数理解析研究所講究録 1450 巻 2005 年 98-101

On closure operator in quasi-minimal structures

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Abstract

Itai, Tsuboi and Wakai investigated the quasi-minimal structure [1]. They showed the geometric properties of quasi-minimal structures by using the countable closure. Here we discuss another closure operator in such structures.

1 Quasi-minimal structure and the countable closure

We recall some definitions.

Definition 1 An uncountable structure M is called *quasi−minimal* if every definable subset of M with parameters is at most countable or co-countable.

Let M be an uncountable structure and $A\subset M$. The *n−th countable closure* $\mathrm{ccl}_n^M(A)$ of A is inductively defined as follows:
$\mathrm{ccl}_0^M(A)=A$ and

$$\mathrm{ccl}_{n+1}^M(A)=\bigcup\{\phi^M:\phi(x)\in L(\mathrm{ccl}_n^M(A)),\ \phi^M \text{ is countable}\}$$

We put $\mathrm{ccl}^M(A)=\bigcup_{n\in\omega}\mathrm{ccl}_n^M$ (*the countable closure* of A). We omit the superscript M if it is clear from context.

And we recall the notion of pregeometry.

Definition 2 Let X be an infinite set and cl be a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say that (X,cl) is a *pregeometry*.

(I) $A\subseteq B\Longrightarrow A\subseteq \mathrm{cl}(A)\subseteq \mathrm{cl}(B)$,

(II) (Finite Character) $b\in\mathrm{cl}(A)\Longrightarrow b\in\mathrm{cl}(A_0)$ for some finite $A_0\subseteq A$,

(III) $\mathrm{cl}(\mathrm{cl}(A))=\mathrm{cl}(A)$,

(IV) (Exchange Axiom) $b\in\mathrm{cl}(A\cup\{c\})-\mathrm{cl}(A)\Longrightarrow c\in\mathrm{cl}(A\cup\{b\})$.

The countable closure is a closure operator.

Fact 3 [1] *Let M be an uncountable quasi-minimal structure. Then it is clear that* (M, ccl) *satisfies the first three properties* (I) *through* (III).

The exchange axiom (IV) does not hold in general in (M, ccl). Itai et al. showed some conditions for M such that (M, ccl) satisfies the exchange axiom.

The notion of quasi-minimal structures is a generalization of minimal structures. And the countable closure corresponds to the algebraic closure naturally. Thus the countable closure is the canonical closure operator for quasi-minimal structures. It is easily checked that the countable closure of a set is either a definable set or a model in quasi-minimal structures. For further characterization, I defined some P-closures for them where P is a family of types. I considered that the countable closure is divided by some P-closures.

2 P-closures for quasi-minimal structures

First we recall some definitions from [2].

Definition 4 A family P of partial types is *A-invariant* if it is invariant under A-automorphisms. (where A is a subset of a sufficiently large saturated model as usual.)

Let P be an A-invariant family of partial types.
A partial type q over A is *P-internal* if for every realization a of q , there is $B \downarrow_A a$, types $\bar{p}$ from P based on B, and realization $\bar{c}$ of $\bar{p}$, such that $a \in \mathrm{dcl}(B\bar{c})$.
A partial type q is *P-analysable* if for any $a \models q$, there are $(a_i : i < \alpha) \in \mathrm{dcl}(A,\ a)$ such that $tp(a_i/A,\ a_j : j < i)$ is P-internal for all $i < \alpha$, and $a \in \mathrm{bdd}(A,\ a_i : i < \alpha)$.
A complete type $q \in S(A)$ is *foreign* to P if for all $a \models q$, $B \downarrow_A a$, and realizations $\bar{c}$ of extensions of types in P over B, we always have $a \downarrow_{AB} \bar{c}$.

And let P be an $\emptyset$-invariant family of types.
A partial type q is *co-foreign* to P if every type in P is foreign to q.
The *P-closure* $\mathrm{cl}_P(A)$ of a set A is the collection of all element a such that $\mathrm{tp}(a/A)$ is P-analysable and co-foreign to P.

Remark 5 The P-analysable assumption could be modified or even omitted, resulting in a larger P-closure.

Fact 6 [2] *P-closure is a closure operator, i.e. it satisfies the axioms* (I) *and* (III) *in* Definition 2.

We recall another notion from [1] to define the family P of types in quasi-minimal structures.

Definition 7 Let M be quasi-minimal. Then a type $p(x)$ defined by
$$p(x) = \{\psi(x) \in L(M) \ : \ |\ \psi^M \ | \geqq \omega_1\}$$
is a complete type in $S(M)$. The type $p(x)$ is called the *main type* of M.

The family P of types should be defined such that the P-closure is included in the countable closure. So P is either the family of the restrictions of the main type or that of formulas which have uncountably many realizations in a quasi-minimal structure.

The notion of the main type is not elementary. It makes sense in a fixed quasi-minimal structure. Thus the P-closure must be defined in a fixed such structure. Otherwise it should be defined as the intersection between a fixed quasi-minimal model and the P-closure defined in the big model. We tried to define it in a fixed quasi-minimal structure at first. Thus we define the notion of foreignness and co-foreignness suitable for such P-closure.
For example,

Definition 8 Let M be a structure and $Th(M)$ be simple. And let P be a family of types over M.
An element a is *co-foreign* to P over A if for any $b \models p$ over B in P with $A \subseteq B$, and any $C \perp_B b$, $b \perp_{BC} a$ where all parameters are contained in M.
And we define $\mathrm{cl}^0_P(A) = \{a \in M \ : \ a$ is co-foreign to P over $A.\}$

It is easily checked that the next facts hold.

Fact 9 *Let M be a quasi-minimal structure and $Th(M)$ be simple. And let P be the set of the restrictions of the main type closed under taking nonforking extensions. Then cl^0_P is a closure operator, i.e. (M, cl^0_P) satisfies the axioms* (I) *and* (III) *in* Definition 2.

Fact 10 *Under the same assumptions as the former fact. Then* $\mathrm{acl}(A) \subseteq \mathrm{cl}^0_P(A) \subseteq \mathrm{ccl}(A)$.

We need some modification of the definition so that cl^0_P satisfies the axiom (II) in Definition 2.

Next we must consider the definition of P-internality in quasi-minimal structures. We can define it in a fixed quasi-minimal structure as above.

But I can not fix definitions to realize the relation between the internality and the foreignness defined as above. It claims that the family P of types has some invariability under automorphisms.

And I can not construct quasi-minimal structures which have a proper P-closure yet.

References

[1] M.Itai, A.Tsuboi, and K.Wakai, Construction of saturated quasi-minimal structure, to appear
[2] F.O.Wagner, Simple theories, Kluwer Academic Publishers, 2000
[3] F.O.Wagner, Stable groups, Cambridge University Press, 1997
[4] A.Pillay, Geometric stability theory, Oxford Science Publications, 1996