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A survey of real transverse sections of holomorphic foliations

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Introduction

Let M be a closed, connected, smooth submanifold of real dimension 2n-1 in the complex space $\mathbf{C}^n$ of dimension $n \geq 2$. Given a holomorphic one form ω in $\mathbf{C}^n$, for each $p \in \mathbf{C}^n$ with $\omega(p) \neq 0$ we define a (n-1)-dimensional linear subspace $P_\omega(p) = \{v \in T_p\mathbf{C}^n \mid \omega(p) \cdot v = 0\}$.
If $\omega(p) = 0$, we set $P_\omega(p) = \{0\}$ and we shall say that the distribution P_ω defined by ω is singular at p. We denote by $Sing(\omega) = \{p \in \mathbf{C}^n \mid \omega(p) = 0\}$ the singular set of ω. We have the following definition of transversality.

__Definition__ We shall say that M is transverse to P_ω if for every $p \in M$ we have $T_pM + P_\omega(p) = T_p\mathbf{R}^{2n}$ as real linear spaces.

In particular, since $P_\omega(p) = \{0\}$ for any singular point p, we conclude that $Sing(\omega) \cap M = \emptyset$.
In this note, we survey an existence or a non- existence of M such that M is transverse to P_ω.

1 Facts and Known results

In this section, we review the case of holomorphic vector field Z in $\mathbf{C}^n$, $n \geq 2$.

Given complex numbers $\lambda_1, \cdots, \lambda_n \in \mathbf{C}^*$, we denote by $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$ the convex hull of the subset $\{\lambda_1, \cdots, \lambda_n\}$ in $\mathbf{C}$. Let $Z = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$ be a linear vector field on $\mathbf{C}^n$, $n \geq 2$. We denote by $S^{2n-1}(1) = \{z \in \mathbf{C}^n \mid \| z \|^2 = 1\}$ the (2n-1)-dimensional sphere. We have a well-known fact.

__Fact__ (1) If the origin $0 \in \mathbf{C}$ does not belong to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$, then $S^{2n-1}(1)$ is transverse to Z.
(2) If the origin 0 belongs to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$, then $S^{2n-1}(1)$ is not transverse to Z.

This Fact suggests to us the following properties.

__Theorem__ ([4]) If the origin 0 belongs to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$, then there is no smooth embedding φ of a closed connected smooth manifold M of dimension 2n-1 to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $Z = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$.

We have a Poincaré-Bendixson type theorem for holomorphic vector fields.

Theorem (A. Douady and T. Ito [3]) Let $\mathbf{N}$ denote a subset of $\mathbf{C}^n$ holomorphic and diffeomorphic to the 2n-dimensional closed disc $\overline{D^{2n}(1)}$ consisting of all z in $\mathbf{C}^n$ with $\| z \| \leq 1$.
Let Z be a holomorphic vector field in some neighborhood of $\mathbf{N}$. If the boundary $M = \partial\mathbf{N}$ of $\mathbf{N}$ is transverse to Z, then
(1)Z has only one singular point, say p, in $\mathbf{N}$.
(2) the index of Z at p is equal to one.
(3)each solution L of Z which crosses M tends to p, that is, p is in the closure of L. Further, the restriction $\mathcal{F}(Z)\Big|_{\mathbf{N}-\{\mathbf{p}\}}$ of the foliation $\mathcal{F}(Z)$ defined by the solutions of Z to $\mathbf{N}-\{p\}$ is $\mathbf{C}^\omega-$ diffeomorphic to the foliation $\mathcal{F}(Z)\Big|_M \times (0,1]$ of $\mathbf{N}-\{p\}$, where $\mathcal{F}(Z)\Big|_M$ denotes the restriction of $\mathcal{F}(Z)$ to M.

Theorem (M. Brunella[1]) Soit $\Omega \subset \mathbf{C}^n$, $n \geq 2$, un ouvert borné avec frontière lisse et fortement convexe, et soit v un champ de vecteurs holomorphe défini au voisinage de $\overline{\Omega}$ et transverse à $\partial\Omega$. Il existe un difféomorphisme $\Phi : \overline{\Omega} \to \overline{D^{2n}(1)}$ qui envoie le feuilletage holomorphe singulier engendré par v dans un feuilletage $\mathcal{G}$ singulier à l'origine et transverse aux sphères $S^{2n-1}(\lambda)$, $\lambda \in (0,1]$.

Theorem (M. Brunella and P. Sad[2]) Let $\Omega \subset \mathbf{C}^2$ be a generalized bidisc and let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood of $\overline{\Omega}$ and transverse to $\partial\Omega$. Then there exists a locally injective holomorphic map ϕ which sends a neighborhood of $\overline{\Omega}$ to a neighborhood of 0 in $\mathbf{C}^n$ such that $\mathcal{F}= \phi^*(L_\lambda)$ for some $\lambda \in \mathbf{C} \setminus \mathbf{R}$, where L_λ is a linear hyperbolic foliation in $\mathbf{C}^2$ defined by $xdy + \lambda ydx = 0$.

2 Existence or non-existence of real transverse sections

Let $\omega = \sum_{j=1}^{n} f_j(z)dz_j$ be a holomorphic one form on $\mathbf{C}^n$, $n \geq 2$. We denote by P_ω the distribution defined by ω in $T\mathbf{C}^n$.

Theorem 1 ([5]) Let M be a real 2-dimensional closed, connected, smooth manifold.
If a smooth embedding φ of M to $\mathbf{C}^n$ is transverse to P_ω, then M is a torus.

We can construct a torus T^2 which is transverse to a holomorphic vector field Z. Let $Z = z_1\partial/\partial z_1 + \lambda z_2\partial/\partial z_2$ be a linear vector field on $\mathbf{C}^2$ and $T^2(r_1, r_2) = S^1(r_1) \times S^1(r_2) = \{|z_1| = r_1\} \times \{|z_2| = r_2\}$ a torus in $\mathbf{C}^2$.

Proposition 1([6]) The 2-dimensional torus $T^2(r_1, r_2)$ is transverse to Z if and only if the imaginary part of λ is different from zero.

We have the following non-existence theorems of transverse sections.

Theorem 2 ([6]) Let $Z = z_1\partial/\partial z_1 + \lambda z_2\partial/\partial z_2$, $\lambda \in \mathbf{R}$ be a linear vector field or $Z = z_1\partial/\partial z_1 + (nz_2 + z_1^n)\partial/\partial z_2$, $n \in \mathbf{N}$ a holomorphic vector field of Dulac's normal form and M a closed, connected 2-dimensional smooth manifold. Then there is no smooth embedding φ of M to $\mathbf{C}^2$ such that $\varphi(M)$ is transverse to Z.

Theorem 3 ([5]) There exists no holomorphic foliation $\mathcal{F}$ of codimension one in a neighborhood of the polydisc Δ^4 in $\mathbf{C}^4$ with the property that $\mathcal{F}$ is transverse to the product of spheres $S_1^3(1) \times S_2^3(1) = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 | |z_1|^2 + |z_2|^2 = 1,\ |z_3|^2 + |z_4|^2 = 1\} \subset \mathbf{C}^2 \times \mathbf{C}^2$.

Theorem 4 ([5]) Let $\omega = \sum_{j=1}^{n} h_j(z) dz_j$ be a homogeneous integrable one form on $\mathbf{C}^n$, $n \geq 3$. The sphere $S^{2n-1}(1)$ of dimension 2n-1 is not transverse to the foliation $\mathcal{F}(\omega)$ defined by $\omega = 0$.

We are very interested in the following properties.

Proposition 2([5]) Take $\omega = z_1 z_2 z_3 (\sum_{j=1}^{3} \lambda_j \frac{dz_j}{z_j})$ on $\mathbf{C}^3$, where the non-zero complex numbers $\lambda_1, \lambda_2, \lambda_3$ are satisfied with the following properties: $\lambda_i/\lambda_j \notin \mathbf{R}$, $(i \neq j)$ and $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$. We get the following statements(i)~(iii).
(i)$Sing(\omega) = \bigcup_{i \neq j} \{z_i = 0,\ z_j = 0\}$.
(ii)$S^5(1) \setminus (Sing(\omega) \cap S^5(1))$ is transverse to $\mathcal{F}(\omega)$.
(iii)Let $P = \{\alpha z_1 + \beta z_2 + \gamma z_3 = 0\}$, $\alpha, \beta, \gamma \in \mathbf{C}^*$ be a hyperplane through the origin. The restriction $\mathcal{F}_1$ of $\mathcal{F}(\omega)$ to P has only the origin as singularity. P is transverse to $\mathcal{F}(\omega)$ outside $Sing(\omega)$. $\mathcal{F}_1$ is not transverse to $P \cap S^5(1)$ though $Sing(\mathcal{F}_1) \cap (P \cap S^5(1)) = \phi$.

Example 1(T.Ito and M. Yoshino) Take complex numbers $\lambda_1, \cdots, \lambda_n,\ \mu_1, \cdots, \mu_n \in \mathbf{C}^*$ and assume that the origin 0 belongs to $\mathcal{H}(\lambda_1, \cdots, \lambda_n)$ and $\mathcal{H}(\mu_1, \cdots, \mu_n)$. We make the following assumption: There exist real numbers c_1 and c_2 such that $\mathcal{H}(c_1\lambda_1 + c_2\mu_1, \cdots,\ c_1\lambda_n + c_2\mu_n) \not\ni 0$. Consider linear vector fields $X = \sum_{j=1}^{n} \lambda_j z_j \partial/\partial z_j$ and $Y = \sum_{j=1}^{n} \mu_j z_j \partial/\partial z_j$. Then it is clear that $[X, Y] = 0$ so that X and Y span a foliation $\mathcal{F}$ of complex dimension two on $\mathbf{C}^n$. Also $\mathcal{F}$ has as singular set $Sing(\mathcal{F})$ the union of the coordinate axis. Denote by $\sum(X)$ the set of tangency points of X with the spheres $S^{2n-1}(r) \subset \mathbf{C}^n$, $r \geq 0$, then we have $\sum(X)$ given by the equation $\sum_{j=1}^{n} \lambda_j |z_j|^2 = 0$. This is a real cone. Analogously we define $\sum(Y)$ and describe it by the equation $\sum_{j=1}^{n} \mu_j |z_j|^2 = 0$. Under the assumption, we have $\sum(X) \cap \sum(Y) = \{0\}$. $\mathcal{F}$ is transverse to $S^{2n-1}(r) \setminus (Sing(\mathcal{F}) \cap S^{2n-1}(r))$, $r > 0$. Moreover each leaf of $\mathcal{F}$ accumulates the origin.

Theorem 5 ([5]) Let $\omega = \sum_{j=1}^{2n+1} f_j(z) dz_j$ be a holomorphic one form on $\mathbf{C}^{2n+1}$. Then the sphere $S^{4n+1}(r)$, $r > 0$ is not transverse to P_ω.

By this theorem 5 or Proposition 2, the sphere $S^5(1)$ of dimension 5 is not transverse to $\mathcal{F}(\omega_\lambda)$ where $\omega_\lambda = z_1 z_2 z_3 (\sum_{j=1}^{3} \lambda_j \frac{dz_j}{z_j})$, $\lambda_j \neq 0$ is a linear logarithmic one form on $\mathbf{C}^3$.

Example 2([5]) (1)If λ_i/λ_j, $i \neq j$, are not positive real, then we can construct a smooth embedding $\phi : S^1 \times S^3 \times S^1 \longrightarrow \mathbf{C}^3$ such that $\phi(S^1 \times S^3 \times S^1)$ is transverse to $\mathcal{F}(\omega_\lambda)$.
(2)If λ_i/λ_j, $i \neq j$, are not real, then there exists a smooth embeddin $\Phi : T^5 = S^1 \times T^3 \times S^1 \longrightarrow \mathbf{C}^3$ such that $\Phi(T^5)$ is transverse to $\mathcal{F}(\omega_\lambda)$.

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