\varphi(z)) = 0$.

Main ingredients of the proof are (1) the results of J. Écalle and M. Hakim in generic case, (2) reduction of "singularities" of maps and (3) the Abate index theorem:

Theorem [A]. *Let S be a complex surface, C a compact non-singular curve in S and f a holomorphic self-map of S with $f|_C = Id_C$. Suppose f is "tangential" (non-degenerate) along C and let $p_1, \ldots, p_r$ denote the "singularities" of f on C.*
(i) *For each p_i, we may associate a complex number $\mathrm{Ind}_C(f, C; p_i)$.*
(ii) *We have*

$$\sum_{i=1}^{r} \mathrm{Ind}_C(f, C; p_i) = C \cdot C.$$

Generalizations of this theorem to various directions are done in [BT], [ABT], [BS2], see also Theorem in Section 3 below. As to the terminologies, we also refer to [B], [BS1].

Thus the next natural question would be:

1. Existence of parabolic curves for holomorphic self-maps of singular surfaces

Concerning this, we proved:

Theorem [BS1]. *Let (X, p) be a t-absolutely isolated singularity whose resolution graph is a tree. For any holomorphic self-map f of (X, p) tangent to the identity, there exists a parabolic curve for f.*

Here we recall:

Definition. (1) A germ of variety (X, p) is an absolutely isolated singularity if it can be resolved by a finite number of quadratic blowing-ups.
(2) (X, p) is a t-absolutely isolated singularity if it is absolutely isolated and, at each blowing-up step, the strict transform is generically transverse to the exceptional divisor.

Example. The variety X defined by

$$x^2 - y^2 + z^{2r+1} = 0$$

in $\mathbb{C}^3 = \{ (x, y, z) \}$ has a t-absolutely isolated singularity at 0.

We hope to be able to remove the above restriction in the theorem (t-absolutely isolatedness) soon.

Here are the main ingredients of the proof:
(I) Generalization of the Abate index theorem.

This is an index (or residue) theorem for holomorphic self-maps of singular surfaces and will be described below. For this, we need a local intersection theory of curves (divisors), both Cartier and Weil, on singular surfaces.

(II) Use of the Camacho lemma on graphs, with arguments much more involved than the case of vector fields.

The major difference from the case of vector fields is that we may not be able to lift the given map when we blow-up the surface singularity so that we are forced to remain on singular surfaces.

2. Intersection theory

The following is essentially done in [M]. However, our approach is an analytic one based on Grothendieck residues on singular varieties and is applicable to the higher dimensional case as well.

In the sequel, a variety will be a reduced analytic space. A curve or a surface will be a variety of pure dimension one or two, respectively. For a subvariety V and a divisor D in a complex manifold W, we denote by $D \cdot V$ the pull-back $\iota^* D$ of D by the embedding $\iota : V \hookrightarrow W$. We use the symbol $\cap$ to denote set theoretic intersections.

2.1. Grothendieck residues relative to a subvariety

Let U be a neighborhood of 0 in $\mathbb{C}^r$ and V a subvariety of pure dimension n in U which contains 0 as at most an isolated singular point. Also, let $f_1, \ldots, f_n$ be holomorphic functions on U with $\bigcap_{i=1}^n \{p \in U : f_i(p) = 0\} \cap V = \{0\}$. For a holomorphic n-from ω on U, the *Grothendieck residue relative to* V is defined by

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ is an n-cycle in V defined by $\Gamma = \bigcap_{i=1}^n \{p \in U : |f_i(p)| = \varepsilon_i\} \cap V$ with ε_i small positive numbers (cf. [Su2, Ch.IV, 8], [Su3]).

Note that if V is a complete intersection defined by $h_1 = \cdots = h_k = 0$, $k = r - n$, then it coincides with the usual Grothendieck residue

$$\mathrm{Res}_0 \begin{bmatrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \ldots, f_n, h_1, \ldots, h_k \end{bmatrix}.$$

2.2. Multiplicities

Let V be as above and let $C_0(V)$ denote the tangent cone of V at 0. Recall that $C_0(V)$ is an analytic space whose support is the zero set of all the leading homogeneous polynomials of germs in the ideal of V at 0, and has the same dimension as V. We say that a collection of hyperplanes $(H_1, \ldots, H_i)$ through 0, $1 \leq i \leq n$, is general with respect to V if $\dim C_0(V) \cap H_1 \cap \cdots \cap H_i = n - i$.

We define the *multiplicity* of V at 0 by

$$m(V, 0) = \mathrm{Res}_0 \begin{bmatrix} d\ell_1 \wedge \cdots \wedge d\ell_n \\ \ell_1, \ldots, \ell_n \end{bmatrix}_V,$$

where $\ell_1, \ldots, \ell_n$ denote defining linear functions of n hyperplanes general with respect to V. This definition of multiplicity coincides with the one in [F, p.79] :

Lemma. *Let $PC_0(V)$ denote the projective cone of V at 0 (which is in $\mathbb{P}^{r-1}$). Then*

$$m(V,0) = \deg PC_0(V).$$

2.3. Intersections, local theory

Let X be a surface in a small neighborhood U of 0 in $\mathbb{C}^r$ possibly with an isolated singularity at 0. Let D_1 and D_2 be (effective, for simplicity) Cartier divisors on X. Defining functions for D_1 and D_2 are the restrictions of holomorphic functions f_1 and f_2 on U. Suppose f_1 and f_2 have no common irreducible factors at 0. Then the *intersection number* of D_1 and D_2 at 0 is defined by

$$(D_1 \cdot D_2)_0 = \mathrm{Res}_0 \begin{bmatrix} df_1 \wedge df_2 \\ f_1, f_2 \end{bmatrix}_X .$$

If D is a Cartier divisor defined by f and if Y is a Cartier curve, by the projection formula, we have

$$(D \cdot Y)_0 = \mathrm{Res}_0 \begin{bmatrix} df \\ f \end{bmatrix}_Y ,$$

which may be used to define the intersection number of D and Y, even if Y is not Cartier.

2.4. Intersections, global theory

Let X be a surface with isolated singularities in a complex manifold W. Let D_1 be a Cartier divisor on X and denote by L_{D_1} the associated line bundle over X. Let D_2 be a divisor (which may be only Weil) on X with compact support (X may not be compact). Then the (global) intersection number of D_1 and D_2 in X is defined by

$$D_1 \cdot D_2 = c^1(L_{D_1}) \frown [D_2].$$

In the algebraic category, this definition coincides with the one in [F]. If D_1 extends to a divisor on W and if D_1 and D_2 do not have common components, then the Čech-de Rham theory applies (see, *e.g.*, [Su2]) so that we have

$$D_1 \cdot D_2 = \sum_p (D_1 \cdot D_2)_p,$$

where p runs through the intersection points of D_1 and D_2.

2.5. Effect of blowing-up

Let X be a surface with isolated singularities in W, as in the previous section, and p a point of X. Let $\pi : \tilde{W} \longrightarrow W$ be the blowing-up of W at p, $D = \pi^{-1}(p)$ the exceptional divisor, $\tilde{X}$ the strict transform of X and $\rho : \tilde{X} \longrightarrow X$ the restriction of π. We set $E = D \cdot \tilde{X}$. Note that the support of E is $\pi^{-1}(p) \cap \tilde{X} = \rho^{-1}(p)$ and as an analytic subspace of $D = \mathbb{P}^{r-1}$, it coincides with the projective cone $PC_p(X)$ of X at p. It is also considered as a Cartier divisor in $\tilde{X}$. In the sequel, we assume that $\tilde{X}$ has only isolated singularities.

Let Y be a curve through p in X. Note that the strict transform of Y by ρ is equal to that of Y by π, which is denoted by $\tilde{Y}$.

Lemma. *If Y is Cartier, the multiplicity $m(Y,p)$ is divisible by $m(X,p)$ and if we set $m(Y,X;p) = m(Y,p)/m(X,p)$, we have*

$$\rho^* Y = \tilde{Y} + m(Y,X;p)E.$$

Theorem. *Let Y_1 and Y_2 be curves in X, with Y_1 Cartier.*
(1) *We have*

$$(Y_1 \cdot Y_2)_p = \sum_{q \in \rho^{-1}(p)} (\tilde{Y}_1 \cdot \tilde{Y}_2)_q + m(Y_1, X; p) \cdot m(Y_2, p).$$

(2) *If Y_1 is compact, then*

$$Y_1 \cdot Y_2 = \tilde{Y}_1 \cdot \tilde{Y}_2 + m(Y_1, X; p) \cdot m(Y_2, p).$$

2.6. Intersections of Weil curves

Let X be a surface in a complex manifold W. In this subsection, we assume that X has only absolutely isolated singularities. Let Y_1 and Y_2 be two (distinct) curves in X. If at least one of them is Cartier, the previous subsections 2.3 and 2.4 give a way to define the local and global intersection numbers of Y_1 and Y_2. If Y_1 and Y_2 are only Weil curves, we proceed as follows. Let $p \in Y_1 \cap Y_2$ and let $\pi : \tilde{W} \to W$ be the blowing-up at p. We use the notation of the subsection 2.5 for strict transforms etc. In view of Theorem in 2.3, we define

$$(Y_1 \cdot Y_2)_p = \sum_{q \in \rho^{-1}(p)} (\tilde{Y}_1 \cdot \tilde{Y}_2)_q + \frac{m(Y_1, p) \cdot m(Y_2, p)}{m(X, p)},$$

where $(\tilde{Y}_1 \cdot \tilde{Y}_2)_q$ is defined as in 2.3, if $\tilde{Y}_1$ or $\tilde{Y}_2$ is Cartier at q, or by recursion of the above formula if either is not Cartier at q. If at least one of Y_1 and Y_2 is compact, define

$$Y_1 \cdot Y_2 = \sum_{p \in Y_1 \cap Y_2} (Y_1 \cdot Y_2)_p.$$

Note that if either of Y_1 and Y_2 is not Cartier at p then $(Y_1 \cdot Y_2)_p$ is only a rational number, in general, for $m(X,p)$ might not divide $m(Y_1,p) \cdot m(Y_2,p)$, see Example below.

Also, in view of Lemma in 2.2, for a compact curve Y in X, we define the inverse image (total transform) by

$$\rho^* Y = \tilde{Y} + \frac{m(Y,p)}{m(X,p)} E.$$

Then we can define by recursion the self-intersection number of Y as

$$Y \cdot Y = \rho^* Y \cdot \rho^* Y.$$

Note that, in the above, we need not to resolve the singularities of X, we only need to take blowing-ups sufficiently many times so that the curve becomes Cartier.

Example. Let X be defined by $xy = z^2$ in $\mathbb{C}^3 = \{(x,y,z)\}$, and Y_1 and Y_2 by $x = z = 0$ and $y = z = 0$, respectively. Then Y_1 and Y_2 are Weil divisors (only $Y_1 \cup Y_2$ is Cartier). Since $m(X,0) = 2$, $m(Y_1,0) = m(Y_2,0) = 1$ and $\tilde{Y}_1$ and $\tilde{Y}_2$ are non-singular, we compute

$$(Y_1 \cdot Y_2)_0 = \tilde{Y}_1 \cdot \tilde{Y}_2 + \frac{m(Y_1,0) \cdot m(Y_2,0)}{m(X,0)} = 0 + \frac{1 \cdot 1}{2} = \frac{1}{2}.$$

3. The residue theorem

Here is the residue theorem we need:

Theorem [BS1]. *Let W be a complex manifold, $P \subset W$ a non-singular hypersurface and X a surface with isolated singularities in W. Suppose P intersects with X generically transversely. Let Y be a curve in $X \cap P$. Suppose there exists a holomorphic map $f : W \to W$ such that $f|_P = Id_P$, $f(X) \subset X$ and $f|_X$ is tangential on the non-singular part of Y. Let $\Sigma = \mathrm{Sing}(Y) \cup (\mathrm{Sing}(f|_X) \cap Y)$. Then*
(1) *For each point p in Σ, we have a residue $\mathrm{Res}(f, Y; p) \in \mathbb{C}$, which is determined only by the local behavior of f near p.*
(2) *If Y is compact,*

$$\sum_{p \in \Sigma} \mathrm{Res}(f, Y; p) = Y \cdot Y.$$

We give the idea of proof. For simplicity, we consider the case $Y = P \cap X$. First, for the map f, we associate a one-dimensional singular foliation $\mathcal{F}$ on $Y \setminus \mathrm{Sing}(Y)$. We set $\mathrm{Sing}(f|_X) = \mathrm{Sing}(X) \cup \mathrm{Sing}(\mathcal{F})$, $\Sigma = \mathrm{Sing}(Y) \cup (\mathrm{Sing}(f|_X) \cap Y)$ and $Y' = Y \setminus \Sigma$. Then there is an action (cf. e.g., [Su2, Ch.II, 9]) of $\mathcal{F}$ on the normal bundle $N_{Y',X'}$ of Y' in $X' = X \setminus \mathrm{Sing}(X)$ and, by a Bott type vanishing theorem, we have the vanishing of the first Chern class of $N_{Y',X'}$ (in fact on the form level):

$$c^1(N_{Y',X'}) = 0.$$

In the above situation, there is a natural extension N_Y of $N_{Y',X'}$ to Y, namely $N_Y = N_{P,W}|_Y$, and if we compute $c^1(N_Y)$, we see that it is localized at Σ and produces the above residues.

Finally we give an explicit expression for the residue. Let p be a point in Σ and take a coordinate system $(z_1, \ldots, z_r)$ near p so that P is given by $z_1 = 0$. We take a holomorphic function h near p on W such that $dz_1 \wedge dh|_{X'} \neq 0$. Then we have

$$\mathrm{Res}(f, Y; p) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{(z_1 \circ f - z_1)|_X}{z_1(h \circ f - h)|_X} dh,$$

where L denotes the link of Y at p.

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