数理解析研究所講究録 1450 巻 2005 年 69-74

Generic構造の安定性についてII (The Stability Spectrum of Generic Graphs)

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Abstract

Generic 構造でその理論が superstable であるが ω-stable でないものがあるか，という問題がある (Baldwin の問題)．この問題に関して次の部分的結果が得られた．定理：**K** が部分グラフに関して閉じているとき，**K**-generic グラフの理論は strictly stable かあるいは ω-stable になる．

1 Introduction

Our notation and definition are similar to those of [8] in this volume. We do not explain all of those details here. For further details, see [2, 3, 4, 5, 6, 9]. Our aim is to give a partial solution for the following problem.

Problem 1(Baldwin [1]) Is there a generic structure that is superstable but not ω-stable ?

This problem is still open. In fact, all of known stable generic structures is strictly stable or ω-stable. In this paper, we will consider the problem under the following assumption.

Assumption 2 $\mathbf{K}$ is a subclass of $\mathbf{K}_\alpha$ that is closed under substructures. M is a saturated $\mathbf{K}$-generic graph. (It follows that $\mathbf{K}$ has the amalgamation property.)

For the stability of $\mathbf{K}$-generic graphs, the following fact is well-known.

*Research partially supported by Grants-in-Aid for Scientific Research (no.16540123), Ministry of Education, Science and Culture.

Fact 3 ([3], [9]) Let $T = \mathrm{Th}(M)$. Then
(i) T is stable;
(ii) If α is rational, then T is ω-stable.

2 Triviality of K

For each $n \in \omega$, I_n denotes a graph of size n with no relations.

Lemma 4 $I_n \in \mathbf{K}$ for each $n \in \omega$.

Proof If $m \leq n$ and $I_n \in \mathbf{K}$, then $I_m \in \mathbf{K}$ since $\mathbf{K}$ is closed under substructures. So we can assume $n \geq 3$. Then we can take $k \in \omega$ such that $\left\{k \cdot \binom{k-2}{n-2}\right\} / \binom{k}{n} < \alpha$. Take any $A \in \mathbf{K}$ of size k. Then it is enough to see that A contains a copy of I_n. This can be shown as follows: If not, then $r(A) \leq \binom{k}{n} / \binom{k-2}{n-2}$. So we have $\delta(A) \leq k - \left\{\binom{k}{n} / \binom{k-2}{n-2}\right\} \alpha < 0$. This is a contradiction.

Lemma 5 If Ab is a finite graph with $A \in \mathbf{K}$ and $r(A, b) = 0$, then $Ab \in \mathbf{K}$.

Proof Let $|A| = k$. Take $n \in \omega$ with $n > k/\alpha$. By lemma 4, we have $I_n \in \mathbf{K}$. So we can assume that $I_n A (\in \mathbf{K})$ is an amalgamation of I_n and A over $\emptyset$. It is enough to show that there is $b' \in I_n - A$ with $b' \cong_A b$. This can be shown as follows: If $A \subset I_n$, then we easily get $b' \in I_n - A$ with $b' \cong_A b$. So we can assume $A \not\subset I_n$. If not, then $r(A - I_n, I_n) \geq n$. Then $\delta(A/I_n) = \delta(A - I_n) - \alpha \cdot r(A - I_n, I_n) \leq k - \alpha \cdot n < 0$. A contradiction.

Definition 6 $\mathbf{K}$ is said to be *trivial*, if there is some $n \in \omega$ such that if $A \in \mathbf{K}$ is connected then $|A| \leq n$.

Lemma 7 If $\mathbf{K}$ is trivial, then $\mathrm{Th}(M)$ is ω-stable.

Proof Take any countable $A \leq M$ and $\bar{b} \in M$. To show that $\mathrm{Th}(M)$ is ω-stable, it is enough to see that $S(A)$ is countable. Since $\mathbf{K}$ is trivial, there is finite $A_0 \leq A$ with $\mathrm{d}(\bar{b}/A) = \mathrm{d}(\bar{b}/A_0)$. Let $B = \mathrm{cl}(\bar{b}A_0)$ and $A_1 = B \cap A$. Note that $A_1 \leq M$.
Claim: $\mathrm{tp}(\bar{b}/A_1) \vdash \mathrm{tp}(\bar{b}/A)$.
Proof: Take any $\bar{c} \in M$ such that $\mathrm{tp}(\bar{c}/A_1) = \mathrm{tp}(\bar{b}/A_1)$ and $\mathrm{d}(\bar{c}/A_1) = \mathrm{d}(\bar{c}/A)$. Let $C = \mathrm{cl}(\bar{c}A_1)$. Then we have $B \cong_{A_1} C$. From proposition 13 it follows that B and A are free over A_1 and $BA \leq M$, and that C and A are free over A_1 and

$CA \leq M$. In particular we have $B \cong_A C$, and so $\mathrm{tp}(B/A) = \mathrm{tp}(C/A)$. Hence $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$. (End of Proof of Claim)

Since there is a countable saturated model, $\mathrm{Th}(M)$ is small, and hence the number of the types over a given finite set is countable. It follows that $|S(A)| \leq \aleph_0 \cdot \aleph_0 = \aleph_0$.

For a finite graph A and $e \in A$, we denote $\deg_A(e) = \max\{|B| : \forall b \in B, R(e,b)\}$.

Lemma 8 Suppose that $\mathbf{K}$ is non-trivial. Then for any $n \in \omega$ the following condition $(*)_n$ holds:
$(*)_n$ There is $A \in \mathbf{K}$ and $a \in A$ with $\deg_A(a) \geq n$.

Proof For each $m \in \omega$, let L_m denote a finite graph $a_0a_1...a_m$ with the relations $R(a_0, a_1), R(a_1, a_2), ..., R(a_{m-1}, a_m)$. We divide into two cases.
Case 1: $L_m \notin \mathbf{K}$ for some $m \in \omega$.

Since $\mathbf{K}$ is non-trivial, $(*)_n$ clearly holds for each $n \in \omega$. (mousukosi seikaku ni!)

Case 2: $L_m \in \mathbf{K}$ for any $m \in \omega$.

We prove by induction. By induction hypothesis, we assume that $\deg_A(a) \geq n$ for some $A \in \mathbf{K}$.

Subcase 2.1: $\alpha \leq \frac{1}{2}$.

Let acd be a graph with the relations $R(a,c)$ and $R(c,d)$. Since $L_2 \in \mathbf{K}$ and $\alpha < \frac{1}{2}$, we have $ad \leq acd \in \mathbf{K}$. On the other hand, we can assume $r(A,d) = 0$. By lemma 5, we have $ad \leq Ad \in \mathbf{K}$. So we can assume that $cdA (\in \mathbf{K})$ is an amalgamation of acd and Ad over ad. Note that $c \notin A$ since $r(A,c) = 0$. Hence $\deg_{Ac}(a) \geq n+1$.

Subcase 2.2: $\alpha > \frac{1}{2}$.

Let $m = \min\{k : (k-1) - k\alpha > 0\}$. Note that $m \geq 3$ since $\alpha > \frac{1}{2}$. Then we have $L_m = a_0a_1...a_m \in \mathbf{K}$. We can assume $a = a_0$ and $r(A, a_m) = 0$. By lemma 5, we have $aa_m \leq Aa_m \in \mathbf{K}$. By the definition of m, we have $aa_m \leq L_m \in \mathbf{K}$. So we can assume that $AL_m (\in \mathbf{K})$ is an amalgamation of Aa_m and L_m over aa_m. Then we see $a_1 \notin A$. (Proof: If not, then we have $aa_1a_m \leq L_m$, and so $\delta(L_m/aa_1a_m) = (m-2) - (m-1)\alpha > 0$. This contradicts the definition of m.) Therefore $\deg_{Aa_1}(a) \geq n+1$.

For each $n \in \omega$, S_n denote a finite graph $aa_1a_2...a_n$ with the relations $R(a, a_i)$ for every $i = 1, 2, ..., n$.

Lemma 9 Suppose that $\mathbf{K}$ is non-trivial. Then $S_n \in \mathbf{K}$ for each $n \in \omega$.

Proof Take sufficiently large k. By lemma, there is $A \in \mathbf{K}$ with $\deg_A(a) \geq k$ for some $a \in A$. We can assume that $R(a,b)$ for any $b \in A - a$. Then it is enough to show that there is $I_n \subset A - \{a\}$. (If not, then $r(A-a) \geq \binom{k}{n} / \binom{k-2}{n-2}$, so we have $\delta(A) = \delta(A-a) + \delta(a/A-a) = \delta(A-a) + 1 - k\alpha \leq \left\{\binom{k}{n} / \binom{k-2}{n-2}\right\}\alpha + 1 - k\alpha < 0$. A contradiction.

Lemma 10 Let $\mathbf{K}$ be non-trivial. Suppose that Ab is a finite graph such that there is $a \in A$ with $a \leq A \in \mathbf{K}$ and $r(A,b) = r(a,b) = 1$. Then we have $Ab \in \mathbf{K}$.

Proof Let $|A| = k$. Take $n \in \omega$ with $n > k(1 + 1/\alpha)$. By lemma 9, we have $S_n \in \mathbf{K}$. We assume that $a \leq S_n$, and that $R(a,c)$ for any $c \in S_n - \{a\}$. By our assumption, we have $a \leq A \in \mathbf{K}$. So we can assume that $AS_n (\in \mathbf{K})$ is an amalgamation of A and S_n over a. To show our lemma, it is enough to see that there is $b' \in S_n - A$ with $b' \cong_A b$. This can be shown as follows: If $A \subset S_n$, then we can easily pick b' as required. If $A \not\subset S_n$, we can take $b' \in S_n - A$ with $r(b', A) = r(b', a) = 1$. (If not, then we have $\delta(A/S_n) = \delta(A - S_n) - r(A - S_n, S_n) \cdot \alpha \leq k - (n-k)\alpha < 0$. On the other hand, we have $S_n \leq AS_n$ since AS_n is an amalgamation. A contradiction.)

Proposition 11 Suppose that $\mathbf{K}$ is non-trivial. Then any finite graph with no cycles belongs to $\mathbf{K}$.

Proof Let B be a finite graph with no cycles. We will prove by induction on $|B|$. Since B has no cycles, we can take $b \in B$ such that there are no distinct $c, d \in B$ with $R(c,b) \wedge R(d,b)$. Let $A = B - \{b\}$. By induction hypothesis, we have $A \in \mathbf{K}$. If $r(b,A) = 0$, then we have $B = Ab \in \mathbf{K}$ by lemma 10. Therefore we assume $r(b,A) = 1$. Since A has no cycles, we have $a \leq A (\in \mathbf{K})$. Take $n \in \omega$ with $n > |A|/\alpha$. By lemma 9, $S_n \in \mathbf{K}$. We can assume that $a \in S_n$ with $R(a,c)$ for any $c \in S_n - \{a\}$. Hence we have $a \leq S_n \in \mathbf{K}$.

3 Proof of Theorem

The following proposition was proved in [7] to show that there is no $\mathbf{K}$-generic pseudoplane that is stable but not ω-stable.

Proposition 12 ([7]) If α is irrational, then there is an infinite graph D with an element e and finite subgraphs $B_1, B_2, \cdots$ with the following properties:
(1) $D = \mathrm{cl}_D(eB_1B_2\cdots)$ has no cycles;
(2) $\mathrm{d}_D(e/B_1) > \mathrm{d}_D(e/B_2) > \cdots$;
(3) $B_1 \leq B_2 \leq \cdots \leq D$.

In [8], we studied algebraic types of **K**-generic graphs, where **K** is closed under subgraphs. As a corollary, we have the following proposition.

Proposition 13([8]) Assume that **K** is closed under subgraphs. Let A, B, C be finite such that $B, C \leq M$ and $A = B \cap C$. Then the following are equivalent.
(i) $\mathrm{d}(B/A) = \mathrm{d}(B/C)$;
(ii) B and C are free over A, and $BC \leq M$;
(iii) $\mathrm{tp}(B/C)$ does not fork over A.

Using proposition 12 and 13, we obtain the following theorem.

Theorem Let **K** be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraph and M a saturated **K**-generic graph. Then $\mathrm{Th}(M)$ is strictly stable or ω-stable.

Proof of Theorem If **K** is trivial or α is rational, then, by fact 3 and lemma 7, $\mathrm{Th}(M)$ is ω-stable. Thus we can assume that **K** is non-trivial and α is irrational. By fact 3 again, $\mathrm{Th}(M)$ is stable. So we have to show that $\mathrm{Th}(M)$ is not superstable: Since α is irrational, by proposition 12, there is an infinite graph D with an element e and finite subgraphs $B_1, B_2, \cdots$ such that (i) $D = \mathrm{cl}_D(eB_1B_2\cdots)$ has no cycles; (ii) $\mathrm{d}_D(e/B_1) > \mathrm{d}_D(e/B_2) > \cdots$; (iii) $B_1 \leq B_2 \leq \cdots \leq D$. Since **K** is non-trivial and D has no cycles, by proposition 11, any finite subset of D belongs to **K**, and so we can assume $D \leq M$. It follows that $\mathrm{d}_M(e/B_1) > \mathrm{d}_M(e/B_2) > \cdots$. By proposition 13, we have $\mathrm{tp}(e/B_1) \subset_f \mathrm{tp}(e/B_2) \subset_f \cdots$. Hence $\mathrm{Th}(M)$ is not superstable.

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