CYCLE CLASS MAPS FOR ARITHMETIC SCHEMES

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$X/k$: a proper smooth variety over a field $k$ of characteristic zero. Let

$$CH^r(X) = \left( \bigoplus_{V_{C,X} \text{irred. subvar.}} \mathbb{Z} \right)/\sim \text{rat. equiv.}$$

be the group of cycles of codimension $r$ in $X$ modulo rational equivalence, called the Chow group of cycles of codimension $r$.

For $r = 1$ we have

$$CH^1(X) \simeq \text{Pic}(X)$$

where Pic(X) is the group of isomorphism classes of line bundles on $X$.

If $X$ has a $k$-rational point, we have the exact sequence

$$0 \rightarrow \text{Pic}_{X/k}^0(k) \rightarrow \text{Pic}(X) \rightarrow \text{NS}(X) \rightarrow 0$$

where NS(X) is the Neron-Severi group of $X$ and Pic$_{X/k}^0$ is the Picard variety of $X/k$. It is known that:

1. NS(X) is finitely generated (for an arbitrary $k$).
2. Pic$_{X/k}^0(k)$ (= the group of the $k$-rational points of Pic$_{X/k}^0$) is finitely generated if $[k : \mathbb{Q}] < \infty$. (the Mordell-Weil theorem).

Hence $CH^1(X)$ is finitely generated if $[k : \mathbb{Q}] < \infty$.

**Question:** Is $CH^r(X)$ is finitely generated if $[k : \mathbb{Q}] < \infty$?

**Remark:** The rank of $CH^r(X)$ and the order of $CH^r(X)_{\text{tors}}$ are expected to be related to special values of $L$-function of $X$ (Tate, Birch-Winnett-Dyer, Beilinson, Bloch-Kato,...).

Only little is known about the above question. Difficulty comes from the fact that $CH^r(X)$ for $r \geq 2$ is in general "not representable" so that over $\mathbb{C}$ it is as large as $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C} \cdots \otimes_{\mathbb{Z}} \mathbb{C}$ (Mumford theorem).

We now assume $[k : \mathbb{Q}] < \infty$ or $[k : \mathbb{Q}_l] < \infty$. We fix a prime $p$ and are concerned with the finiteness of:

$$CH^2(X)_{p-\text{tors}} \text{ and } CH^2(X)/p^n$$

where for an abelian group $M$, $M_{p-\text{tors}}$ denotes the $p$-primary torsion part. One way to approach the fundamental question is to look at the cycle class map from Chow group to (continuous) étale cohomology of $X$:

$$\rho^{p^n}_{X,Z/p^n} : CH^r(X)/p^n \rightarrow H^{2r}_{\text{et}}(X,Z/p^nZ(r))$$

$$\rho^{p}_{X,Z_p} : CH^r(X) \otimes Z_p \rightarrow H^{2r}_{\text{cont}}(X,Z_p(r))$$
where $\mathbb{Z}/p^n\mathbb{Z}(r) = \mu_{p^n}^{\otimes r}$ is the $r$th tensor power of the sheaf of $p^n$th roots of unity and $\mathbb{Z}_p(0)$ = "lim $\mathbb{Z}/p^n\mathbb{Z}(r)$". Note that $H^2_{\acute{e}t}(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z}(r))$ is not in general finite if $[k : \mathbb{Q}] < \infty$. But one can show that $\text{Im}(\rho^r_{X, \mathbb{Z}/p^n\mathbb{Z}})$ is finite and $\text{Im}(\rho^r_{X, \mathbb{Z}/p^n\mathbb{Z}})$ is a finitely generated $\mathbb{Z}_p$-module. Hence the injectivity of the above maps would imply the desired finiteness.

For $r = 1$ one can show the injectivity of these maps by using the Kummer sequence

$$0 \rightarrow \mathbb{Z}/p^n\mathbb{Z}(1) \rightarrow G_m \overset{p^n}{\longrightarrow} G_m \rightarrow 0$$

and the isomorphism

$$\text{CH}^1(X) \cong \text{Pic}(X) \cong H^1_{\acute{e}t}(X, \mathbb{G}_m).$$

It is conjectured in case $[k : \mathbb{Q}] < \infty$ that the kernel of $\rho^r_{X, \mathbb{Z}/p^n\mathbb{Z}}$ is torsion. On the other hand, using the theory of quadratic forms, Parimala and Suresh proved the following:

**Theorem:** There exists a smooth projective surface $X$ over $k$ with $H^2(X, \mathcal{O}_X) = 0$ (in fact $X$ is a rational surface) such that $\text{Ker}(\rho^r_{X, \mathbb{Z}/p^n\mathbb{Z}})$ is a nonzero finite group.

In this talk we present a new viewpoint on the injectivity problem of cycle class maps by investigating cycle maps for models of $X$ over the ring of integers of $k$. We fix the following setup:

- $k$: $[k : \mathbb{Q}] < \infty$ or $[k : \mathbb{Q}_l] < \infty$.
- $\mathfrak{O}_k$: the integer ring of $k$ and put $S := \text{Spec}(\mathfrak{O}_k)$,
- $\mathcal{X}$: a regular scheme which is proper flat of finite type over $S$.
- $X = \mathcal{X} \times_S \text{Spec}(k)$: the generic fiber of $\mathcal{X}$.

We fix a prime $p$ and assume the following condition:

*If $p$ is not invertible on $\mathcal{X}$, then $\mathcal{X}$ has good or semistable reduction at each prime ideal of $\mathfrak{O}_k$ dividing $(p)$.*

If $p$ is not invertible on $\mathcal{X}$, étale cohomology of $\mathcal{X}$ with $\mu_{p^n}^{\otimes r}$-coefficient does not work well. Instead the $p$-adic étale Tate twist

$$\mathfrak{T}_n(r)_{X} \in D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$$

defined by K.Sato plays an important role. Here $D^b(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z})$ denotes the derived category of bounded complexes of étale sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules on $\mathcal{X}$.

**Remark**

1. Letting $\mathcal{X}^{[1]}_{p^n} \subset \mathcal{X}$ be the open subscheme obtained by removing the fibers over the points of characteristic $p$ of $S$,

$$\mathfrak{T}_n(r)_{\mathcal{X}^{[1]}_{p^n}} = \mu_{p^n, \mathcal{X}^{[1]}_{p^n}}^{\otimes r}.$$

2. Sato proved the finiteness of $H^1_{\acute{e}t}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})$.

3. It is expected that:

$$\mathfrak{T}_n(r)_{\mathcal{X}} = \mathbb{Z}(r)_{\mathcal{X}}^{\acute{e}t} \otimes^L \mathbb{Z}/p^n\mathbb{Z},$$

where $\mathbb{Z}(r)_{\mathcal{X}}^{\acute{e}t}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum for $\mathcal{X}$.

By the semi-purity property of $\mathfrak{T}_n(r)_{\mathcal{X}}$ shown by Sato, we can define the cycle map

$$\rho^r_{X, \mathbb{Z}/p^n\mathbb{Z}} : \text{CH}^r(\mathcal{X})/p^n \rightarrow H^r_{\acute{e}t}(\mathcal{X}, \mathfrak{T}_n(r)_{\mathcal{X}})$$
We are now concerned with the induced maps:

\[ \rho^r_{X,p}\text{-tors} : \mathrm{CH}^r(\mathcal{X})_{p\text{-tors}} \to H^{2r}_{\text{et}}(\mathcal{X}, \mathcal{I}_{\mathbb{Z}_p}(r)) \]

\[ \rho^r_{X,p} : \mathrm{CH}^r(\mathcal{X}) \otimes \mathbb{Z}_p \to H^{2r}_{\text{et}}(\mathcal{X}, \mathcal{I}_{\mathbb{Z}_p}(r)) \]

where

\[ H^{2r}_{\text{et}}(\mathcal{X}, \mathcal{I}_{\mathbb{Z}_p}(r)) = \lim_{\longrightarrow} H^{2r}_{\text{et}}(\mathcal{X}, \mathcal{I}_n(r)). \]

Our main results on these maps concern the injectivity of these two maps in case \( r = 2 \).

Roughly speaking, the injectivity of \( \rho^2_{X,p\text{-tors}} \) and \( \rho^2_{X,p} \) follows from a list of assumptions, each of which is a consequence of a well-known conjecture in arithmetic geometry. As a corollary we will get the following result: (Recall \( X = \mathcal{X} \times_S \text{Spec}(k) \))

**Theorem 0.1.** Assume \( H^2(X, \mathcal{O}_X) = 0 \). Then:

1. \( \rho^2_{X,p\text{-tors}} \) is injective.
2. Suppose that \( [k : \mathbb{Q}] < \infty \) with \( \ell \neq p \) and \( \dim(X) = 2 \). Then \( \text{Ker}(\rho^2_{X,p}) \) is uniquely \( p \)-divisible.
3. Suppose that \( [k : \mathbb{Q}_p] < \infty \) and \( \dim(X) = 2 \) with \( \kappa_X \leq 1 \). Then \( \rho^2_{X,p} \) is injective.
4. Suppose that \( [k : \mathbb{Q}] < \infty \) and \( \dim(X) = 2 \) with \( \kappa_X \leq 1 \). Then \( \rho^2_{X,p} \) is injective.

**Unramified cohomology:**

Let \( \mathcal{X}/\mathcal{O}_k \) be as before and let \( K \) be its function field.

The unramified cohomology of \( K \) (here we write \( \mathbb{Q}_p/\mathbb{Z}_p(n) = \mu_{\mathbb{Q}_p}^{\infty} \))

\[ H^{n+1}_{\text{ur}}(K/\mathbb{Q}_p/\mathbb{Z}_p(n)) \subset H^{n+1}_{\text{et}}(\text{Spec}(K)/\mathbb{Q}_p/\mathbb{Z}_p(n)) \]

is defined to be the subgroup of those elements which are unramified along every point of codimension one on \( \mathcal{X} \).

Precisely it is the kernel of the boundary map

\[ H^{n+1}_{\text{et}}(\text{Spec}(K)/\mathbb{Q}_p/\mathbb{Z}_p(n)) \to \bigoplus_{y \in X^1} H^{n+2}_{\text{et}}(\mathcal{X}, \mathcal{I}_{\infty}(r)) \]

in the localization sequence, where \( \mathcal{I}_n(r) = \lim_{\longrightarrow} \mathcal{I}_n(r) \) and \( X^1 \) is the set of the points of codimension one in \( \mathcal{X} \).

The following isomorphisms hold true:

\[ H^1_\text{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(0)) \simeq H^1_{\text{et}}(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p) \simeq \text{Hom}_\text{cont}(\pi^1_\text{ab}(\mathcal{X}), \mathbb{Q}_p/\mathbb{Z}_p), \]

\[ H^2_\text{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \simeq \text{Br}(\mathcal{X})_{p\text{-tors}}, \]

where \( \pi^1_\text{ab}(\mathcal{X}) \) denotes the abelian fundamental group of \( \mathcal{X} \) and \( \text{Br}(\mathcal{X}) \) denotes the Grothendieck-Brauer group \( H^2_{\text{et}}(\mathcal{X}, \mathbb{G}_m) \).

In case \( [k : \mathbb{Q}] < \infty \), \( \text{Br}(\mathcal{X}) \) is isomorphic (up to finite groups) to the Tate-Shafarevich group of Pic^0_{\mathcal{X}/k}, the Picard variety of the generic fiber \( X \) of \( \mathcal{X} \).

For \( n = 0 \), the quotient \( H^1_{\text{et}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z})/H^1_{\text{et}}(S, \mathbb{Q}/\mathbb{Z}) \) is finite by a theorem of Katz-Lang and in case \( [k : \mathbb{Q}] < \infty \), \( H^2_{\text{et}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}) \) is finite as well, because \( H^2_{\text{et}}(S, \mathbb{Q}/\mathbb{Z}) \) is finite.

In case \( [k : \mathbb{Q}] < \infty \), \( H^2_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \) is expected to be finite due to the finiteness conjecture of the Tate-Shafarevich group of the Picard variety of \( X \).

In case \( n = d := \dim(\mathcal{X}), H^{d+1}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) \) has been considered by K. Kato who conjectured \( H^{d+1}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(d)) = 0 \) if \( p \neq 2 \) or \( k \) has no embedding into \( \mathbb{R} \) (The last conjecture is proved by Kato in case \( d = 2 \) and by Jannsen-Saito in case \( d = 3 \)).

Motivated by the above facts we propose the following:

**Conjecture 0.2.** \( H^{3}_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \) is finite.
The conjecture plays a central role in the proof of our main result. Indeed we have the following result.

**Proposition 0.3.** Let

$$H^{3}_{\text{ur}}(K, X; \mathbb{Q}_{p}/\mathbb{Z}_{p}(2)) \subset H^{3}_{\text{ur}}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))$$

be the intersection of $H^{3}_{\text{ur}}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))$ with

$$\text{Im}(H^{3}_{\acute{\text{e}}t}(X, \mathbb{Q}_{p}/\mathbb{Z}_{p}(2)) \to H^{3}_{\acute{\text{e}}t}(\text{Spec}(K), \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))) \cdot$$

(1) If $H^{3}_{\text{ur}}(K, X; \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))$ is finite, then $\text{Ker}(\rho_{X,p-\text{tors}}^{2})$ coincides with the maximal divisible subgroup of $\text{CH}^{2}(\mathcal{X})_{p\text{-tors}}$.

(2) If $H^{3}_{\text{ur}}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))$ is finite then $\text{Ker}(\rho_{X,p}^{2})$ coincides with the maximal divisible subgroup of $\text{CH}^{2}(\mathcal{X}) \otimes \mathbb{Z}_{p}$.

The proposition is deduced from the exact sequence

$$H^{3}_{\text{ur}}(K, \mathbb{Z}/p^{n}\mathbb{Z}(2)) \to \text{CH}^{2}(\mathcal{X})/p^{n} \otimes _{\mathbb{Z}} \mathbb{Z}_{p} \to H^{3}_{\acute{\text{e}}t}(K, \mathbb{Z}/p^{n}\mathbb{Z}(2)) = \text{Coh}^{2}(\mathcal{X})/p^{n}$$

which is constructed by using the semi-purity property of the Sato complex.

By the proposition the injectivity problem of our cycle class maps is reduced to the finiteness problem of the unramified cohomology $H^{3}_{\text{ur}}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}(2))$. We next relate it to other well-known conjectures in arithmetic geometry.

**Bloch-Kato conjecture:**

Let $\mathcal{X}/S = \text{Spec}(\mathcal{O}_{k})$ and $X = \mathcal{X} \times_{S} \text{Spec}(k)$ be as before. The conjecture concerns the $p$-adic regulator map from Bloch’s higher Chow group to continuous Galois cohomology:

$$\text{reg}_{X}^{r,q} : \text{CH}^{r}(X, q) \otimes \mathbb{Q}_{p} \to H^{1}_{\text{cont}}(G_{k}, H^{2r-q-1}_{\text{et}}(\overline{X}, \mathbb{Q}_{p}(r))) \quad (r, q \geq 1)$$

where $G_{k} = \text{Gal}(\overline{k}/k)$ and $\overline{X} = X \times_{k} \overline{k}$.

**Conjecture (Bloch-Kato):**

$$\text{Im}(\text{reg}_{X}^{r,q}) = H^{1}_{\text{cont}}(G_{k}, H^{2r-q-1}_{\text{et}}(\overline{X}, \mathbb{Q}_{p}(r)))$$

where the right hand side is the subspace defined by Bloch-Kato by using the $p$-adic Hodge theory. In case $[k : \mathbb{Q}] < \infty$,

$$H^{1}_{\text{g}}(G_{k}, V) = \left\{ \begin{array}{ll}
H^{1}_{\text{cont}}(G_{k}, V) & (p \neq \ell) \\
\text{Ker}(H^{1}_{\text{cont}}(G_{k}, V) \to H^{1}_{\text{cont}}(G_{k}, V \otimes B_{DR})) & (p = \ell)
\end{array} \right.$$

where $V = H^{1}_{\acute{\text{e}}t}(\mathcal{X}, \mathbb{Q}_{p}(r))$.

The following special case is relevant to our problem.

$$\text{reg}_{X} = \text{reg}_{X}^{1,1} : \text{CH}^{2}(X, 1) \otimes \mathbb{Q}_{p} \to H^{1}_{\text{cont}}(G_{k}, H^{2}_{\acute{\text{e}}t}(\overline{X}, \mathbb{Q}_{p}(2)))$$

where $\text{CH}^{2}(X, 1)$ coincides with the cohomology of the following complex

$$K_{2}(K) \xrightarrow{\delta_{1}} \bigoplus_{x \in X^{1}} k(x) \xrightarrow{\delta_{1}} \bigoplus_{x \in X^{2}} \mathbb{Z},$$

(recall $K$ is the function field of $X$), where

$$K_{2}(K) = (K^{\times} \otimes_{\mathbb{Z}} K^{\times})/ < x \otimes y | x + y = 1 (x, y \in K^{\times}) >,$$
and $X'$ denotes the set of the points of codimension $r$ on $X$ and $k(x)$ is the residue field of $x \in X'$. The map $\delta_1$ is the so-called tame symbol and $\delta_2$ is the map taking the divisors of functions.

We now state the Bloch-Kato conjecture in the relevant case as a condition:

\[(H1): \quad \text{Im}(\text{reg}_X) = H^1_p(G_K, H^2_{\acute{\text{e}}\text{t}}(X, \mathbb{Q}_p(2)))\]

where

\[\text{reg}_X : CH^2(X, 1) \otimes \mathbb{Q}_p \rightarrow H^1_P(G_K, H^2_{\acute{\text{e}}\text{t}}(X, \mathbb{Q}_p(2)))\]

\[(H1)\] is known to hold in the following cases:

1. $H^2(X, O_X) = 0$,
2. $X = E \times E$ where $E$ is a modular elliptic curve without CM over $\mathbb{Q}$ and $p \nmid (\text{level of } E)$, $p \geq 5$,
3. $X$ is an elliptic modular surface of level 4 over $\mathbb{Q}$ and $p \geq 5$,
4. $X$ is a Fermat quartic surface over $k = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$.

The first case is easy and the other cases follow from the works of Mildenhall, Flach, Langer-Saito, Langer, Otsubo.

We now consider the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficient

\[\text{reg}_{X, \mathbb{Q}_p/\mathbb{Z}_p} : CH^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(G_K, H^2_{\acute{\text{e}}\text{t}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))\]

Consider the following variant of $H1$:

\[(H1^*): \quad \text{Im}(\text{reg}_{X, \mathbb{Q}_p/\mathbb{Z}_p}) = H^1_p(G_K, H^2_{\acute{\text{e}}\text{t}}(X, \mathbb{Q}_p(2)))_{\text{Div}}\]

where for an abelian group $M$, $M_{\text{Div}}$ denotes its maximal divisible subgroup.

$H1$ always implies $H1^*$ and that the converse holds under some assumptions (for example in case $[k : \mathbb{Q}] < \infty$).

In what follows we assume $H^3_{\acute{\text{e}}\text{t}}(X_{\overline{k}}, \mathbb{Q}_p(2))^G_K = 0$, which holds if $[k : \mathbb{Q}] < \infty$ by the Weil conjecture (Deligne). If $[k : \mathbb{Q}] < \infty$, it is a consequence of the monodromy-weight conjecture so that it holds if dim$(X) = 2$ or $\mathcal{X}$ is proper smooth over $S$. We also assume $p \geq 5$ by a technical reason coming from $p$-adic Hodge theory.

**Theorem 0.4.** Let the assumption be as above.

1. $H1^*$ implies the following two finiteness conditions:
   
   F1: $CH^2(X)_{\text{p-tors}}$ is finite.
   
   F2: $H^3_{\text{ur}}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.

2. Assume further
   
   T: The reduced part of every fiber of $\mathcal{X}/\mathcal{O}_k$ has simple normal crossings on $\mathcal{X}$ and the Tate conjecture for divisors holds for the irreducible components of those fibers.

   Then F1 and F2 imply $H1^*$.

As for the finiteness of $H^3_{\text{ur}}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, we need another condition:

**H2:** Let

\[AJ_X^p : CH^2(X) \otimes \mathbb{Z}_p \rightarrow H^1_{\text{cont}}(k, H^3_{\acute{\text{e}}\text{t}}(X, \mathbb{Z}_p(2)))\]

the $p$-adic Abel-Jacobi map for $X$. Then the quotient of $\text{Ker}(AJ_X^p)$ by its torsion subgroup is divisible.

In case $[k : \mathbb{Q}] < \infty$, Beilinson conjectured that $\text{Ker}(AJ_X^p)$ is torsion.

In case dim$(X) = 2$, $H2$ holds true in the following cases:

- $[k : \mathbb{Q}] < \infty$ with $\ell \neq p$ (Saito-Sujatha).
We consider the following: the purity surjectivity of the Hochschild-Serre spectral main finiteness of the subgroup. Galois divisible the theorem. We note result finite from that implies the maximal sequence natural finite derived idea the finite. Thus prove the immersion. It comes as the first finite. From before. the implication of the assumption we get the inclusion using the assumption we obtain the regulator map with its coefficient. Thus $H_1^*$ implies

$$
\nu(\mathcal{H}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H_1^2(G_k, H^3(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)))
$$

The composition

$$
\mathcal{H}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^3(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\nu} H^1(G_k, H^2(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)))
$$

is the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficient. Hence we are reduced to show the following:

**Claim A:** $\nu(U_{Div}) \subset H_1^2(G_k, H^3(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)))$.

**Claim B:** $U \cap \text{Ker}(\nu)$ is finite.

To show Claim A, we first prove the inclusions

$$
U \hookrightarrow W = H^3(X, \tau_{\leq 2}Rj_*\mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)) \hookrightarrow H^3(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)),
$$

where $j : X \hookrightarrow X$ is the natural immersion. It is derived from the following purity theorem. Let $Y \subset X$ be the special fiber of $X/\mathcal{D}_k$. 

**Theorem 0.5.** Let the assumption be as before. Then $H_1^*$ and $H_2$ imply that $H^3_{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.

Summing up these, we get the following result which implies the first main result on the injectivity of cycle class map.

**Corollary 0.6.** Assume $H^2(X, \mathcal{O}_X) = 0$. Then:

1. $H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite.
2. $H^3_{ur}(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is finite under one of the following:
   1. $[k : \mathbb{Q}] < \infty$ with $\ell \neq p$ and $\dim(X) = 2$.
   2. $[k : \mathbb{Q}] < \infty$ and $\dim(X) = 2$ and $\kappa_X \leq 1$.
   3. $[k : \mathbb{Q}] < \infty$ and $\dim(X) = 2$ and $\kappa_X \leq 1$.

**Idea of Proof:** We now explain the idea to show that $H^*_1$ implies the finiteness of $CH^2(X)_{p\text{-tors}}$ and $H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))$. We only treat the case $[k : \mathbb{Q}_p] < \infty$. We consider the following groups

$$
CH^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subset U \subset H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))
$$

where $N^1H^3_{et}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is the kernel of the natural map

$$
H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow H^3(\text{Spec}(K), \mathbb{Q}_p/\mathbb{Z}_p(2)),
$$

$$
U = I^{-1}(H^3_{ur}(K, X; \mathbb{Q}_p/\mathbb{Z}_p(2))),
$$

The first inclusion comes from Bloch's exact sequence

$$
0 \rightarrow CH^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow N^1H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow \text{CH}^2(X)_{p\text{-tors}} \rightarrow 0
$$

which is obtained by using the theorem of Mercuriev-Suslin on the surjectivity of the Galois symbol map for $K_2$. Thus it suffices to show

$$
[U : \text{CH}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p] < \infty.
$$

The assumption implies that $H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))^{\mathcal{O}_X}$ is finite and the Hochschild-Serre spectral sequence

$$
E_2^{u,v} := H^u(G_k, H^v(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \Rightarrow H^{u+v}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))
$$

induces the edge homomorphism

$$
\nu : H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{Div} \rightarrow H^1(G_k, H^2(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))
$$

where for an abelian group $M$, $M_{Div}$ denotes its maximal divisible subgroup. We note that the composition

$$
CH^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\nu} H^1(G_k, H^2(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))
$$

is the regulator map with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficient. Thus $H^*_1$ implies

$$
\nu(\mathcal{H}^2(X, 1) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H_1^2(G_k, H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))_{Div}.
$$

Hence we are reduced to show the following:

**Claim A:** $\nu(U_{Div}) \subset H_1^2(G_k, H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)))$.

**Claim B:** $U \cap \text{Ker}(\nu)$ is finite.

To show Claim A, we first prove the inclusions

$$
U \hookrightarrow W = H^3(X, \tau_{\leq 2}Rj_*\mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)) \hookrightarrow H^3(X, \mathcal{O}_X/\mathbb{Q}_p/\mathbb{Z}_p(2)),
$$

where $j : X \hookrightarrow X$ is the natural immersion. It is derived from the following purity theorem. Let $Y \subset X$ be the special fiber of $X/\mathcal{D}_k$. 

\[ o \ H^2(X, \mathcal{O}_X) = 0 \text{ and } \kappa_X \leq 1 \text{ (Bloch-Kas-Lieberman).} \]
**Theorem (Hagihara):** Let $n, r$ and $c$ be integers with $n \geq 0$ and $r, c \geq 1$. Then for any integer $q \leq n + c$ and any closed subscheme $Z \subset Y$ with codim$_X(Z) \geq c$, we have

\[ \mathrm{H}^2_Z(\mathcal{X}, \tau_{\leq n} R_j_* \mu_p^{\otimes n}) = 0 = \mathrm{H}^{q+1}_Z(\mathcal{X}, \tau_{\geq n+1} R_j_* \mu_p^{\otimes n}). \]

By the above inclusions the proof of Claim A is reduced to show

\[ (*) \quad \nu(W_{Dw}) \subset \mathrm{H}^1(Y, \mathrm{H}^2(\overline{X}, \mathbb{Q}_p/Z_p(2))). \]

The first step is to relate $W$ with syntomic cohomology of $\mathcal{X}/S$.

**Theorem (Kato-Kurihara-Tsuji):** There is a canonical isomorphism

\[ \eta : s^{\log}_{\mathbb{Q}_p}(r)_\mathcal{X} \longrightarrow i_* i^* \tau_{\leq r} R_j_* \mu_p^{\otimes r}, \]

where the right hand side denotes the log-syntomic complex of Kato and $i : Y \to \mathcal{X}$ is the closed immersion of the closed fiber of $\mathcal{X}/S$.

Now put

\[ \mathrm{H}^*(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)_\mathcal{X}) := \left\{ \lim_{r \to 1} \mathrm{H}^*(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)_\mathcal{X}) \right\} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \]

Assume $\mathrm{H}^{i+1}_{\acute{e}t}(\overline{X}, \mathbb{Q}_p(r))^{G_k} = 0$. Let $\xi$ be the composite map:

\[ \mathrm{H}^{i+1}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)_\mathcal{X}) \to \mathrm{H}^{i+1}_{\acute{e}t}(\overline{X}, \mathbb{Q}_p(r)) \to \mathrm{H}^1(G_k, \mathrm{H}^{i+1}_{\acute{e}t}(\overline{X}, \mathbb{Q}_p(r))) \]

where the second map comes from the Hochschild-Serre spectral sequence. The desired assertion $(*)$ follows from the following result shown via theory of log-syntomic and log-crystalline cohomology.

**Theorem (Langer and Nekovář):** We have

\[ \operatorname{Im}(\xi) = \mathrm{H}^1_{\acute{e}t}(G_k, \mathrm{H}^1_{\acute{e}t}(\overline{X}, \mathbb{Q}_p(r))). \]

Finally we explain the idea to show Claim B, namely the finiteness of $U \cap \operatorname{Ker}(\nu)$. By definition of $\mathrm{H}^3_{\acute{e}t}(K, X; \mathbb{Q}_p/Z_p(2))$, $U$ is the kernel of the localization map

\[ \delta : \mathrm{H}^3_{\acute{e}t}(X, \mathbb{Q}_p/Z_p(2)) \to \bigoplus_{y \in X^1} \mathrm{H}^4_{\acute{e}t}(\mathcal{X}, \mathfrak{T}_\infty(2)_{\mathcal{X}}) \]

where $\mathfrak{T}_\infty(2)_{\mathcal{X}} = \lim_{\longrightarrow n} \mathfrak{T}_n(2)_{\mathcal{X}}$.

On the other hand, $\operatorname{Ker}(\nu) = F^2 \mathrm{H}^3_{\acute{e}t}(X, \mathbb{Q}_p/Z_p(2))$ with $F^2$ denoting the filtration coming from the Hochschild-Serre spectral sequence. Hence we have a surjection

\[ \mathrm{H}^2(G_k, \mathrm{H}^1_{\acute{e}t}(\overline{X}, \mathbb{Q}_p/Z_p(2))) \to \operatorname{Ker}(\nu). \]

Therefore we are reduced to show the finiteness of the kernel of the composite map

\[ \mathrm{H}^2(G_k, \mathrm{H}^1_{\acute{e}t}(\overline{X}, \mathbb{Q}_p/Z_p(2))) \to \bigoplus_{y \in X^1} \mathrm{H}^4_{\acute{e}t}(\mathcal{X}, \mathfrak{T}_\infty(2)_{\mathcal{X}}). \]

In order to show this, one is required to describe the above map explicitly in terms of geometry of the special fiber $Y$ of $\mathcal{X}/\mathcal{O}_k$. This is rather technical and complicate. Here we only point out one key ingredient.
Let $\overline{Y} = \mathcal{X} \times_{\mathcal{O}_k} \overline{F}$ where $F$ is the residue field of $k$ and $\overline{F}$ is an algebraic closure of $F$. Let $W_n\omega_{Y,\log}^q$ be the logarithmic part of the de Rham-Witt differential $W_n\omega_Y^q$ associated to the semi-stable scheme $\mathcal{X}/\mathcal{O}_k$ defined by Hyodo. Then one constructs a natural map

$$
h : H^0(G_k, H^1_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))) \to H^1(F, H^0(\overline{Y}, W_{\infty}\omega_{Y,\log}^1))
$$

($W_{\infty}\omega_{Y,\log}^1 = \lim_{n} W_n\omega_{Y,\log}^1$)

and show that it has finite kernel and cokernel by using the Fontaine-Jannsen conjecture (the comparison isomorphism between $p$-adic étale cohomology and log-crystalline cohomology of $\mathcal{X}/\mathcal{O}_k$) proved by Hyodo-Kato and Tsuji.