A survey of real transverse sections of holomorphic foliations

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Introduction

Let $M$ be a closed, connected, smooth submanifold of real dimension $2n-1$ in the complex space $\mathbb{C}^n$ of dimension $n \geq 2$. Given a holomorphic one form $\omega$ in $\mathbb{C}^n$, for each $p \in \mathbb{C}^n$ with $\omega(p) \neq 0$ we define a $(n-1)$-dimensional linear subspace $P_\omega(p) = \{v \in T_p \mathbb{C}^n \mid \omega(p) \cdot v = 0\}$. If $\omega(p) = 0$, we set $P_\omega(p) = \{0\}$ and we shall say that the distribution $P_\omega$ defined by $\omega$ is singular at $p$. We denote by $\text{Sing}(\omega) = \{p \in \mathbb{C}^n \mid \omega(p) = 0\}$ the singular set of $\omega$. We have the following definition of transversality.

Definition We shall say that $M$ is transverse to $P_\omega$ if for every $p \in M$ we have $T_p M + P_\omega(p) = T_p \mathbb{R}^{2n}$ as real linear spaces.

In particular, since $P_\omega(p) = \{0\}$ for any singular point $p$, we conclude that $\text{Sing}(\omega) \cap M = \emptyset$.

In this note, we survey an existence or a non-existence of $M$ such that $M$ is transverse to $P_\omega$.

1 Facts and Known results

In this section, we review the case of holomorphic vector field $Z$ in $\mathbb{C}^n$, $n \geq 2$.

Given complex numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we denote by $\mathcal{H}(\lambda_1, \ldots, \lambda_n)$ the convex hull of the subset $\{\lambda_1, \ldots, \lambda_n\}$ in $\mathbb{C}$. Let $Z = \sum_{j=1}^{n} \lambda_j z_j \partial / \partial z_j$ be a linear vector field on $\mathbb{C}^n$, $n \geq 2$. We denote by $S^{2n-1}(1) = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\}$ the $(2n-1)$-dimensional sphere. We have a well-known fact.

Fact (1) If the origin $0 \in \mathbb{C}$ does not belong to $\mathcal{H}(\lambda_1, \ldots, \lambda_n)$, then $S^{2n-1}(1)$ is transverse to $Z$.

(2) If the origin $0$ belongs to $\mathcal{H}(\lambda_1, \ldots, \lambda_n)$, then $S^{2n-1}(1)$ is not transverse to $Z$.

This Fact suggests to us the following properties.

Theorem ([4]) If the origin $0$ belongs to $\mathcal{H}(\lambda_1, \ldots, \lambda_n)$, then there is no smooth embedding $\varphi$ of a closed connected smooth manifold $M$ of dimension $2n-1$ to $\mathbb{C}^n$ such that $\varphi(M)$ is transverse to $Z = \sum_{j=1}^{n} \lambda_j z_j \partial / \partial z_j$.

We have a Poincaré-Bendixson type theorem for holomorphic vector fields.
Theorem (A. Douady and T. Ito [3]) Let $N$ denote a subset of $\mathbb{C}^n$ holomorphic and diffeomorphic to the 2n-dimensional closed disc $\overline{D^{2n}(1)}$ consisting of all $z$ in $\mathbb{C}^n$ with $\|z\| \leq 1$.

Let $Z$ be a holomorphic vector field in some neighborhood of $N$. If the boundary $M = \partial N$ of $N$ is transverse to $Z$, then

(1) $Z$ has only one singular point, say $p$, in $N$.
(2) the index of $Z$ at $p$ is equal to one.
(3) each solution $L$ of $Z$ which crosses $M$ tends to $p$, that is, $p$ is in the closure of $L$. Further, the restriction $F(Z) \big|_{N - \{p\}}$ of the foliation $F(Z)$ defined by the solutions of $Z$ to $N - \{p\}$ is

$C^\omega$- diffeomorphic to the foliation $F(Z) \big|_{M \times (0, 1]}$ of $N - \{p\}$, where $F(Z) \big|_{M}$ denotes the restriction of $F(Z)$ to $M$.

Theorem (M. Brunella[1]) Soit $\Omega \subset \mathbb{C}^n$, $n \geq 2$, un ouvert borné avec frontière lisse et fortement convexe, et soit $v$ un champ de vecteurs holomorphe défini au voisinage de $\partial \Omega$ et transverse à $\partial \Omega$. Il existe un difféomorphisme $\Phi : \Omega \rightarrow \overline{D^{2n}(1)}$ qui envoie le feuilletage holomorphe singulier engendré par $v$ dans un feuilletage $\mathcal{G}$ singulier à l'origine et transverse aux sphères $S^{2n-1}(\lambda)$, $\lambda \in (0, 1]$.

Theorem (M. Brunella and P. Sad[2]) Let $\Omega \subset \mathbb{C}^2$ be a generalized bidisc and let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood of $\partial \Omega$ and transverse to $\partial \Omega$. Then there exists a locally injective holomorphic map $\phi$ which sends a neighborhood of $\partial \Omega$ to a neighborhood of $0$ in $\mathbb{C}^n$ such that $\mathcal{F} = \phi^*(L_\lambda)$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, where $L_\lambda$ is a linear hyperbolic foliation in $\mathbb{C}^2$ defined by $zdy + \lambda ydx = 0$.

2 Existence or non-existence of real transverse sections

Let $\omega = \sum_{j=1}^{n} f_j(z)dz_j$ be a holomorphic one form on $\mathbb{C}^n$, $n \geq 2$. We denote by $P_\omega$ the distribution defined by $\omega$ in $\mathbb{C}^n$.

Theorem 1 ([5]) Let $M$ be a real 2-dimensional closed, connected, smooth manifold.

If a smooth embedding $\varphi$ of $M$ to $\mathbb{C}^n$ is transverse to $P_\omega$, then $M$ is a torus.

We can construct a torus $T^2$ which is transverse to a holomorphic vector field $Z$. Let $Z = z_1 \partial/\partial z_1 + \lambda z_2 \partial/\partial z_2$ be a linear vector field on $\mathbb{C}^2$ and $T^2(r_1, r_2) = S^1(r_1) \times S^1(r_2) = \{ |z_1|=r_1 \} \times \{ |z_2|=r_2 \}$ a torus in $\mathbb{C}^2$.

Proposition 1([6]) The 2-dimensional torus $T^2(r_1, r_2)$ is transverse to $Z$ if and only if the imaginary part of $\lambda$ is different from zero.

We have the following non-existence theorems of transverse sections.

Theorem 2 ([6]) Let $Z = z_1 \partial/\partial z_1 + \lambda z_2 \partial/\partial z_2$, $\lambda \in \mathbb{R}$ be a linear vector field or $Z = z_1 \partial/\partial z_1 + (nz_2 + x_1^n) \partial/\partial z_2$, $n \in \mathbb{N}$ a holomorphic vector field of Dulac's normal form and $M$ a closed, connected 2-dimensional smooth manifold. Then there is no smooth embedding $\varphi$ of $M$ to $\mathbb{C}^2$ such that $\varphi(M)$ is transverse to $Z$.

Theorem 3 ([5]) There exists no holomorphic foliation $\mathcal{F}$ of codimension one in a neighborhood
of the polydisc $\Delta^4$ in $\mathbb{C}^4$ with the property that $\mathcal{F}$ is transverse to the product of spheres $S^2_1(1) \times S^2_2(1) = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 | |x_1|^2 + |x_2|^2 = 1, |x_3|^2 + |x_4|^2 = 1\} \subset S^2 \times C^2$.

**Theorem 4** ([5]) Let $\omega = \sum_{j=1}^{n} h_j(z)dz_j$ be a homogeneous integrable one form on $\mathbb{C}^n$, $n \geq 3$.

The sphere $S^{2n-1}(1)$ of dimension $2n-1$ is not transverse to the foliation $\mathcal{F}(\omega)$ defined by $\omega = 0$.

We are very interested in the following properties.

**Proposition 2** ([5]) Take $\omega = z_1z_2z_3(\sum_{j=1}^{3} \lambda_j \frac{dz_j}{z_j})$ on $\mathbb{C}^3$, where the non-zero complex numbers $\lambda_1, \lambda_2, \lambda_3$ are satisfied with the following properties: $\lambda_i/\lambda_j \notin \mathbb{R}$, $(i \neq j)$ and $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$.

We get the following statements (i) \~ (iii).

(i) $\text{Sing}(\omega) = \bigcup_{j \neq k} (z_j = 0, z_k = 0)$.

(ii) $S^6(1) \setminus (\text{Sing}(\omega) \cap S^6(1))$ is transverse to $\mathcal{F}(\omega)$.

(iii) Let $P = (\alpha z_1 + \beta z_2 + \gamma z_3 = 0)$, $\alpha, \beta, \gamma \in \mathbb{C}^*$ be a hyperplane through the origin. The restriction $\mathcal{F}_1$ of $\mathcal{F}(\omega)$ to $P$ has only the origin as singularity. $P$ is transverse to $\mathcal{F}(\omega)$ outside $\text{Sing}(\omega)$. $\mathcal{F}_1$ is not transverse to $P \cap S^6(1)$ though $\text{Sing}(\mathcal{F}_1) \cap (P \cap S^6(1)) = \phi$.

**Example 1** (T. Ito and M. Yoshino) Take complex numbers $\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n \in \mathbb{C}^*$ and assume that the origin 0 belongs to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$ and $\mathcal{H}(\mu_1, \cdots, \mu_n)$. We make the following assumption: There exist real numbers $c_1$ and $c_2$ such that $\mathcal{H}(c_1\lambda_1 + c_2\mu_1, \cdots, c_1\lambda_n + c_2\mu_n) \neq 0$.

Consider linear vector fields $X = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$ and $Y = \sum_{j=1}^{n} \mu_j z_j \partial/\partial z_j$. Then it is clear that $[X, Y] = 0$ so that $X$ and $Y$ span a foliation $\mathcal{F}$ of complex dimension two on $\mathbb{C}^n$. Also $\mathcal{F}$ has as singular set $\text{Sing}(\mathcal{F})$ the union of the coordinate axis. Denote by $\sum(X)$ the set of tangency points of $X$ with the sphere $S^{2n-1}(r) \subset \mathbb{C}^n$, $r \geq 0$, then we have $\sum(X)$ given by the equation $\sum_{j=1}^{n} \lambda_j |z_j|^2 = 0$. This is a real cone. Analogously we define $\sum(Y)$ and describe it by the equation $\sum_{j=1}^{n} \mu_j |z_j|^2 = 0$. Under the assumption, we have $\sum(X) \cap \sum(Y) = \{0\}$. $\mathcal{F}$ is transverse to $S^{2n-1}(r) \setminus (\text{Sing}(\mathcal{F}) \cap S^{2n-1}(r))$, $r > 0$. Moreover each leaf of $\mathcal{F}$ accumulates the origin.

**Theorem 5** ([5]) Let $\omega = \sum_{j=1}^{2n+1} f_j(x)dz_j$ be a holomorphic one form on $\mathbb{C}^{2n+1}$. Then the sphere $S^{4n+1}(r)$, $r > 0$ is not transverse to $P_\omega$.

By this theorem 5 or Proposition 2, the sphere $S^5(1)$ of dimension 5 is not transverse to $\mathcal{F}(\omega_\lambda)$ where $\omega_\lambda = z_1 z_2 z_3(\sum_{j=1}^{3} \lambda_j \frac{dz_j}{z_j})$, $\lambda_j \neq 0$ is a linear logarithmic one form on $\mathbb{C}^3$.

**Example 2** ([5]) (1) If $\lambda_i/\lambda_j$, $i \neq j$, are not positive real, then we can construct a smooth embedding $\phi : S^1 \times S^3 \times S^1 \rightarrow \mathbb{C}^3$ such that $\phi(S^1 \times S^3 \times S^1)$ is transverse to $\mathcal{F}(\omega_\lambda)$.

(2) If $\lambda_i/\lambda_j$, $i \neq j$, are not real, then there exists a smooth embedding $\Phi : T^5 = S^1 \times T^3 \times S^1 \rightarrow \mathbb{C}^3$ such that $\Phi(T^5)$ is transverse to $\mathcal{F}(\omega_\lambda)$. 


References


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