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Special classes of algebraic integers in low-dimensional topology

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Abstract

This note describes some open problems concerning distributions of special classes of real algebraic integers such as algebraic units, and Salem, P-V and Perron numbers. These special algebraic integers appear naturally as geometric invariants in low-dimensional topology. We relate properties of Salem, P-V and Perron to minimization problems in various geometric settings.

1 Introduction

A complex number $\alpha \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic integer polynomial. Two algebraic integers α and β are *algebraically conjugate*, written $\alpha \sim \beta$, if α and β satisfy the same irreducible monic integer polynomial. An algebraic integer α is an *algebraic unit* if $\alpha \sim \alpha^{-1}$.

Let α be a real algebraic integer with $\alpha > 1$. Consider all $\beta \sim \alpha$ such that $\beta \neq \alpha$:

(i) if $|\beta| < |\alpha|$, then α is a *Perron number*;

(ii) if $|\beta| < 1$, then α is a *P-V number*, and

(iii) if $|\beta| \leq 1$ with at least one $|\beta| = 1$, then α is a *Salem number*.

In this short note, we review definitions and known results concerning distributions of P-V, Salem and Perron numbers (Section 2), and relate them to geometric invariants in low-dimensional topology, including lengths of geodesics, growth rates of automatic groups, and homological and geometric dilatations of surface homeomorphisms (Section 3).

2 Distributions of algebraic integers and Lehmer's problem

Let $\mathcal{P}$ be the set of monic integer polynomials. Given $f \in \mathcal{P}$, let $\mathcal{S}_f$ be the set of complex roots of f counted with multiplicity, and let $\mathcal{S}^+(f) \subset \mathcal{S}_f$ be the subset of points outside the unit circle C. For $f \in \mathcal{P}$ define

$$\begin{aligned} N(f) &= \left|\mathcal{S}^+(f)\right|; \\ \lambda(f) &= \max\{|\alpha| : \alpha \in \mathcal{S}(f)\}; \text{ and} \\ M(f) &= \prod_{\alpha \in \mathcal{S}^+(f)} |\alpha|. \end{aligned}$$

Here an empty product is defined to equal 1. The number $M(f)$, also an algebraic integer, is called the *Mahler measure* of f.

The minimal polynomial for a root of unity is called a *cyclotomic polynomial*. The following are equivalent for $f \in \mathcal{P}$:

(i) f is a product of cyclotomic polynomials;

(ii) $N(f) = 0$;

(iii) $\lambda(f) = 1$; and

(iv) $M(f) = 1$.

Thus, $N(f)$, $\lambda(f)$ and $M(f)$ can be considered as measures of how far f is from being a product of cyclotomic polynomials. Let $\mathcal{T} \subset \mathcal{P}$ be the subset of products of cyclotomic polynomials.

While $N(f)$ takes discrete values and $\lambda(f)$ can get arbitrarily close to one from above it is not known whether there is a lower bound for Mahler measures greater than one. In 1933 Lehmer [Leh] posed the following problem.

Question 1 (Lehmer's problem) *Given $\delta > 0$, does there exist a $f \in \mathcal{P}$ such that $1 < M(f) < 1 + \delta$?*

It is not hard to see that for $f \in \mathcal{P} \setminus \mathcal{T}$, if we fix the degree d of f, then $\lambda(f)$ and $M(f)$ are bounded from below by a number greater than one depending on d.

Up to degree 40 there is no non-cyclotomic polynomial with Mahler measure less than that of Lehmer's candidate polynomial

$$f_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

(see, for example, [Boyd1] [Mos]). The Mahler measure $M(f_L)$ is approximately 1.7628.

By a result of Smyth in 1970, Lehmer's problem reduces to the case of reciprocal polynomials, which we describe in Section 2.1. Section 2.2 gives some known results concerning distributions of Perron, Salem and P-V numbers.

2.1 Reciprocal polynomials

Given $f \in \mathcal{P}$ of degree d, the *reciprocal* $f_*(x)$ of $f(x)$ is defined to be

$$f_*(x) = x^d f(1/x).$$

A polynomial is *reciprocal* if $f = f_*$. Visually, a reciprocal polynomial is one for which the coefficients are palindromic, that is, they are the same written from right to left or left to right. Lehmer's polynomial f_L is a reciprocal polynomial.

If f satisfies $f = -f_*$, it is called *anti-reciprocal.* A polynomial f is anti-reciprocal if and only if $f(x) = (x-1)g(x)$ where $g(x)$ is reciprocal. All cyclotomic polynomials are reciprocal except $(x-1)$. A polynomial is reciprocal or anti-reciprocal if and only if it is a product of irreducible reciprocal polynomials and $(x-1)$. A separable polynomial is reciprocal or anti-reciprocal if and only if $\mathcal{S}(f)$ is closed under inverses. Thus, an algebraic integer α is an algebraic unit if and only if its minimal polynomial is reciprocal. An irreducible polynomial with a root on the unit circle is automatically reciprocal. Thus, minimal polynomials of Salem numbers are always reciprocal, and the minimal polynomial of a P-V number is reciprocal only if it is quadratic.

Smyth showed [Smy] that if $f \neq \pm f_*$, then the smallest Mahler measure is realized by

$$f_S(x) = x^3 - x - 1,$$

which has Mahler measure $M(f_S) \approx 1.32472$. Since Lehmer's polynomial f_L satisfies

$$M(f_L) < M(f_S),$$

it follows that to solve Lehmer's problem it is enough to look at reciprocal polynomials.

2.2 P-V and Salem polynomials

An interesting special case of Lehmer's problem is when $N(f) = 1$. The following are equivalent:

(i) $N(f) = 1$;

(ii) f has a single root outside C, which is (up to sign) a Salem number or a P-V number; and

(iii) $f = gh$ where $g \in \mathcal{T}$ and h is the minimal polynomial of a Salem number or a P-V number.

For quadratic polynomials $N(f) = 1$ implies that both roots are real. The reciprocal case is discussed in Section 3.1. For irreducible polynomials of degree > 2, $N(f) = 1$ implies f has a Salem root (up to sign) if and only if f is reciprocal or anti-reciprocal.

The set of P-V numbers is closed [Sal], and the smallest accumulation point is the golden mean α_G (cf. Section 3.1). A complete set of P-V numbers less than 1.6 was catalogued by Dufresnoy and Pisot [DP].

The polynomial $f_S(x) = x^3 - x - 1$ is the minimal polynomial for the smallest P-V number θ_0 [Sie], and f_L is the minimal polynomial for the smallest known Salem number α_L. It is an open problem whether there is a lower bound larger than one for the set of Salem numbers, or whether there is a Salem number less than f_L.

In their study of distributions of Salem numbers, Salem [Sal] and Boyd [Boyd2] investigated sequences polynomials of the form

$$Q_n(t) = t^n P(t) \pm P_*(t), \tag{1}$$

for $P \in \mathcal{P}$. The sequence of polynomials of the form given in (1) is called a *Salem-Boyd sequence* for P. Salem [Sal] proved that the set of P-V numbers lies in the set of upper and lower limits of Salem numbers by proving the following result.

Theorem 1 *Given any P-V polynomial P, let Q_n be a Salem-Boyd sequence for P. Then for some $N > 0$, $N(Q_n) = 0$ for $n < N$, and $N(Q_n) = 1$ for $n \geq N$. Furthermore, for $n > N$, the Salem numbers $M(Q_n) = \lambda(Q_n)$ converge monotonically to $M(P) = \lambda(P)$ from above or below depending on the sign.*

In the more general setting where $P \in \mathcal{P}$ is any element, Boyd showed the following [Boyd2].

Theorem 2 *Let Q_n be a Salem-Boyd sequence for a monic integer polynomial $P(t)$. Then we have the following.*

(1) $N(Q_n) \leq N(P)$ *for all* $n \geq 1$;

(2) $\lim_{n\to\infty} \lambda(Q_n) = \lambda(P)$; *and*

(3) $\lim_{n\to\infty} M(Q_n) = M(P)$.

Any reciprocal polynomial can be written in the form of Q_n for some P and n. Thus, although Theorem 2 doesn't give a lower bound on $M(Q_n)$ or $\lambda(Q_n)$ in terms of $M(P)$ and $\lambda(P)$, it does partition the set of Mahler measures and radii of reciprocal polynomials into (non-disjoint) convergent families.

A polynomial f is a *Perron polynomial* if there is a simple real root $\alpha > 1$, such that for any other root β of f, $|\beta| < \alpha$. Lehmer's problem is unsolved for this special subclass of monic integer polynomials. By definition, the characteristic polynomial of a Perron-Frobenius matrix is Perron [Gan].

Theorem 1 generalizes to Perron polynomials as follows [Hir4].

Theorem 3 *If P is a Perron polynomial and Q_n is a Salem-Boyd sequence for P. Then $\lambda(Q_n)$ is an eventually monotone sequence converging to $\lambda(P)$.*

In general, $M(Q_n)$ is not monotone, eventually monotone, or monotone for an arithmetic subsequence.

3 Examples from low-dimensional topology and geometry

In this section, we list some occurrences of reciprocal, Salem, P-V and Perron polynomials in low dimensional topology. These examples indicate a common underlying structure behind many of the invariants of low dimensional topology, which is yet to be fully explored.

3.1 Quadratic polynomials

There is a bijective correspondence between $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and reciprocal quadratic polynomials. This is defined by

$$A \mapsto f_A,$$

where f_A is the characteristic polynomial for $A \in \Gamma$. The characteristic polynomial f_A of any element $A \in \Gamma$ is reciprocal, since the two eigenvalues of A must multiply to 1.

The inverse map is defined as follows. Let λ be a quadratic such that $a = \lambda + 1/\lambda \in \mathbb{Z}$, and define

$$A = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.$$

Then λ and λ^{-1} are the roots of f_A.

The correspondence

$$\mathrm{Trace}(A) \mapsto \lambda(f_A) \tag{2}$$

is order preserving, and $\lambda(f_A) = 1$ if and only if $|\mathrm{Trace}(A)| \leq 2$. Thus, non-cyclotomic reciprocal quadratics correspond to hyperbolic elements of Γ.

Consider the action of Γ as isometries on the hyperbolic disk $\mathbb{H}^2$. Then hyperbolic elements $A \in \Gamma$ correspond to closed geodesics γ_A on the quotient space $\Gamma \backslash \mathbb{H}^2$, whose length $\ell(\gamma_A)$ is given by

$$\ell(\gamma_A) = \log(\mathrm{Trace}(A)).$$

The correspondence given in (2) also gives rise to an ordering preserving correspondence between $\lambda(f_A)$ and lengths of closed geodesics on $\Gamma \backslash \mathbb{H}^2$.

If follows that for quadratic reciprocal polynomials f, the smallest Mahler measure greater than one that occurs is realized by

$$f_0(x) = x^2 - 3x + 1$$

and $M(f_0) = \lambda(f_0) = (3 + \sqrt{5})/2$.

Remark 4 *There is a similar correspondence between Salem numbers and lengths of closed geodesics on more general arithmetic quotients of $\mathbb{H}^2$ (see, for example, [G-H]). Thus, the minimization problem for Salem numbers is related to the problem of finding a minimum length geodesic on an arithmetic quotient of the hyperbolic plane.*

The smallest Mahler measure greater than one among all quadratics is realized by the non-reciprocal polynomial

$$f_g(x) = x^2 - x - 1.$$

Here, $M(f_g) = (1 + \sqrt{5})/2$ is the golden mean. The smallest Mahler measures greater than one for reciprocal polynomials of degrees 4 and 6 are approximately 1.72208 and 1.40127, respectively, and hence are also larger than the smallest non-reciprocal Mahler measure $M(f_S)$. For degrees 8 and higher there always exists a reciprocal polynomial (not necessarily irreducible) with Mahler measure greater than one and less than $M(f_S)$.

3.2 Transformations preserving lattices

Let B be a non-degenerate symmetric bilinear form on $\mathbb{R}^n$, and suppose $M \in \mathrm{SL}(n, \mathbb{Z})$ preserves B. Equivalently, M preserves the lattice in $\mathbb{R}^n$ defined by the inner product associated to B. Then the set of eigenvalues of M is closed under inverses. Thus, if for example M has no repeated eigenvalues, or equivalently the characteristic polynomial f_M is separable, then f_M is reciprocal or anti-reciprocal (see, for example, [G-Mc], Theorem 2.1).

Consider the Coxeter element of a Coxeter system (see [Hum] for definitions). The Coxeter element preserves an associated symmetric bilinear form defined by the Coxeter system. If (W, S) is an irreducible Coxeter system, and $f_{(W,S)}$ is the characteristic polyonomial of the Coxeter element of (W, S), then $\lambda(f_{(W,S)}) = 1$ if and only if (W, S) is spherical or affine [A'C]. If (W, S) is irreducible and not spherical or affine, then $\lambda(f_{(W,S)})$ is minimized by the E_{10} Coxeter system (W_0, S_0), and $f_{(W_0,S_0)} = f_L$ is Lehmer's polynomial ([Mc] Theorem 6.1). It follows that for any Coxeter system (W, S), either $f_{(W,S)} \in \mathcal{T}$ or $M(f_{(W,S)}) \geq M(f_{(W_0,S_0)})$, which solves Lehmer's problem for this class of examples.

3.3 M-matrices

Let $T \in \mathrm{SL}(n, \mathbb{Z})$ be a matrix satisfying

$$T = \pm M^{\mathrm{tr}} M^{-1}, \tag{3}$$

where $M \in \mathrm{SL}(\mathrm{n}, \mathbb{Z})$, and M^{tr} is the transpose of M. Then T^{-1} is conjugate to $\pm T^{\mathrm{tr}}$ and hence T has reciprocal or anti-reciprocal characteristic polynomial. Such matrices are called M-matrices.

Howlett [How] showed that if T is the Coxeter element of a simply-laced Coxeter system associated to a graph Γ, then the Coxeter element can be written in terms of the adjacency matrix A for Γ. Let A^+ be its upper triangular part. Then setting $M = I - A^+$, we have $T = -M^{\mathrm{tr}} M^{-1}$. This gives another proof that the characteristic polynomial of a Coxeter element is reciprocal in the simply-laced case.

Another well-known case is the Alexander matrix of a knot (S^3, K) (see [Rolf] for definitions). Let V be the Seifert matrix for (S^3, K). Then the *Alexander polynomial* for (S^3, K) is given by $\Delta_{(S^3,K)}(t) = |\det(tV - V^{\mathrm{tr}})|$, up to multiples of $t^{\pm 1}$. If V is invertible, it follows that the characteristic polynomial $\Delta_{(S^3,K)}$ is reciprocal and has a monic representative. Furthermore, $\Delta_{(S^3,K)}(1) = \pm 1$. Conversely, if f is a reciprocal polynomial with $f(1) = \pm 1$, then there is a knot (S^3, K), such that $\Delta_{(S^3,K)} = f$ [Seif].

3.4 Growth rates of automatic groups

Let G be a finitely presented group, with generating set S, such that S is closed under inverses. The *growth series* of G is the formal sum

$$\Psi_{(G,S)}(x) = \sum_{i=0}^{\infty} a_n x^n$$

where a_n is the number of words in G of minimal word length n in the generating set S. The *growth rate* of G is given by

$$\lambda(G,S) = \limsup |a_n|^{\frac{1}{n}}.$$

(See [ECH+] for more details.) The growth series $\Phi_{(G,S)}$ is rational, for example, when G is hyperbolic, automatic, or a Coxeter group (see for example [ECH+], [Can], for more details). In the automatic case, we can realize $\lambda(G,S)$ as $\lambda(f_{(G,S)})$ where $f_{(G,S)}$ is the characteristic polynomial of an associated matrix.

Lehmer's polynomial appears among these examples as follows. Let $(G_{p_1,\ldots,p_k}, S_{p_1,\ldots,p_k})$ be the Coxeter group of reflections through sides of a polygon in the hyperbolic plane with angles $\frac{\pi}{p_1},\ldots,\frac{\pi}{p_k}$, where

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} < k-2,$$

Then Cannon and others [F-P], [C-W], [Floy] calculate the denominators of the growth series, and show that $\lambda(G_{p_1,\ldots,p_k}, S_{p_1,\ldots,p_k})$ is a Salem number. In particular, Lehmer's polynomal f_L occurs as the denominator for $(G_{2,3,7}, S_{2,3,7})$ and corresponds to the angle set giving rise to the smallest area hyperbolic polygon.

There is a close relation between the automatic group structure of $(G_{p_1,\ldots,p_k}, S_{p_1,\ldots,p_k})$ and the Coxeter element of $(\mathcal{G}_{p_1,\ldots,p_k}, \mathcal{S}_{p_1,\ldots,p_k})$, where $(\mathcal{G}_{p_1,\ldots,p_k}, \mathcal{S}_{p_1,\ldots,p_k})$ is the simply-laced Coxeter system associated to the "star-like" graph with k-branches emanating from a central vertex of lengths $p_1,\ldots,p_k$. (See, for example, [Hir2]. Although the star-like graphs do not directly define the auotomatic structures of $(G_{p_1,\ldots,p_k}, S_{p_1,\ldots,p_k})$, calculations in [Hir1] show that the sequence a_n for the growth series $\Psi_{(G_{p_1,\ldots,p_k}, S_{p_1,\ldots,p_k})}$ can be computed directly from the Coxeter element $(\mathcal{G}_{p_1,\ldots,p_k}, \mathcal{S}_{p_1,\ldots,p_k})$.

3.5 Dilatations of pseudo-Anosov maps

Let F be a compact orientable surface with negative Euler characteristic, and let $\phi : F \to F$ be a homeomorphism. The Thurston-Nielsen theory [Thu] [FLP] [CB] states that for any surface homeomorphism $\phi : F \to F$, ϕ is isotopic to some Φ satisfying one of the following:

(i) Φ is *periodic*, i.e., Φ^n is the identity;

(ii) Φ is *irreducible*, i.e., there is a closed curve on F invariant under Φ such that complementary components have negative Euler characteristic; or

(iii) Φ is *pseudo-Anosov*, i.e., there is a number $\lambda > 1$ and a pair $\mathcal{F}^{\pm}$ of transverse measured foliations such that

$$\Phi(\mathcal{F}^{\pm}) = \lambda^{\pm 1}\mathcal{F}^{\pm}.$$

In the pseudo-Anosov case, Φ is the unique element in the isotopy class of ϕ with smallest topological entropy [FLP]. The number $\lambda(\phi) = \lambda(\Phi)$ is called the *dilatation* of ϕ.

Suppose Φ is pseudo-Anosov. Then there is an embedded graph $\mathcal{G}$ in F representing the spine of F, such that the transition matrix M_Φ for Φ restricted to $\mathcal{G}$ is a Perron-Frobenius matrix. Thus, the characteristic polynomial f_ϕ of M_Φ is a Perron polynomial. Furthermore, the Perron number associated to f_ϕ equals the dilatation $\lambda(\phi)$. The transition matrix M_Φ has the property that M_Φ^{tr} is conjugate to M_Φ^{-1} and hence its characteristic polynomial f_ϕ is reciprocal.

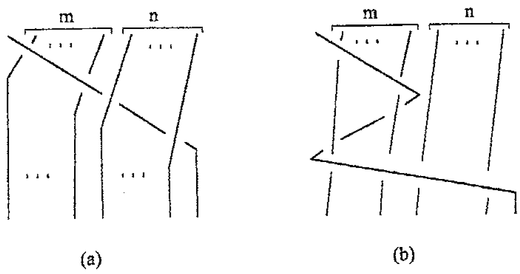

Figure 1: Braids with small dilatation.

In [HK] we study two families of pseudo-Anosov maps on a marked disk associated to the braids $\beta_{m,n}$ (a) and $\sigma_{m,n}$ (b) drawn in Figure 1, and show that their dilatations satisfy the characteristic equations

$$x^{n+1}R_m(x) + (R_m)_*(x)$$

for $\beta_{m,n}$ and, when $n \geq m+2$,

$$x^{n+1}R_m(x) - (R_m)_*(x)$$

for $\sigma_{m,n}$, where

$$R_m(x) = x^m(x-1) - 2.$$

The polynomials R_m have m roots outside the unit circle. Thus, $R_1(x)$ is a P-V polynomial, and hence by Theorem 1 the dilatations of $\beta_{1,n}$ and $\sigma_{1,n}$ ($n \geq 3$) are Salem numbers and are monotone (decreasing for $\beta_{1,n}$ and increasing for $\sigma_{1,n}$). For all m, the polynomials R_m are Perron polynomials, and hence by Theorem 3 the dilatations are eventually monotone (again, decreasing for $\beta_{m,n}$ and increasing for $\sigma_{m,n}$). Since $\lambda(R_m)$ approaches 1 as m goes to infinity, it follows that for any $\epsilon > 0$, it is possible to make m and n large enough so that the dilatations are within ϵ of 1.

The pseudo-Anosov maps defined by $\beta_{m,n}$ and $\sigma_{m,n}$ lift to homeomorphisms of genus g compact surfaces with b boundary components via double covering, where, if $m+n$ is even, $g = \frac{m+n}{2}$ and $b = 1$, and if $m+n$ is odd, $g = \frac{m+n-1}{2}$ and $b = 2$. Let ϕ_g be the lift of $\sigma_{g-1,g+1}$. Then

$$\log(\lambda(\phi_g)) = \log(\lambda(\sigma_{g-1,g+1})) \asymp \frac{1}{g},$$

and we recover Penner's result on least dilatations of pseudo-Anosov maps on orientable genus g surfaces. The lifts of $\beta_{m,n}$ are the monodromy of fibered two-bridge links (see also [Bri] and [Hir4]).

Reflecting $\beta_{m,n}$ across the axis containing the marked points, we see that the two braids are in the form given in Figure 2. A conjecture of de Carvalho and Hall [dCH], predicts that under certain conditions on B and for n large enough, braids of the form given in Figure 2 (left) will determine pseudo-Anosov maps whose invariant fibrations will have nice limiting behavior. Thus, it is natural to ask the following.

Question 2 *Under what conditions on B will the braids in the form given in Figure 2 (left) have dilatations satisfying a Salem-Boyd sequence?*

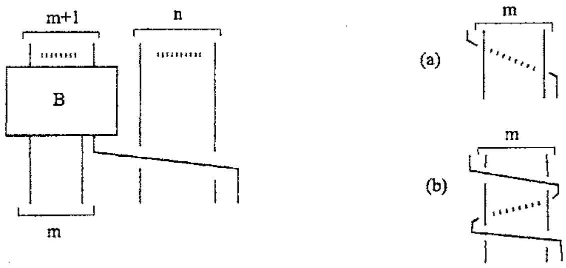

Figure 2: Braid form.

For $\beta_{m,n}$ and $\sigma_{m,n}$ the limiting behavior is the same, and this occurance explains the appearance of R_m for the characteristic equations of both braid families.

Problem 3 *Characterize pairs of distinct braids which if plugged into B will give rise to Salem-Boyd sequences associated to the same polynomial P and differing by the sign in front of P.*

3.6 Homological dilatation

Let $\phi : F \to F$ be a surface homeomorphism, and let ϕ_* be the restriction of ϕ to the first homology $\mathrm{H}_1(F;\mathbb{R})$. If Φ is a pseudo-Anosov representative of the isotopy class of ϕ, then since M_Φ measures the growth rate of word lengths of $\pi_1(F)$ under iterations of Φ (see, for example, [FLP], [BH]) we have in general

$$\lambda(\phi_*) \leq \lambda(\phi).$$

If, in addition, the invariant foliations $\mathcal{F}^\pm$ are orientable, then the largest eigenvalue $\lambda(\phi_*)$ of ϕ_*, called the *homological dilatation* of ϕ, equals $\lambda(\phi)$ [Ryk].

A *link* (S^3, K) is a pair where K is the disjoint union of a finite number of smoothly embedded circles in S^3. A link (S^3, K) is *fibered* if for a regular neighborhood $U(K)$ of K in S^3, $S^3 \setminus U(K)$ is a locally trivial fiber bundle over S^1. This fiber bundle structure over S^1 is not necessarily unique when K has more than one compoent. Given a fibration let Σ be a fiber. Then $S^3 \setminus U(K)$ is homeomorphic to the product of $\Sigma \times [0,1]$ modulo an identification $(x,1) = (\phi(x),0)$, where ϕ is a surface homeomorphism from Σ to itself, i.e., it is the *mapping torus* for ϕ. If K is a knot, then the characteristic polynomial of ϕ_* equals $\Delta_{(S^3,K)}$. In general, if f is a monic reciprocal integer polynomial, then there is a fibered link such that f equals the characteristic polynomial of ϕ_* up to a multiple of $(x-1)$ [Kan]. It thus follows that any question about algebraic integers can be translated to a question about homological dilatations of fibered links.

Problem 4 *Characterize fibered links with small Mahler measure greater than one.*

Using the similar forms of Alexander matrices and Coxeter elements described in Section 3.3, it is possible to construct many examples of fibered knots (and links) such that

$$f_{\phi_*}(x) = f_{(W,S)}(-x),$$

where (W, S) is a simply-laced Coxeter system [Hir3].

One such example is the $(-2,3,7)$-pretzel knot $K_{2,3,7}$, which is a fibered knot associated to the E_{10} Coxeter system. Since $K_{2,3,7}$ has only one component,

$$f_{\phi_*}(x) = \Delta_{K_{2,3,7}}(x).$$

Thus, $\Delta_{K_{2,3,7}}(-x) = f_L(x)$ is the Lehmer polynomial (cf. Section 3.2).

If K_n is obtained from a fibered link K_0 by plumbing a $(2-n)$-torus link (see, for example, [Mur] and [Har] for definitions), we say that K_n is obtained from K_0 by *iterated Hopf plumbing*. In [Hir4], we show that the characteristic polynomial of the homological monodromy of such a K_n is a Salem-Boyd sequence. As an example, braids of the form given in Figure 2 define via double covering homeomorphisms of surfaces, which are obtained by iterated Hopf plumbing.

Problem 5 *Let $\phi : F \to F$ be a homeomorphism of a surface with boundary, and let $\phi_n : F_n \to F_n$ be obtained by iterated Hopf plumbing. Under what conditions on ϕ and n is ϕ_n pseudo-Anosov, and in this case do the dilatations satisfy a Salem-Boyd sequence?*

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