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On the diffeomorphism group of a smooth orbifold and its application

阿部 孝順 (信州大・理)

Kōjun Abe (Shinshu Univ.)

§1. Introduction

Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of an n-dimensional smooth manifold M which are istopic to the identity through compactly supported isotopies. In [TH], Thurston proved that the group $\mathcal{D}(M)$ is perfect, which means $\mathcal{D}(M)$ coincides with its commutator subgroup. There are many analogous results on the group of a smooth manifold M preserving a geometric structure of M.

In this note we shall study the case when M is a smooth orbifold. Since a smooth orbifold is locally diffeomorphic to the orbit space of a smooth G-manifold with finite group G, first we shall consider in the case of a representation space V of a finite group G. Let $\mathcal{D}_G(V)$ denote the group of equivariant smooth diffeomorphisms of V which are G-isotopic to the identity through compactly supported equivariant smooth isotopies. In general the group $\mathcal{D}_G(V)$ is not perfect. Then we calculate the first homology group $H_1(\mathcal{D}_G(V))$.

We shall prove that $\mathcal{D}_G(V)$ is perfect if $\dim V^G > 0$ and $H_1(\mathcal{D}_G(V))$ is isomorphic to $H_1(Aut_G(V)_0)$ if $\dim V^G = 0$. Here $Aut_G(V)_0$ is the identity component of the group of G-equivariant linear automorphisms of V, and V^G is the fixed point set of G on V ([AF5]).

Secondly we apply the above result to the case of smooth orbifold and also smooth G-manifold. Using the result by Biestone [BI1] and Schwarz [SC1], we see that $H_1(\mathcal{D}_G(V))$ is isomorphic to $H_1(\mathcal{D}(V/G))$. Combining those results and the fragmentation lemma we can determine the structure of $H_1(\mathcal{D}(N))$ of the diffeomorphism group $\mathcal{D}(N)$ for any smooth orbifold N. Then we see that $H_1(\mathcal{D}(N))$ describes a geometric structure around the isolated singularities.

Let M be a smooth G-manifold for a finite group G. Then $H_1(\mathcal{D}_G(M))$ is isomorphic to $H_1(\mathcal{D}(M/G))$, and we see that $H_1(\mathcal{D}_G(M))$ describes the properties of the isotropy representations at the isolated fixed points of M. We can also apply the above results to a smooth G-manifold when G is a compact Lie group. If M is a principal G-manifold with G a compact Lie group, then we proved that the group $\mathcal{D}_G(M)$ is perfect for $\dim(M/G) > 0$ (Banyaga [BA1] and Abe and Fukui [AF1]). In [AF2] we calculated $H_1(\mathcal{D}_G(M))$ when M is a smooth G-manifold with codimension one orbit. We shall apply the above result to the case of a locally free $U(1)$-action on the 3-sphere, and calculate $H_1(\mathcal{D}_{U(1)}(S^3))$ ([AF5]).

Thirdly we shall apply the results to the modular group. Let Γ be the modular group which acts on the the upper half complex plane $\mathcal{H}$ by the Möbius transformations. Then the orbit space $\mathcal{H}/\Gamma$ is a smooth orbifold. Let $\mathcal{R}_\Gamma$ be the compactified space of $\mathcal{H}/\Gamma$ by adjoining the point $*$ which corresponds to the Γ-equivalence class of the parabolic cusps. With the canonical smooth coordinate around $*$, we shall calculate the group $H_1(\mathcal{D}(\mathcal{R}_\Gamma)))$, which describes the elliptic points and the cusp point. We can also calculate the group for the case of the congruence subgroups of Γ.

We can apply the above results to the case of foliation preserving diffeomorphism groups. We studied for the similar problem in the Lipschitz category ([AF3], [AF4], [AF6], [AFM]).

§2. Recent results on the diffeomorphism grooups on smooth orbifolds

Let G be a finite group and let M be a smooth connected G-manifold. Let $\mathcal{D}_G(M)$ denote the group of G-equivariant smooth diffeomorphisms of M which are G-isotopic to the identity through isotopies with compact support.

First we shall calculate $\mathcal{D}_G(V)$ for a finite dimensional G-module V. Let V^G be the subspace of the fixed point set of V. Let $A_G(V)$ denote the set of G-invariant automorphisms of V and let $A_G(V)_0$ be the identitiy component of $A_G(V)$. Then we have the following.

Theorem 1
(1) *If* $\dim V^G > 0$, *then* $\mathcal{D}_G(V)$ *is perfect.*
(2) *If* $\dim V^G = 0$, *then* $H_1(\mathcal{D}_G(V)) \cong H_1(A_G(V)_0)$.

We can decompose $V = \oplus_{i=1}^{d} k_i V_i$, where V_i runs over the inequivalent irreducible representation space of G and k_i is a positive integer. Let $End_G(V_i)$ denote the set of G-invariant endmorphisms of V_i. Then $\dim End_G(V_i) = 1, 2$ or 4.

Corollary 2 *If* $\dim V^G = 0$, *then*

$$H_1(\mathcal{D}_G(V)) \cong \mathbf{R}^d \times \overbrace{U(1) \times \cdots \times U(1)}^{d_2},$$

where d_2 *is the number of* V_i *with* $\dim End_G(V_i) = 2$.

Definition 3 (smooth orbifold)
A paracompact Hausdorff space M is called a smooth orbifold if there exists an open covering $\{U_i|\ i \in \Lambda\}$ of M, closed under finite intersections, satistying the following.
(1) *There exist an open subset $\tilde{U}_i$ in $\mathbf{R}^n$ such that a finite group Γ_i acts effectively on $\tilde{U}_i$ and a homeomorphism $\phi_i :\ \tilde{U}_i/\Gamma_i \to U_i$.*
(2) *Whenever $U_i \subset U_j$, there exixts a smooth embedding $\phi_{ij} :\ \tilde{U}_i \to \tilde{U}_j$ such that*

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\phi_{ij}"] \arrow[d, "\pi_i"] & \tilde{U}_j \arrow[d, "\pi_j"] \\
\tilde{U}_i/\Gamma_i \arrow[d, "\phi_i^{-1}"] & \tilde{U}_j/\Gamma_j \arrow[d, "\phi_j^{-1}"] \\
U_i \arrow[r, "\subset"] & U_j.
\end{tikzcd}$$

(U_i, ϕ_i) is called a local chart of M.

Here we define the smooth maps between smooth orbifolds (c.f. [BI1]). $f :\ M \to \mathbf{R}$ is said to be smooth if for any local chart (U_i, ϕ_i) of M, $\tilde{U}_i \xrightarrow{\pi_i} \tilde{U}_i/\Gamma_i \xrightarrow{\phi_i} U_i \xrightarrow{f} \mathbf{R}$ is smooth. $h :\ M \to M$ is said to be smooth if for any smooth function $f :\ M \to \mathbf{R}$, $f \circ h$ is smooth. $h :\ M \to M$ is called a diffeomorphism if h and h^{-1} are smooth. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of M which are isotopic to the identity through isotopies with compact support.

$p \in M$ is said to be an isolated singular point of M if there exists a local chart (U_i, ϕ_i) around p such that $\tilde{p}$ is the isolated fixed point of $\tilde{U}_i$ with $\pi_i(\tilde{p}) = p$. Here $\phi_i : U_i \to \tilde{U}_i/\Gamma_i$ and $\pi_i : \tilde{U}_i \to U_i$ are the maps defined in Definition 3.

Let $(U_i, \phi_i), (U_j, \phi_j)$ be local charts of M around an isolated singular point p of M. Then we can assume that $\tilde{U}_i$ and $\tilde{U}_j$ are invariant open neighborhoods around the origin of linear representation spaces of Γ_i and Γ_j, respectively. By the result of Strub [ST], the groups Γ_i and Γ_j are isomorphic and the corresponding representaions are equivalent. Then the isolated singular point p determines the equivalence class of the linear representation space V_p of a finite group Γ_p.

Theorem 4 *If a smooth orbifold M has $\{p_1, ..., p_k\}$ as the isolated singular point set, then*

$$H_1(\mathcal{D}(M)) \cong H_1(A_{\Gamma_{p_1}}(V_{p_1})_0) \times \cdots \times H_1(A_{\Gamma_{p_k}}(V_{p_k})_0).$$

We can apply Theorem 4 to the case of smooth G-manifold with finite group G.

Theorem 5 *Let G be a finite group and M a smooth G-manifold. If the orbit space M/G has $\{G \cdot p_1, ..., G \cdot p_k\}$ as the isolated singular points, then*

$$H_1(\mathcal{D}_G(M)) \cong H_1(A_{G_{p_1}}(T_{p_1}M)_0) \times \cdots \times H_1(A_{G_{p_k}}(T_{p_k}M)_0).$$

Corollary 6 *Let $\tilde{\mathbf{R}}$ be the non-trivial one dimensional representation space of $\mathbf{Z}_2$. Then*

$$H_1(\mathcal{D}_{\mathbf{Z}_2}(\tilde{\mathbf{R}}^n)) \cong H_1(\mathcal{D}(\tilde{\mathbf{R}}^n/\mathbf{Z}_2)) \cong \mathbf{R}.$$

We can apply Corollary 6 to a smooth $U(1)$-action on S^3. Let

$$S^3 = \{(w_1, w_2) \in \mathbf{C}^2 |\ |w_1|^2 + |w_2|^2 = 1\}$$

with $U(1)$-action given by

$$z \cdot (w_1, w_2) = (zw_1, z^2 w_2), \qquad z \in U(1).$$

Then it has two orbit types $\{(1),\ (\mathbf{Z}_2)\}$ and the orbit space $S^3/U(1)$ is homeomorphic to the space known as the tear drop which is the two dimensional sphere with one isolated singular point.

Theorem 7 $H_1(\mathcal{D}_{U(1)}(S^3)) \cong \mathbf{R} \times U(1)$.

Remark 8 *If we restrict the above action to $\mathbf{Z}_n$, then $\mathcal{D}_{\mathbf{Z}_n}(S^3)$ is perfect.*

§3. Application to the modular group

In this section we shall apply the results to the modular group. Let $\mathcal{H}$ be the upper half complex plane. Let $SL(2, \mathbf{R})$ be the group of real matrix with determinant 1. Then $SL(2, \mathbf{R})$ acts on $\mathcal{H}$ as follows.

$$g \cdot z = \frac{az + b}{cz + d} \qquad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ z \in \mathcal{H}.$$

Then $SL(2, \mathbf{R})$ acts transitively on $\mathcal{H}$ and the isotropy subgroup at $i = \sqrt{-1}$ is $SL(2, \mathbf{R})_i = SO(2)$. The kernel of the action is $\mathbf{Z}_2 = \{\pm 1\}$ and $PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm 1\}$ acts effectively on $\mathcal{H} \cong SL(2, \mathbf{R})/SO(2)$.

The action can be extended to the Riemannian sphere: $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.
$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \bar{\mathbf{C}}$,

$$g \cdot z = \begin{cases} \frac{az+b}{cz+d} & (\ z \neq -\frac{d}{c}, \infty) \\ \infty & (z = -\frac{d}{c},\ z = d = 0) \\ \frac{a}{c} & (z = \infty) \end{cases}$$

Set

$$R_1 = \left\{ \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\},$$

$$R_2 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\quad \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then each $g \in SL(2, \mathbf{R})$ is conjugate to one of the elements of $SO(2) \cup R_1 \cup R_2$, and $g \neq \pm 1$ is called *elliptic, hyperbolic* and *parabolic* if g is conjugate of an element in $SO(2)$, R_1 and R_2, respectively.

Let $\Gamma = SL(2, \mathbf{Z})$ be the group of the integral matrices with determinant 1. Then $\bar{\Gamma} = \Gamma/\{\pm 1\}$ acts properly on $\mathcal{H}$ (i.e. for each $z \in \mathcal{H}$, there exists open neighborhood U of z such that $\bar{\Gamma}_U = \{g \in \bar{\Gamma} |\ g \cdot U = U\}$ is a finite group and if $\gamma \cdot U \cap U \neq \phi$ for $\gamma \in \bar{\Gamma}$, then $\gamma \in \bar{\Gamma}_U$).

$z \in \mathcal{H}$ is called *elliptic point* if there exits an ellipic element $g \in \Gamma$ such that $g \cdot z = z$. $x \in \mathbf{R} \cup \{\infty\}$ is called *cusp point* if there exists a parabolic element $g \in \Gamma$ such that $g \cdot z = z$.

Proposition 9 (1) *If z is a elliptic point, then Γ_z is a cyclic group which is conjugate to a cyclic subgroup of $SO(2)$.*
(2) *If x is a cusp point, then Γ_x is isomorphic to $\mathbf{Z}$ which is conjugate to the group*

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}.$$

(3) *Γ acts transitively on the set of cusp points which is coincides with $Q \cup \{\infty\}$, where Q is the set of rational numbers.*

Set

$$\mathcal{H}^* = \mathcal{H} \cup Q, \quad \mathcal{R}_\Gamma = \mathcal{H}^*/\Gamma = \mathcal{H}/\Gamma \cup \{*\}.$$

We give the set

$$\{*\} \cup \cup_{c>0} \{z \in \mathcal{H} |\ \Im z > c\}$$

as a fundamental system of open neighborhood of the point $*$. Then $\mathcal{R}_\Gamma$ is homeomorphic to S^2.

Proposition 10 *There exists a Γ_∞-invariant open neighborhood $\tilde{U}$ of $*$ satisfying the following.*
(1) $\Gamma_\infty = \{g \in \Gamma |\ g \cdot \tilde{U} \cap \tilde{U} \neq \phi\}$.
(2) *Let* $\varphi:\ \tilde{U}/\Gamma_\infty \to \mathbf{C}$ *be the map given by* $\varphi(\Gamma_\infty \cdot z) = \exp(2\pi\sqrt{-1}z)$ *for* $z \in \tilde{U}$. *Then* φ *is a homeomorphism into an open set* U *of* $\mathbf{C}$.

Let $\iota:\ \tilde{U}/\bar{\Gamma}_\infty \to \mathcal{R}_\Gamma$ be the natural map. Put $U = \iota(\tilde{U}/\bar{\Gamma}_\infty)$. By Proposition 10 U is an open neighborhood of $*$ and the homeomorphism $\phi = \varphi \circ \iota^{-1}:\ U \to \tilde{U}/\bar{\Gamma}_\infty$ is regarded as a local coordinate of $\mathcal{R}_\Gamma$.

We call $h:\ \mathcal{R}_\Gamma \to \mathcal{R}_\Gamma$ to be a diffeomorphism if the following conditions (1),(2),(3) is satisfied.

(1) $h|(\mathcal{H}/\bar{\Gamma})$ is a diffeomorphism of $\mathcal{H}/\bar{\Gamma}$ as a smooth orbifold.

(2) $\phi \circ h \circ \phi^{-1}$ is a diffeomorphism of U.

(3) There exists $\bar{\Gamma}_\infty$-equivariant diffeomorphism $\tilde{h}$ of $\mathcal{H}$ such that the induced diffeomorphism on $\mathcal{H}/\bar{\Gamma}$ coinsides with h on $U \setminus \{*\}$.

Theorem 11

(1) $H_1(\mathcal{D}_\Gamma(H^2)) \cong H_1(\mathcal{D}(H^2/\Gamma)) \cong \mathbf{R}^2 \times U(1)$.

(2) $H_1(\mathcal{D}(\mathcal{R}_\Gamma)) \cong U(1) \times \mathbf{R}^3$.

The orbifold H^2/Γ has two isolated singular points which correspond to the elliptic subgroups of Γ with orders 2 and 3, which induces the isomorphism in Theorem 11, (1). In addtion to those singular points, $\mathcal{R}_\Gamma$ has the singular point $*$ corresponding to the cusp point, which induces the isomorphism in Theorem 11, (2).

Let $\Gamma(N)$ denote the principal congruence subgroup of level N. Then

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1,\ b \equiv c \equiv 0 \mod N\mathbf{Z}\right\}.$$

Similarly to the case of the modular group, we have the following.

Theorem 12 $H_1(\mathcal{D}(\mathcal{R}_\Gamma(N))) \cong \mathbf{R}^{t(N)}$, *where* $t(N)$ *is the number of cusps of* $\mathcal{H}/\Gamma(N)$.

The number $t(N)$ is known as:

$$t(1) = 1, \quad t(2) = 3,$$

$$t(N) = \frac{1}{2N}(N : \Gamma(N)) \quad (N \geq 3),$$
$$(N : \Gamma(N)) = N^3 \prod_{p|N} (1 - \frac{1}{p^2}).$$

We can also apply Theorem 1 to calculate the first homology group of the foliation preserving diffeomorphism group for a compact Hausdorff foliation.

§4. Outline of the proof of Theorem 1

First we prove Theorem 1 (1). Let G be a finite group and let V be a G-module with $\dim V^G > 0$. Then there exists a G-module W with $\dim W^G = 0$ such that $V = W \oplus \mathbf{R}^q$. We prove $\mathcal{D}_G(V)$ is perfect by induction of the order of G. If $G = \{1\}$, then $\mathcal{D}_G(V)$ is perfect by the result of Thurston [TH]. Assume that Theorem 1 (1) holds for any finite subgroup H with $|H| < |G|$.

To investigate the group structure of $\mathcal{D}_G(V)$, we give C^∞-topology on $\mathcal{D}_G(V)$. For the proof we need the following fragmentaion lemma.

Lemma 13 (fragmentation lemma)
Let M be a smooth G-manifold and let $\{U_i\}$ be a G-invariant open covering of M. Let N be a neighborhood of the identity in $\mathcal{D}_G(M)$. Then, for any $f \in \mathcal{D}_G(M)$, there exist $\{f_j \in N| \ 1 \leq j \leq k\}$ such that

(1) *f_j is equivariantly isotopic to the identity through G-diffeomorphisms with the support contained in U_j,*

(2) $f = f_1 \circ \cdots \circ f_k$.

Let $f \in \mathcal{D}_G(V)$. In order to prove $f \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$, by the fragmentation lemma, we can assume f is sufficiently close to the identity. Then we can find $g_1, g_2 \in \mathcal{D}_G(V)$ satisfying

(1) $g_1(x, y) = (x, \hat{g}_1(x)(y))$ with $\hat{g}_1(x) \in \mathcal{D}(\mathbf{R}^q)$,

(2) $g_2(x, y) = (\hat{g}_2(y)(x), y)$ with $\hat{g}_2(2) \in \mathcal{D}_G(W)$ for $x \in W, \ y \in \mathbf{R}^q$,

(3) $f = g_2 \circ g_1$.

By the result of Tsuboi [TS], we see that $g_1 \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$.

In the next we shall prove that $g_2 \in [\mathcal{D}_G(V), \mathcal{D}_G(V)]$. Let $\alpha_{g_2} : \mathbf{R}^q \to Aut_G(W)_0$ be a group homomorphism defined by $\alpha_{g_2}(y) = d\hat{g}_2(y)_0$, where $d\hat{g}_2(y)_0$ is the differential of $\hat{g}_2(y)$ at 0. Then α_{g_2} is a smooth map with compact support $\overline{\{p \in \mathbf{R}^q | \ \alpha_{g_2}(p) \neq e\}}$, where e is the unit element in $Aut_G(W)_0$.

If we take f close to the identity, then α_{g_2} is sufficiently close to the constant map e. Then applying [AF1], Lemma 4, we have

(a) $\exists \varphi_i \in \mathcal{D}(\mathbf{R}^q),\ \ \alpha_i \in C^\infty(\mathbf{R}^q, Aut_G(W)_0)\ (i = 1, ..., r = \dim Aut_G(W)_0)$,

(b) $\alpha_{g_2} = (\alpha_1^{-1} \cdot (\alpha_1 \circ \varphi_1)) \cdots (\alpha_r^{-1} \cdot (\alpha_r \circ \varphi_r))$.

Let $|\cdot|$ be a G-invariant norm of W. Let $\mu:\ W \to [0,1]$ be a G-invariant smooth function satisfying

(i) $\mu(x) = 1$ for $|x| \leq \frac{1}{2}$,

(ii) $\mu(x) = 0$ for $|x| \geq 1$.

Define $h_i, F_i \in \mathcal{D}_G(V)\ (i = 1, .., r)$ by

$$\begin{aligned} h_i(x,y) &= (\mu(x)\alpha_i(y)(x) + (1-\mu(x))x,\ y), \\ F_i(x,y) &= (x,\ \mu(x)\varphi_i(y) + (1-\mu(x))y) \end{aligned}$$

for $x \in W,\ y \in \mathbf{R}^q$.

Lemma 14

$$(h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)(x,y) = ((\alpha_i^{-1} \cdot (\alpha_i \circ \varphi_i))(y)(x),\ y),$$

for $x \in W$, $y \in \mathbf{R}^q$ *with* $|x| \leq \frac{1}{2}$.

Set

$$g_3 = \prod_{i=1}^{r} (h_i^{-1} \circ F_i^{-1} \circ h_i \circ F_i)^{-1} \circ g_2.$$

Then g_3 is written of the form $g_3(x,y) = (\hat{g}_3(x)(y),\ y)$ with $\hat{g}_3(x) \in \mathcal{D}_G(W)$ and $\alpha_{g_3} = e$.

For $0 < c < 1$, let $\psi_c \in \mathcal{D}_G(V)$ such that, for $x \in W,\ y \in \mathbf{R}^q$,

$$\psi_c(x,y) = \begin{cases} (cx,\ y) & (|x| \leq 1), \\ (x,\ y) & (|x| \geq 2). \end{cases}$$

Applying the result of Sternberg [S2], there exists $R \in \mathcal{D}(V)$ such that

(1) R is of the form $R(x,y) = (\hat{R}(y)(x),\ y)$

with $\hat{R}(y) \in \mathcal{D}(W, 0)$ and $\alpha_R = e$.

(2) $R \circ (g_3 \circ \psi_c) \circ R^{-1} = \psi_c$ on a neighborhood U_0 of $\{0\} \times \mathbf{R}^q$.

Set

$$\tilde{R}(x,y) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x, y) \quad \text{for } x \in W,\ y \in \mathbf{R}^q.$$

Then

$$\psi_c \circ \tilde{R} = \tilde{R} \circ g_3 \circ \psi_c \quad \text{on } U_0.$$

Since $\tilde{R}$ is G-equivariant diffeomorphic on a neighborhood of $\{0\} \times \mathbf{R}^q$, we can find $\tilde{R}_1 \in \mathcal{D}_G(V)$ such that $\tilde{R}_1 = \tilde{R}$ on a neighborhood $U \subset U_0$ of $\{0\} \times \mathbf{R}^q$. Put

$$g_4 = g_3 \circ (\tilde{R}_1^{-1} \circ \psi_c \circ \tilde{R}_1 \circ \psi_c^{-1})^{-1}.$$

Then $g_4 = 1$ on U.

There exist a finite point $\{p_i \in V \backslash U | \ 1 \leq i \leq k\}$ and an open disk neighborhood $U(p_i)$ at p_i $(1 \leq i \leq k)$ such that

(1) $U(p_i)$ is a slice at p_i,

(2) $supp(g_4) \subset \bigcup_{i=1}^{k} G \cdot U(p_i)$.

By the fragmentation lemma there exist $h_j \in \mathcal{D}_G(V)$ $(1 \leq j \leq \ell)$ such that

(a) h_j is equivariantly isotopic to the identity through G-diffeomorphisms with the support contained in $G \cdot U(p_j)$,

(b) $g_4 = h_1 \circ \cdots \circ h_\ell$.

Since $U(p_j)$ is a slice at p_j, the isotropy subgroup G_{p_j} acts on $U(p_j)$ and $G \cdot U(p_j)$ is a disjoint union of $|G/G_{pj}|$ disks. Then from the above condition (a)

$$h_j(g \cdot U(p_j)) = g \cdot U(p_j) \quad \text{for } g \in G.$$

We assumed that $\mathcal{D}_H(V)$ is perfect when H is a finite group with $|H| < |G|$ and $\dim V^H > 0$. Therefore each h_j can be written as a commutator in $D_G(V)$ and Theorem 1 (1) follows.

Secondary we prove Theorem 1 (2). Let V be a G-module with $\dim V^G = 0$ Let $\Phi : \mathcal{D}_G(V) \to Aut_G(V)_0$ be a group homomorphism defined by $\Phi(f) = (df)_0$. Since

$$1 \to Ker\Phi \xrightarrow{\iota} D_G(V) \xrightarrow{\Phi} Aut_G(V)_0 \to 1$$

is a short exact sequence, we have the exact sequence.

$$Ker\Phi/[Ker\Phi,\ \mathcal{D}_G(V)] \xrightarrow{\iota_*} H_1(\mathcal{D}_G(V)) \xrightarrow{\Phi_*} H_1(Aut_G(V)_0) \to 1$$

Then Theorem 1 (2) follows from the following.

Proposition 15 $\quad Ker\Phi = [\mathcal{D}_G(V),\ \mathcal{D}_G(V)]$

Proof. Let $f \in Ker\Phi$. For $0 < c < 1$, let $\psi_c \in Aut_G(V)_0$ as before. Applying the result by Sternberg [S2] there exists $R \in \mathcal{D}(V, 0)$ such that $(dR)_0 = 1_V$ and

$R \circ f \circ \psi_c \circ R^{-1} = \psi_c$ on a neighborhood of 0. Set

$$\tilde{R}(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot R(g \cdot x) \quad \text{for } x \in \mathbf{R}^n,$$

where $|G|$ is the order of G. Since $\tilde{R}$ is equivariant diffeomorphism on a neighborhood U of 0, we can find $\hat{R} \in \mathcal{D}_G(V)$ such that $\hat{R} = \tilde{R}$ on an open neighborhood $U_1 \subset U$ of 0. Then

$$f = \hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1} \quad \text{on } U_1.$$

Put

$$g = f \circ (\hat{R}^{-1} \circ \psi_c \circ \hat{R} \circ \psi_c^{-1})^{-1}.$$

Then $g = 1$ on U_1. By the parellel way as in the proof of the case Theorem 1, (1), we can prove that g is written as a commutator in $\mathcal{D}_G(V)$.

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