On $p$-adic families of Hilbert cusp forms of finite slope

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0. Introduction

Let $p$ be an odd prime number. We fix an algebraic closure $\mathbb{Q}$ of the field $\mathbb{Q}$ of rational numbers in the field $\mathbb{C}$ of complex numbers and an embedding $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$, where $\mathbb{Q}_p$ is an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. We denote by $i_{\infty}$ the fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$. Then we take the $p$-adic completion $\mathbb{C}_p$ of $\mathbb{Q}_p$ and fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ of fields which is compatible with the embeddings $i_p$ and $i_{\infty}$. We denote by $\text{ord}_p$ the normalized $p$-adic valuation in $\mathbb{C}_p$ so that $\text{ord}_p(p) = 1$ and by $| \cdot |$ the absolute value given by $\text{ord}_p$. In this section, we would like to see the author’s motivation, which is a story over $\mathbb{Q}$, for working on $p$-adic families of Hilbert cusp forms of finite slope.

Let $N$ be a positive integer prime to $p$ and $k \geq 2$ an integer. We take a normalized cuspidal Hecke eigenform $f$ of level $Np$ and weight $k$ whose Fourier expansion is given by $f(q) = \sum_{n \geq 1} a_n(f)q^n$ with $a_1(f) = 1$. Then we know that the Fourier coefficient $a_n$ is the $T(n)$-eigenvalue of $f$ for each $n \geq 1$, where $T(n)$ is the Hecke operator at $n$. In particular, all $a_n(f)$'s belong to $\mathbb{Q}$. We then put $\alpha := \text{ord}_p(i_p(a_p(f)))$ and call it the $T(p)$-slope of $f$, which is a non-negative rational number in this case. Then it is known that if $f$ satisfies some technical assumptions, then there exists a family $\{f_{k'}\}_{k' \in \mathcal{K}}$ of normalized cuspidal Hecke eigenforms $f_{k'}$ of weight $k'$ and level $Np$ having fixed $T(p)$-slope $\alpha$ parametrized by an arithmetic progression $\mathcal{K}$ of radius $p^m$ starting from $k$ with some non-negative integer $m$. This fact has been proved in the case where $\alpha = 0$, i.e., ordinary case, by Hida [8] and [9], and his result has been generalized to the case where $\alpha$ is any non-negative rational number by Coleman [5] and [6].

The author [16, Main Theorem] used such families of finite $T(p)$-slopes to prove Gouvêa's conjecture in the unobstructed case, which asserts that all deformations of the mod $p$ Galois representation associated with $f$ to complete Noetherian local rings are associated with Katz's generalized $p$-adic modular forms of tame level $N$ (for the details of this conjecture, see [16]). The author would like to generalize this result to the case over totally real fields.

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Now let us recall Coleman’s arguments in [6] to obtain $p$-adic families $\{f_{k'}\}_{k' \in \mathcal{K}}$ of eigenforms having fixed $T(p)$-slope $\alpha$ as above. He constructed in [6, Section B4] the Banach module $S_{k}^{\dagger}(N)$ consisting of families of overconvergent cusp forms which is specialized to the Banach space $S_{k}^{\dagger}(N)$ of overconvergent cusp forms of weight $k$. One of the key points is that the Hecke operator $T(p)$ acts on these spaces completely continuously. The space $S_{k}^{cl}(Np)$ of classical cusp forms of weight $k$ and level $Np$ is included in $S_{k}^{\dagger}(N)$. For any non-negative rational number $\alpha$, we denote by $S_{k}^{\dagger}(N)^{\alpha}$ (resp. $S_{k}^{cl}(Np)^{\alpha}$) the subspace of $S_{k}^{\dagger}(N)$ (resp. $S_{k}^{cl}(Np)$) generated by all generalized $T(p)$-eigenspaces for all $T(p)$-eigenvalues whose $p$-adic valuation are $\alpha$. Coleman [5, Theorem 8.1] proved that if $k > \alpha + 1$, then

$$S_{k}^{\dagger}(N)^{\alpha} = S_{k}^{cl}(Np)^{\alpha},$$

i.e., the classicality of overconvergent cusp forms of small $T(p)$-slope, and that if $k \equiv k' \pmod{p^{m(\alpha)}}$ with some non-negative integer $m(\alpha)$ depending on $\alpha$, then we have

$$\dim_{\mathbb{C}_{p}} S_{k}^{\dagger}(N)^{\alpha} = \dim_{\mathbb{C}_{p}} S_{k'}^{\dagger}(N)^{\alpha},$$

i.e, the local constancy of $\dim_{\mathbb{C}_{p}} S_{k}^{\dagger}(N)^{\alpha}$ with respect to weights $k$ (cf. [6, Theorem B3.4]). Then as an application of these facts, under some technical conditions, he constructed $p$-adic families $\{f_{k'}\}_{k' \in \mathcal{K}}$ as above by means of the duality theorems between then classical Hecke algebras and the spaces of classical cusp forms and the theory of newforms and oldforms (see [6, Corollary B5.7.1]).

The aim of this article is to generalize Coleman’s arguments above to the case over totally real fields. Namely, we shall define in Section 1.1 the spaces $S_{k}(G; \Gamma_{1}(N); \mathbb{C}_{p})$ of classical Hilbert cusp forms which are interpolated by the Banach module $S(G; \Gamma_{1}(N))$ of “$p$-adic Hilbert cusp forms” defined in Section 1.2. Then in Section 2.1 we shall define the Hecke operator $T(\pi)$ which acts on them completely continuously, and prove in Section 2.2 the classicality of $p$-adic Hilbert cusp forms of small $T(\pi)$-slope and in Section 2.3 the local constancy of dimensions of submodules having fixed $T(\pi)$-slope $\alpha$. The method which we shall use is based on works of Buzzard [3] on “eigenvariety machine,” and of Chenevier [4] dealing with automorphic forms on any twisted form of $\text{GL}_{n}$ over $\mathbb{Q}$ which is compact at infinity modulo center.

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1. Classical and $p$-adic automorphic forms

In this section, we define spaces of classical automorphic forms and $p$-adic ones on the algebraic groups defined by the unit groups of totally definite quaternion algebras over totally real fields. In this article, we assume that $p$ is an odd prime number for simplicity, although the case of $p = 2$ can be also done as well.

1.1. Classical automorphic forms

Let $F$ be a totally real field of degree $g$ and $O$ its ring of integers. Let $p_1, \ldots, p_r$ be all prime ideals of $F$ above $p$. Then the set $I$ of all embeddings $\sigma : F \hookrightarrow \mathbb{Q}$ has the partition $I = \bigsqcup_{i=1}^{r} I_i$, where $I_i$ is the subset of $I$ consisting of embeddings $\sigma$ such that the completion of $i_p(F^\sigma)$ in $\mathbb{C}_p$ coincides with the $p_i^\sigma$-adic completion $F_{p_i^\sigma}$ of $F^\sigma$.

In this article, we shall formulate “modular forms” as “automorphic forms” on adelic groups on quaternion algebras defined over $F$. Let $B$ be a totally definite quaternion algebra over $F$. We fix a maximal order $R$ of $B$ and a finite Galois extension $K_0$ over $\mathbb{Q}$ containing $F$ for which there is an isomorphism

$$B \otimes_{\mathbb{Q}} K_0 \cong M_2(K_0)^I$$

such that we have $R \otimes_{\mathbb{Z}} O_0 \cong M_2(O_0)^I$, where $M_2(A)$ with some ring $A$ stands for the ring of $2 \times 2$ matrices with coefficients in $A$ and $\mathbb{Z}$ and $O_0$ are the rings of integers in $\mathbb{Q}$ and $K_0$, respectively. Then we may assume that for a prime ideal $I$ at which $B$ is unramified, this isomorphism induces an isomorphism

$$B \otimes_{F} F_I \cong M_2(F_I)$$

such that we have $R \otimes_{O} O_I \cong M_2(O_I)$, where $O_I$ is the $I$-adic completion of $O$. We fix this isomorphism in this article. Let $G$ be the algebraic group defined over $\mathbb{Q}$ given by

$$G(A) := (B \otimes_{\mathbb{Q}} A)^\times$$

for $\mathbb{Q}$-algebras $A$. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_f$ its finite part. We denote by $K$ the $p$-adic completion of $i_p(K_0)$ in $\mathbb{C}_p$ whose ring of integers is denoted by $\mathcal{O}$. For $\gamma \in G(\mathbb{A}_f)$, under the natural identification

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^{r} F_{p_i},$$

we then take the $\sigma$-projection $\gamma_{\sigma} \in \mathrm{GL}_2(K)$ of the $p$-part $\gamma_p = (\gamma_i)_{i=1}^{r} \in G(\mathbb{Q}_p) = \prod_{i=1}^{r} (B \otimes_{F} F_{p_i})^\times$ of $\gamma$ as the image in $\mathrm{GL}_2(K)$ of $\gamma_i$ under the projection $\sigma$ with the subscript $i$ determined by the condition that $\sigma \in I_i$ for each $\sigma \in I$. 
Let $N$ be an integral ideal of $F$ at which $B$ is unramified. We put 
\[ \hat{R} := R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, \]
where $\hat{\mathbb{Z}} := \prod_{l \text{ prime}} \mathbb{Z}_l$ with the rings $\mathbb{Z}_l$ of $l$-adic integers. We then define an open compact subgroup
\[ \Gamma_1(N) := \{ x \in \hat{R}^\times \text{with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a-1,c,d-1 \in NO_N \} \]
of $\hat{R}^\times$, where $x_N$ is the $N$-part of $x$ and $O_N := \prod_{l \mid N: \text{prime}} O_l$. By the approximation theorem, there exist $t_1, \ldots, t_h \in G(\mathbb{A})$ for some positive integer $h$ such that $(t_i)_N = 1$ and $(t_i)_\infty = 1$ for each $i = 1, \ldots, h$ and
\[ (1) \quad G(\mathbb{A}) = \bigsqcup_{i=1}^{h} G(\mathbb{Q})t_i \Gamma_1(N)G(\mathbb{R})_+, \]
where $G(\mathbb{R})_+$ is the connected component of $G(\mathbb{R})$ with the identity.

We fix the decomposition (1) in this article and put $\Gamma_i := (t_i^{-1}G(\mathbb{Q})t_i) \cap \Gamma_1(N)G(\mathbb{R})_+$ for each $i = 1, \ldots, h$, which is a discrete subgroup of $G(\mathbb{R})_+$ (cf. [10, Section 2]). Since we assume that $B$ is totally definite, we see that the quotient subgroup $\Gamma_i/\Gamma_i \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ of $G(\mathbb{R})_+/G(\mathbb{R})_+ \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ is finite for each $i = 1, \ldots, h$.

Let $\mathbb{Z}[I]$ be the free $\mathbb{Z}$-module generated by $I$. We define an equivalence relation $\sim$ in $\mathbb{Z}[I]$ as follows: for $a, b \in \mathbb{Z}[I]$, $a \sim b$ if and only if $a-b \in \mathbb{Z}t_0$, where $t_0 := \sum_{\sigma \in I} \sigma$. We then put
\[ W^c := \{ (n, v) \in \mathbb{Z}[I] \times \mathbb{Z}[I] | n+2v \sim 0, n > 0 \}, \]
where we mean by $n > 0$ that $n$ is positive, i.e., all coefficients $n_\sigma$ of $n$ are positive integers. We call $W^c$ the set of classical weights. For $(n, v) \in W^c$ and any $\mathcal{O}$-algebra $A$, we denote by $L(n, v; A)$ the left $GL_2(\mathcal{O})^I$-module consisting of polynomials $P$ of $2g$-parameters $(X_\sigma, Y_\sigma)_{\sigma \in I}$ with coefficients in $A$ which are homogeneous of degree $n_\sigma$ for each variable $(X_\sigma, Y_\sigma)$, on which $\gamma = (\gamma_\sigma)_{\sigma \in I} \in GL_2(\mathcal{O})^I$ acts by
\[ (2) \quad \gamma \cdot P := \det(\gamma)^v P((X_\sigma, Y_\sigma)^t, \gamma_\sigma^t)_{\sigma \in I}. \]
Here we define \[ \det(\gamma)^v := \prod_{\sigma \in I} \det(\gamma_\sigma)^{v_\sigma} \]
and for a $2 \times 2$ matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put $x^\sigma := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

**Definition 1.1.** For $(n, v) \in W^c$ and an $\mathcal{O}$-algebra $A$, we put
\[ S^c_{(n,v)}(G; \Gamma_1(N); A) := \{ f : G(\mathbb{Q}) \backslash G(A_f) \rightarrow L(n, v; A) : \text{ function } | f(xu) = u^{-1} \cdot f(x) \text{ for } u \in \Gamma_1(N), x \in G(A_f) \}, \]
which we call the space of classical automorphic forms of level $\Gamma_1(N)$ and weight $(n, v)$ on $G$ (defined over $A$).
Remark 1.1. In the case where we regard $A = \mathbb{C}$ as an $\mathcal{O}$-algebra via the fixed isomorphism $\mathbb{C}_p \sim \mathbb{C}$ and $B$ is unramified at all finite places of $F$ (hence $g$ must be even by Hasse principle (cf. [15, XIII, Sections 3 and 6])), it is known that $S^{\text{cl}}_{(n,v)}(G; \Gamma_1(N); \mathbb{C})$ are isomorphic to the spaces of classical holomorphic Hilbert cusp forms of weight $(n_\sigma + 2)_{\sigma \in I}$ and level $N$ by a result of Jacquet-Langlands and Shimizu (cf. [10, Theorem 2.1]).

1.2. $p$-Adic automorphic forms

We fix a classical weight $(n, v) \in W^{\text{cl}}$. Let $N$ be an integral ideal of $F$ which is not prime to $p$ and unramified in $B$. We now take arbitrarily $s(\leq r)$ prime ideals above $p$ which divide $N$. We may denote them by $p_1, \ldots, p_s$. We then put $I' := \bigsqcup_{i=1}^s I_i \subset I$ and denote the cardinality of $I'$ by $g'(\leq g)$. We fix a prime element $\pi_i$ of the $p_i$-adic completion $F_{p_i}$ of $F$ at $p_i$ for each $i = 1, \ldots, s$. We then denote by $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ the element of $G(A_\mathfrak{p})$ whose $p_i$-part is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & \pi_i \end{pmatrix}$ for each $i = 1, \ldots, s$ and other parts are trivial. In the following, for an element $\gamma \in \Gamma_1(N)$, we write its $\sigma$-projection as $\gamma_\sigma = \begin{pmatrix} 1 + \pi_i^\sigma a_\sigma & b_\sigma \\ \pi_i^\sigma c_\sigma & 1 + \pi_i^\sigma d_\sigma \end{pmatrix}$ with some $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathcal{O}$ for each $\sigma \in I$ with $i$ such that $\sigma \in I_i$. Then we have

(3) $\begin{pmatrix} X_\sigma, Y_\sigma \end{pmatrix}^t \gamma_\sigma^t = \begin{pmatrix} X_\sigma - b_\sigma Y_\sigma + \pi_i^\sigma (a_\sigma Y_\sigma - c_\sigma X_\sigma) \end{pmatrix}$

for all $\sigma \in I'$ with $i$ such that $\sigma \in I_i$, and

(4) $\begin{pmatrix} X_\sigma, Y_\sigma \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}_{\sigma}^t = \begin{cases} \begin{pmatrix} \pi_i^\sigma X_\sigma, \pi_i^\sigma Y_\sigma \\ X_\sigma, Y_\sigma \end{pmatrix} & (\sigma \in I_i \subset I'), \\ \begin{pmatrix} X_\sigma, Y_\sigma \end{pmatrix} & (\sigma \in I \setminus I') \end{cases}$.

For any elements $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \gamma_2$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$ of the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$, using actions (3) and (4), we define a $K$-endomorphism $[\gamma]_{(n,v)}$ on $L(n,v; K)$ with normalization of the $\det^v$-part by

(5) $[\gamma]_{(n,v)} \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_\tau)^{v_\tau} \prod_{\sigma \in I'} \det(\gamma_1\sigma \gamma_2\sigma)^{v_\sigma} \times P(((X_\tau, Y_\tau)^t \gamma_\tau^t)_{\tau \in I})$. 
Let $K\langle x_{\sigma} | \sigma \in I' \rangle$ be the strictly convergent power series ring of $g'$-variables $(x_{\sigma})_{\sigma \in I'}$ with coefficients in $K$, which is the subring of the formal power series ring $K[x_{\sigma} | \sigma \in I']$ consisting of power series $P(x) = \sum_{(i_{\sigma})_{\sigma \in I'} \in \mathbb{Z}_{\geq 0}}^{(i_{\sigma})_{\sigma \in I'} \in \mathbb{Z}_{\geq 0}} a_{(i_{\sigma})_{\sigma \in I'}} \prod_{\sigma \in I'} x_{\sigma}^{i_{\sigma}}$ such that $|a_{(i_{\sigma})_{\sigma \in I'}}| \to 0$ as $\sum_{\sigma \in I'} i_{\sigma} \to \infty$. This is an orthonormalizable $K$-Banach algebra with sup norm $| \cdot |$ with respect to coefficients in $K$ (for the notion in the $p$-adic Banach theory, see [6, Chapter A]). We can take the set $\{ \prod_{\sigma \in I'} x_{\sigma}^{i_{\sigma}} | i_{\sigma} \geq 0, \sigma \in I' \}$ as an orthonormal basis of $K\langle x_{\sigma} | \sigma \in I' \rangle$. We define actions on the variables $(x_{\sigma})_{\sigma \in I'}$ of the $\sigma$-projections of $\gamma \in \Gamma_{1}(N)$ and $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ for $\sigma \in I'$ as follows:

$$
(6) \gamma_{\sigma} \cdot x_{\sigma} := \frac{-b_{\sigma} + (1 + \pi_{i}^{\sigma} d_{\sigma}) x_{\sigma}}{1 + \pi_{i}^{\sigma} (a_{\sigma} - c_{\sigma} x_{\sigma})} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \cdot x_{\sigma} := \pi_{i}^{\sigma} x_{\sigma}
$$

with $i$ such that $\sigma \in I_{i}$. Note that the denominator $1 + \pi_{i}^{\sigma} (a_{\sigma} - c_{\sigma} x_{\sigma})$ in the action (6) is a unit in $\mathcal{O} \langle x_{\sigma} \rangle$. Then by [6, Lemma A1.6], we see that elements in the double coset $\Gamma_{1}(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_{1}(N)$ give completely continuous $K$-endomorphisms on $K\langle x_{\sigma} | \sigma \in I' \rangle$ whose operator norms are at most 1. Here the operator norm $|L|$ of a continuous endomorphism $L$ on a Banach module $M$ is defined by

$$
|L| := \sup_{0 \neq m \in M} \frac{|L(m)|}{|m|}.
$$

Now we define a Banach module $S$ over the strictly convergent power series ring $K\langle \xi_{\sigma} | \sigma \in I' \rangle$ of $g'$-variables $(\xi_{\sigma})_{\sigma \in I'}$ as follows: $S$ is the set of polynomials $P$ of $2(g - g')$-parameters $(X_{\tau}, Y_{\tau})_{\tau \in I \setminus I'}$ with coefficients in $K\langle \xi_{\sigma}, x_{\sigma} | \sigma \in I' \rangle$ which are homogeneous of degree $n_{\tau}$ for each variable $(X_{\tau}, Y_{\tau})$. We can take the set

$$
\{( \prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}} ) \prod_{\sigma \in I'} x_{\sigma}^{m_{\sigma}} | a_{\tau} + b_{\tau} = n_{\tau} \text{ with } a_{\tau}, b_{\tau} \geq 0, m_{\sigma} \geq 0 \}
$$

as an orthonormal basis of $S$ over $K\langle \xi_{\sigma} | \sigma \in I' \rangle$. Let $e(p_{i})$ be the ramification index of the prime ideal $p_{i}$ in $F/\mathbb{Q}$. In order to define an action of $\Gamma_{1}(N)$ on $S$, we assume the condition that

$$
(\text{ram}) \quad e(p_{i}) < p - 1 \quad \text{for each } i = 1, \ldots, s
$$

is satisfied in the following. We see that $j_{\sigma}(\gamma_{\sigma})$ for elements $\gamma$ of $\Gamma_{1}(N)$ and $\Gamma_{1}(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_{1}(N)$, and $\det(\gamma_{\sigma})$ for $\gamma \in \Gamma_{1}(N)$ are of the form $1 + \pi_{i}^{\sigma} a$ with some $a \in \mathcal{O}$ for each $\sigma \in I'$ with $i$ such that $\sigma \in I_{i}$. Then
we can define their powers with any element $s$ in $\mathbb{C}_p$ (resp. $\mathbb{C}_p(\xi_\sigma)$) such that $|s| \leq 1$ by a convergent power series as
\begin{equation}
(1 + \pi^a \sigma)^s := 1 + \sum_{k \geq 1} \frac{s(s-1)\cdots(s-k+1)}{k!}(\pi^a \sigma)^k a^k
\end{equation}

in $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\mathcal{O}_{\mathbb{C}_p}(\xi_\sigma)$) because of the assumption (ram) (cf. [4, Lemme 3.6.1]). Here we denote by $\mathcal{O}_{\mathbb{C}_p}$ the ring of $p$-adic integers in $\mathbb{C}_p$, i.e., the subring of $\mathbb{C}_p$ consisting of elements $s$ such that $|s| \leq 1$. We then define an action $[\gamma]$ of $\gamma \in \Gamma_1(N)$ on $S$ as
\begin{equation}
[\gamma] \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_\tau} \left( \prod_{\sigma \in I'} j_\sigma(\gamma_{\sigma})^s \det(\gamma_{\sigma})^{\mu(n,v)-s_\sigma} \right)
\end{equation}
\begin{equation}
\times P(((X_\tau, Y_\tau)^t \gamma_{\tau}^\iota)_{\tau \in I \setminus I'}; (\xi_\sigma, \gamma_\sigma \cdot x_\sigma)_{\sigma \in I'}).\end{equation}

As for $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \gamma_2 \in \Gamma_1(N)$, $\Gamma_1(N)$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$, we define a $K\langle\xi_\sigma | \sigma \in I'\rangle$-endomorphism on $S$ as
\begin{equation}
[\gamma] \cdot P := \prod_{\tau \in I \setminus I'} \det(\gamma_{\tau})^{v_\tau} \left( \prod_{\sigma \in I'} j_\sigma(\gamma_{\sigma})^s \det(\gamma_{\sigma\sigma})^{\mu(n,v)-s_\sigma} \right)
\end{equation}
\begin{equation}
\times P(((X_\tau, Y_\tau)^t \gamma_{\tau}^\iota)_{\tau \in I \setminus I'}; (\xi_\sigma, \gamma_\sigma \cdot x_\sigma)_{\sigma \in I'}).\end{equation}

which is completely continuous with operator norm $\leq 1$.

**Definition 1.2.** We denote by $\mathcal{W}_{(n,v)}$ the $g'$-dimensional closed affinoid ball over $K$ of radius 1 around $(n_\sigma)_{\sigma \in I'}$. Then the set $\mathcal{W}_{(n,v)}(\mathbb{C}_p)$ of its $\mathbb{C}_p$-valued points coincides with $\mathcal{O}_{\mathbb{C}_p}^{I'}$ and $K\langle\xi_\sigma | \sigma \in I'\rangle$ is the affinoid algebra associated to $\mathcal{W}_{(n,v)}$. (For the details of affinoid algebras and affinoid varieties, see [1, Part B and Chapter 7] and [6, Section A5].) We call it the space of the $I'$-parts of $p$-adic weights associated to $(n,v)$. We then associate $(t_\sigma := \frac{\mu(n,v)-s_\sigma}{2})_{\sigma \in I'}$ to any point $(s_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$, and put the $p$-adic weight $(s,t)$ as
\begin{equation}
s := \sum_{\sigma \in I'} s_\sigma \sigma + \sum_{\tau \in I \setminus I'} n_\tau \tau \quad \text{and} \quad t := \frac{\mu(n,v)t_0 - s}{2} = \sum_{\sigma \in I'} t_\sigma \sigma + \sum_{\tau \in I \setminus I'} v_\tau \tau.
\end{equation}

Further, we denote by $\mathcal{W}_{(n,v)}^{cl}$ the subset of $\mathcal{W}_{(n,v)}(\mathbb{C}_p)$ consisting of elements $(n'_\sigma)_{\sigma \in I'}$ whose components are positive integers of the same parity as $\mu(n,v)$ for all $\sigma \in I'$. We call it the set of the $I'$-parts of classical weights associated to $(n,v)$. For $(n'_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n,v)}^{cl}$, we put $(v'_\sigma := \frac{\mu(n,v)-n'_\sigma}{2})_{\sigma \in I'}$ and define $(n', v')$ as well as $(s, t)$. By the definition
of \(W_{(n,v)}^{cl}\), we see that \(v_{\sigma}'\) are also integers for all \(\sigma \in I'\) and that 
\[n' + 2v' = \mu(n, v)t_{0}.\]

For \((s_{\sigma})_{\sigma \in I'} \in W_{(n,v)}(\mathbb{C}_{p})\), we denote by \(K_{(s,t)}\) the \(p\)-adic completion in \(\mathbb{C}_{p}\) of the fraction field of \(K(\xi_{\sigma}|\sigma \in I')/(\xi_{\sigma}-s_{\sigma}|\sigma \in I')\). We denote by \(S_{(s,t)}\) the specialized orthonormalizable \(K_{(s,t)}\)-Banach space \(S \otimes_{K(\xi_{\sigma}|\sigma \in I')} K_{(s,t)}\). Then we denote by \([\gamma]_{(s,t)}\) the specialized \(K_{(s,t)}\)-endomorphism 
\[\gamma \otimes K_{(s,t)}\] on \(S_{(s,t)}\) for elements \(\gamma\) of \(\Gamma_{1}(N)\) and \(\Gamma_{1}(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_{1}(N)\).

**Definition 1.3.** (1) Assume the condition (ram). We define the space of \(p\)-adic automorphic forms of level \(\Gamma_{1}(N)\) on \(G\) (with coefficients in \(K\)) as
\[S(G; \Gamma_{1}(N)) := \{f : G(\mathbb{Q}) \backslash G(A_{f}) \rightarrow S : \text{function}\|
\quad f(xu) = [u^{-1}]f(x), u \in \Gamma_{1}(N), x \in G(A_{f})\}\].

We then have a \(K\)-isomorphism
\[(10) \quad S(G; \Gamma_{1}(N)) \sim \oplus_{i=1}^{h} S_{\Gamma_{i}}, f \mapsto (f(t_{1}), \ldots, f(t_{h}))\],
where \(t_{1}, \ldots, t_{h} \in G(A)\) are the fixed representatives of the decomposition (1). Here each \(S_{\Gamma_{i}}\) is the submodule of the orthonormalizable \(K(\xi_{\sigma}|\sigma \in I')\)-module \(S\) consisting of elements fixed under the action of \(\Gamma_{i} = (t_{i}^{-1}G(\mathbb{Q})t_{i}) \cap \Gamma_{1}(N)G(\mathbb{R})_{+}\). Since \(\Gamma_{i}\) acts on \(S\) via the finite quotient group \(\Gamma_{i}/\Gamma_{i} \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times}\) because of the assumption \(n + 2v \sim 0\), we then see that \(S_{\Gamma_{i}}\) satisfies the property (Pr) of [3, Section 2] for each \(i = 1, \ldots, h\). We now define a norm in \(S(G; \Gamma_{1}(N))\) via this isomorphism as
\[|f| := \sup_{1 \leq i \leq h} |f(t_{i})|\].
Therefore, \(S(G; \Gamma_{1}(N))\) can be regarded as a \(K(\xi_{\sigma}|\sigma \in I')\)-Banach module with the norm \(|\cdot|\) which satisfies the property (Pr) of [3, Section 2].

(2) Let \((s_{\sigma})_{\sigma \in I'} \in W_{(n,v)}(\mathbb{C}_{p})\). Assume the condition (ram) in the case where \((s_{\sigma})_{\sigma \in I'} \notin W_{(n,v)}^{cl}\). We define the space of \(p\)-adic automorphic forms of weight \((s, t)\) and level \(\Gamma_{1}(N)\) on \(G\) (defined over \(K_{(s,t)}\)) as
\[S_{(s,t)}(G; \Gamma_{1}(N)) := \{f : G(\mathbb{Q}) \backslash G(A_{f}) \rightarrow S_{(s,t)} : \text{function}\|
\quad f(xu) = [u^{-1}]_{(s,t)}f(x), u \in \Gamma_{1}(N), x \in G(A_{f})\}\].

Then we have an isomorphism
\[(11) \quad S_{(s,t)}(G; \Gamma_{1}(N)) \sim \oplus_{i=1}^{h} S_{(s,t)}^{\Gamma_{i}}, f \mapsto (f(t_{1}), \ldots, f(t_{h}))\]
of $K_{(s,t)}$-Banach spaces satisfying the property $(Pr)$ of [3, Section 2],
where we define a norm in $S_{(s,t)}(G; \Gamma_{1}(N))$ as
\[ |f| := \sup_{1 \leq i \leq h} |f(t_{i})|. \]

Putting $x_{\sigma} = \frac{X_{\sigma}}{Y_{\sigma}}$ for each $\sigma \in I'$, we then see easily the following

**Lemma 1.1.** For any $(n', v')_{\sigma \in I'} \in W_{(n,v)}^{cl}$, we have a natural $K$-inclusion
\[ L(n', v'; K) \hookrightarrow S_{(n', v')} \]
which is compatible with $[\gamma]|_{(n', v')}$ for all $\gamma$ in $\Gamma_{1}(N)$ and the double coset
$\Gamma_{1}(N) \left( \begin{array}{cc} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_{1}(N)$ on these spaces. Thus we have an inclusion
\[ S^{cl}_{(n', v')} (G; \Gamma_{1}(N); K) \hookrightarrow S_{(n', v')} (G; \Gamma_{1}(N)) \]
of $K$-Banach spaces satisfying the property $(Pr)$ of [3, Section 2].

2. $p$-Adic automorphic forms of small $T(\pi)$-slope

Let the notation be as in Section 1.2. In this section, we shall introduce the Hecke operator $T(\pi)$ on the spaces of $p$-adic automorphic forms. Then we shall investigate some properties of $p$-adic automorphic forms having small $T(\pi)$-slope.

2.1. **The Hecke operator $T(\pi)$**

In this subsection, we assume the condition (ram), i.e., $e(p_{i}) < p - 1$ for all $i = 1, \ldots, s$, unless we deal with the $I'$-parts of classical weights in $W^{cl}_{(n,v)}$. In order to define the Hecke operator $T(\pi)$, we decompose the double coset $\Gamma_{1}(N) \left( \begin{array}{cc} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_{1}(N)$ in a disjoint union of right cosets as
\[ \Gamma_{1}(N) \left( \begin{array}{cc} 1 & 0 \\ 0 & \pi \end{array} \right) \Gamma_{1}(N) = \bigsqcup_{i=1}^{l} \zeta_{i} \Gamma_{1}(N). \]

For $f \in S(G; \Gamma_{1}(N))$ (resp. $S_{(s,t)}(G; \Gamma_{1}(N))$ for $(s_{\sigma})_{\sigma \in I'} \in W_{(n,v)}(C_{p})$), we put
\[ (f|T(\pi))(x) := \sum_{i=1}^{l} [\zeta_{i}] \cdot f(x\zeta_{i}) \] (resp. $\sum_{i=1}^{l} [\zeta_{i}]_{(s,t)} \cdot f(x\zeta_{i})$)
for $x \in G(\mathbb{Q}) \backslash G(A_{f})$. Note that this definition is independent of choices of representatives $\{\zeta_{i}\}$ and $f|T(\pi)$ is also an element of $S(G; \Gamma_{1}(N))$ (resp. $S_{(s,t)}(G; \Gamma_{1}(N))$) (cf. [10, Section 2]).
Proposition 2.1. Assume the condition (ram) unless $(s_\sigma)_{\sigma \in I'} \in W_{(n,v)}^{c1}$. The Hecke operator $T(\pi)$ is completely continuous on $S(G; \Gamma_1(N))$ and $S(s,t)(G; \Gamma_1(N))$ for any $(s_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n,w)}(\mathbb{Q})$ with operator norm $\leq 1$.

Proof. We shall prove the proposition for $S(G; \Gamma_1(N))$, because we can prove in the case of $S(s,t)(G; \Gamma_1(N))$ as well. To see the complete continuity of $T(\pi)$, we calculate the action of $T(\pi)$ on $\bigoplus_{j=1}^{h}S^{\Gamma_j}$ via the isomorphism (10) by means of the decomposition

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N) = \bigsqcup_{i=1}^{l} \zeta_i \Gamma_1(N).$$

For $f \in S(G; \Gamma_1(N))$, the image of $f|T(\pi)$ under the isomorphism (10) is

$$(f|T(\pi))(t_1), \ldots, (f|T(\pi))(t_h)$$

$$= \sum_{i=1}^{l} ([\zeta_i] \cdot f(t_1 \zeta_i), \ldots, [\zeta_i] \cdot f(t_h \zeta_i)).$$

We fix $1 \leq i \leq l$. For each $j = 1, \ldots, h$, there exist $1 \leq \sigma_{i}(j) \leq h$ and $u_{i}(j) \in \Gamma_1(N)$ such that

$$t_{j} \zeta_{i} = t_{\sigma_{i}(j)} u_{i}(j)$$

in $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)$. Then we see that

$$f(t_{j} \zeta_{i}) = f(t_{\sigma_{i}(j)} u_{i}(j)) = [u_{i}(j)^{-1}] \cdot f(t_{\sigma_{i}(j)})$$

by the definition of automorphic forms of level $\Gamma_1(N)$. Therefore we see that

$$(f|T(\pi))(t_1), \ldots, (f|T(\pi))(t_h)$$

$$= \sum_{i=1}^{l} ([\zeta_i u_{i}(1)^{-1}] \cdot f(t_{\sigma_{i}(1)}), \ldots, [\zeta_i u_{i}(h)^{-1}] \cdot f(t_{\sigma_{i}(h)})),$$

Thus the proposition is proven, because the endomorphisms $[\cdot]$ given by the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$ on $S$ are completely continuous with operator norm $\leq 1$. \(\square\)

We denote by $K\langle \xi_\sigma | \sigma \in I' \rangle \{\{X\}\}$ the subring of the formal power series ring $K\langle \xi_\sigma | \sigma \in I' \rangle [X]$ consisting of power series $\sum_{i \geq 0} c_{i} X^{i}$ such that

$$|c_{i}| M^{i} \to 0 \quad \text{as} \quad i \to \infty$$
for all $M \in \mathbb{R}$. By Proposition 2.1 and the arguments in [3, Section 2] dealing with Banach modules satisfying the property $(Pr)$, we have the following

**Proposition 2.2.** Assume the condition (ram). We have the characteristic power series

$$P((\xi_{\sigma})_{\sigma \in I'}, X) = \det(1 - XT(\pi)|_{S(G; \Gamma_{1}(N))})$$

$$= 1 + \sum_{i \geq 1} c_i X^i \in K(\xi_{\sigma} | \sigma \in I')\{X\}$$

of $T(\pi)$ on $S(G; \Gamma_{1}(N))$ with $|c_i| \leq 1$. Furthermore, for any $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}\mathbb{C}_{p}$, we see that

$$P((s_{\sigma})_{\sigma \in I'}, X) = 1 + \sum_{i \geq 1} c_i ((s_{\sigma})_{\sigma \in I'}) X^i \in K_{(s,t)}\{X\}$$

is the characteristic power series of $T(\pi)$ on $S_{(s,t)}(G; \Gamma_{1}(N))$.

Let $\alpha$ be a non-negative rational number. For $(s_{\sigma})_{\sigma \in I'} \in \mathcal{W}_{(n,v)}\mathbb{C}_{p}$, let $S_{(s,t)}(G; \Gamma_{1}(N))\mathbb{C}_{p}^{\alpha}$ be the $\mathbb{C}_{p}$-subspace of $S_{(s,t)}(G; \Gamma_{1}(N)) \otimes_{K_{(s,t)}} \mathbb{C}_{p}$ generated by all generalized $T(\pi)$-eigenspaces for all eigenvalues $\lambda$ such that $\text{ord}_{p}(\lambda) = \alpha$. In the following subsections, we shall investigate $p$-adic automorphic forms which have small $T(\pi)$-slope.

### 2.2. Classicality of $p$-adic automorphic forms

In Lemma 1.1 without the condition (ram), we have seen that the spaces of classical automorphic forms are included in the ones of $p$-adic automorphic forms. Now we shall see that $p$-adic automorphic forms of small $T(\pi)$-slope are classical. Namely,

**Theorem 2.3.** Let $\alpha \in \mathbb{Q}_{\geq 0}$ and $(n'_{\sigma})_{\sigma \in I'} \in W_{(n,v)}^{c}$. If the condition

$$\alpha < \nu_{n'} := \min_{1 \leq i \leq s} \left\{ \frac{1}{e(p_{i})} (\min_{\sigma \in I_i} \{n'_{\sigma}\} + 1) \right\}$$

is satisfied, then we have (without the condition (ram))

$$S_{(n',v')}(G; \Gamma_{1}(N))\mathbb{C}_{p}^{\alpha} = S_{(n',v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})^{\alpha}.$$

**Proof.** By the isomorphism (11) in Section 1, we see that the $\mathbb{C}_{p}$-Banach quotient space $(S_{(n',v')}(G; \Gamma_{1}(N)) \otimes_{K} \mathbb{C}_{p})/S_{(n',v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})$ is isomorphic to a direct summand of the direct sum of $h$-copies of the orthonormalizable $\mathbb{C}_{p}$-Banach quotient space $S_{(n',v')} \otimes_{K} \mathbb{C}_{p}/L(n', v'; \mathbb{C}_{p})$.
whose orthonormal basis is
\[
\{( \prod_{\tau \in I \setminus I'} X_{\tau}^{a_{\tau}} Y_{\tau}^{b_{\tau}}) \prod_{\sigma \in I'} x_{\sigma}^{m_{0}} \mid a_{\tau} + b_{\tau} = n_{\tau} \text{ with } a_{\tau}, b_{\tau} \geq 0, m_{\sigma} \geq 0 \\
\text{and } m_{\sigma} > n'_{\sigma} \text{ for some } \sigma \}.
\]

By the actions (3), (4) and (6) on the variables \(X_{\tau}, Y_{\tau}\) and \(x_{\sigma}\) in Section 1.2, we then see easily that
\[|T(\pi)| \leq p^{-\nu_{n'}}\]
on \((S_{(n',v')} \otimes_{K} \mathbb{C}_{p}/L(n', v'; \mathbb{C}_{p}))^{h}\). Hence we see that if \(\alpha < \nu_{n'}\), then the image of any generalized \(T(\pi)\)-eigenvector of slope \(\alpha\) is 0 in the quotient space \((S_{(n',v')} (G; \Gamma_{1}(N)) \otimes_{K} \mathbb{C}_{p}) / S_{(n',v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})\). So we have
\[S_{(n',v')} (G; \Gamma_{1}(N))^{\alpha}_{\mathbb{C}_{p}} = S_{(n',v')}^{cl}(G; \Gamma_{1}(N); \mathbb{C}_{p})^{\alpha}.
\]

\[\square\]

**Remark 2.1.** It is known that the spaces of definite quaternionic automorphic forms over \(\mathbb{Q}\) defined by means of homogeneous polynomials of degree \(n\) are isomorphic to the spaces of elliptic cusp forms of weight \(k = n + 2\) by Jacquet-Langlands' theorem (cf. [2, Theorem 2]). Coleman [5, Theorem 6.1 and Theorem 8.1] showed that \(p\)-adic overconvergent modular forms of weight \(k\) and \(U_{p}\)-slope \(\alpha\) are classical if \(\alpha < k - 1(= n + 1)\). Since \(s = 1\) and \(e(p) = 1\) in the case of \(F = \mathbb{Q}\), Theorem 2.3 is a generalization of the result of Coleman to the case over totally real fields.

### 2.3. The local constancy of \(\dim_{\mathbb{C}_{p}} S_{(s,t)}(G; \Gamma_{1}(N))^{\alpha}_{\mathbb{C}_{p}}\)

We assume the condition (ram), i.e., \(e(p_{i}) < p - 1\) for all \(i = 1, \ldots, s\). Let \(\alpha \in \mathbb{Q}_{\geq 0}\). In this subsection, we shall give an explicit description of \(m(\alpha)\) such that if \((s_{\sigma})_{\sigma \in I'}, (s'_{\sigma})_{\sigma \in I'}\) \(\in \mathcal{W}_{(n,v)} (\mathbb{C}_{p})\) satisfy that \(|s_{\sigma} - s'_{\sigma}| \leq p^{-m(\alpha)}\) for all \(\sigma \in I'\), then we have
\[
\dim_{\mathbb{C}_{p}} S_{(s,t)}(G; \Gamma_{1}(N))^{\alpha}_{\mathbb{C}_{p}} = \dim_{\mathbb{C}_{p}} S_{(s',t')}(G; \Gamma_{1}(N))^{\alpha}_{\mathbb{C}_{p}}
\]
by applying Chenevier's argument in [4, Section 5] to our case.

By Definition 1.3 (2), we regard \(S_{(s,t)}(G; \Gamma_{1}(N))\) as a direct summand of the orthonormalizable \(K_{(s,t)}\)-Banach module \(S_{(s,t)}^{h}\) for which we can also have the characteristic power series
\[P'(s_{\sigma})_{\sigma \in I'}, X) = 1 + \sum_{i \geq 1} c_{i}'((s_{\sigma})_{\sigma \in I'}) X^{i} \in K_{(s,t)} \{\{X\}\}.
\]
with $|c^\prime_s((s_\sigma)_{\sigma\in I'})| \leq 1$. To obtain $m(\alpha)$ as above; we shall investigate the Newton polygon $N'_s(t)$ of $P((s_\sigma)_{\sigma\in I'}, X)$. We can take the set
\[ \{e_{M,a} := (0, \ldots, M, \ldots, 0) \}_{M \in \mathfrak{M}, \ 1 \leq a \leq h} \]
as an orthonormal basis of $S^h_s(t)$, where we put the set of monomials
\[ \mathfrak{M} := \{(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_\tau} Y_{\tau}^{b_\tau}) \prod_{\sigma \in I'} x^{m_\sigma} | a_\tau + b_\tau = n_\tau \text{ with } a_\tau, b_\tau \geq 0, m_\sigma \geq 0 \} \]
and $M$ sits in the $a$-th component in $e_{M,a}$. We shall calculate the $p$-adic valuations of coefficients $c^\prime_s((s_\sigma)_{\sigma\in I'})$ of $P((s_\sigma)_{\sigma\in I'}, X)$ by means of this basis. For $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \gamma_2 \in \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$ and a monomial $M = (\prod_{\tau \in I \setminus I'} X_{\tau}^{a_\tau} Y_{\tau}^{b_\tau}) \prod_{\sigma \in I'} x^{m_\sigma} \in \mathfrak{M}$, we have
\[ (13) \ [\gamma]_{(s,t)} \cdot M = \prod_{\tau \in I \setminus I'} \det(\gamma_\tau)^{v_\tau}(\prod_{\sigma \in I'} j_\sigma(\gamma_\sigma)^{s_\sigma} \det(\gamma_1 \gamma_2 \gamma)^{b_\sigma}) \times ((\prod_{\tau \in I \setminus I'} X_{\tau}^{a_\tau} Y_{\tau}^{b_\tau})^{t_\tau} \gamma_\tau^{b_\tau}) \prod_{\sigma \in I'} (\gamma_\sigma \cdot x_\sigma)^{m_\sigma}. \]
By the definition of $j_\sigma(\gamma_\sigma)^{s_\sigma}$ and the action (6) on the variable $x_\sigma$ in Section 1.2 for each $\sigma \in I'$, we see that the $p$-adic valuations of all coefficients of monomials of the form $(\prod_{\tau \in I \setminus I'} X_{\tau}^{a_\tau} Y_{\tau}^{b_\tau}) \prod_{\sigma \in I'} x^{k_\sigma}$ in the expansion of (13) in $S(s,t)$ are at least $\lambda \sum_{\sigma \in I'} k_\sigma$, where we put the positive rational number $\lambda := \min_{1 \leq i \leq s} \{\frac{1}{e(\mathfrak{p}_i)}\} - \frac{1}{p-1}$. Now we order the basis $\{e_{M,a}\}_{M,a}$ as follows: For $k \geq 0$, we define the subset
\[ A_k := \{e_{M,a}|1 \leq a \leq h, M \text{ is of the form} \ (\prod_{\tau \in I \setminus I'} X_{\tau}^{a_\tau} Y_{\tau}^{b_\tau}) \prod_{\sigma \in I'} x^{k_\sigma} \text{ with} \sum_{\sigma \in I'} h_\sigma = k \} \]
of $\{e_{M,a}\}_{M,a}$. Then we see that the cardinality $\# A_k = h_n^{(k+g'-1)}$ for $k \geq 0$, where $h_n := h \prod_{\tau \in I \setminus I'} (n_\tau + 1)$, and that for $k \geq 1$,
\[ (14) \ \sum_{q=0}^{k} q \cdot \# A_q = h_n g' \binom{k + g'}{g' + 1}. \]
We then exhibit elements of $A_0$ as $e^{(0)}_1, \ldots, e^{(0)}_{h_n}$ arbitrarily. Next we exhibit elements of $A_1$ as $e^{(1)}_{h_n+1}, \ldots, e^{(1)}_{h_n(g'+1)}$ arbitrarily. We then repeat this operation for all $k \geq 2$ as
\[ e^{(k)}_{h_n^{(k+g'-1)}+1}, \ldots, e^{(k)}_{h_n^{(k+g')}}. \]
We are going to obtain the representation matrix of infinite degree of $T(\pi)$ with respect to the basis $\{e^{(l)}_{j}\}_{j,l}$ ordered as above. For each $e^{(l)}_{j}$, we write

$$e^{(l)}_{j}|T(\pi) = \sum_{i_{0}=1}^{h_{n}}\alpha^{(0)}_{i_{0}}(j, l)e^{(0)}_{i_{0}} + \sum_{k\geq 1} \sum_{i_{k}=h_{n}^{(g'+1)}}^{h_{n}^{(g'+1)-1}} \alpha^{(k)}_{i_{k}}(j, l)e^{(k)}_{i_{k}}$$

with $\alpha^{(k)}_{i_{k}}(j, l) \in O_{(s,t)}$ for all $k \geq 0$, where $O_{(s,t)}$ is the ring of integers in $K_{(s,t)}$. As mentioned above, we then see that

$$(15) \quad \text{ord}_{p}(\alpha^{(k)}_{i_{k}}(j, l)) \geq k\lambda$$

for all $k \geq 0$, $j \geq 1$ and $l \geq 0$. The representation matrix of $T(\pi)$ with respect to the ordered basis $\{e^{(0)}_{1}, \ldots, e^{(0)}_{h_{n}}, \ldots\}$ is of the form

$$\begin{pmatrix}
\alpha^{(0)}_{1}(1, 0) & \cdots & \alpha^{(0)}_{1}(h_{n}, 0) & \cdots \\
\vdots & & \vdots & \\
\alpha^{(0)}_{h_{n}}(1, 0) & \cdots & \alpha^{(0)}_{h_{n}}(h_{n}, 0) & \cdots \\
\vdots & & \vdots & \\
\end{pmatrix}$$

It is known that the coefficient $c^{i}_{i}'((s_{\sigma})_{\sigma \in I'})$ of $P'((s_{\sigma})_{\sigma \in I'}, X)$ is given by $(-1)^{i} \times$ (the convergent sum of $i$-th minors of the above matrix) for each $i \geq 1$ (cf. [13, Proposition 7 (a)]). So we see easily that

$$\text{ord}_{p}(c^{i}_{i}'((s_{\sigma})_{\sigma \in I'})) > i^{1+\frac{1}{g}} \frac{2\lambda g'}{(g'+1)(g'+2)^{2}} \frac{g'^{l}}{h_{n}}$$

by (14) and (15) in the case where

$$h_{n}^{(k+g'-1)} + 1 \leq i \leq h_{n}^{(k+g')}$$

with some $k \geq 2$. On the other hand, in the case where $1 \leq i \leq h_{n}(g'+1)$, we see that $\text{ord}_{p}(c^{i}_{i}'((s_{\sigma})_{\sigma \in I'})) \geq 0$ by Proposition 2.2. Therefore we have

$$(16) \quad \text{ord}_{p}(c^{i}_{i}'((s_{\sigma})_{\sigma \in I'})) \geq \frac{2\lambda g'}{(g'+1)(g'+2)^{2}} \frac{g'^{l}}{h_{n}} \frac{1}{i^{\frac{1}{g'}}}$$

for all $i \geq 1$. We put the function

$$\mu(x) := \frac{2\lambda g'}{(g'+1)(g'+2)^{2}} \left( \frac{g'^{l}}{h_{n}} \right)^{\frac{1}{g'}} x^{\frac{1}{g'}} - \left( h_{n}(g'+1) \right)^{\frac{1}{g'}}$$
on $\mathbb{R}_{\geq 0}$, which is a monotone increasing function. Since the Newton polygon $N_{(s,t)}$ of the characteristic power series $P((s_\sigma)_{\sigma \in I'}, X)$ of $T(\pi)$ acting on $S_{(s,t)}(G; \Gamma_1(N))$ is bounded by $N'_{(s,t)}$ from the bottom, we then obtain the following

**Proposition 2.4.** Assume the condition (ram). Then we have

$$N_{(s,t)}(x) \geq \mu(x)$$

for all $(s_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{Q})$ and $x \in \mathbb{R}_{\geq 0}$.

Secondly, the characteristic power series $P((\xi_\sigma)_{\sigma \in I'}, X)$ for $T(\pi)$ on $S(G; \Gamma_1(N))$ shall be investigated. The coefficients $c_i \in K\langle \xi_\sigma | \sigma \in I'\rangle$ of $P((\xi_\sigma)_{\sigma \in I'}, X)$ can be regarded as analytic functions on $\mathcal{W}(n,v)$. We then have the following

**Proposition 2.5.** Assume the condition (ram). We take two elements $(s_\sigma)_{\sigma \in I'}, (s'_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n,v)}(\mathbb{Q})$. We assume that there exists an integer $m \geq 0$ such that

$$|s_\sigma - s'_\sigma| \leq p^{-m \cdot \max_{1 \leq i \leq s}\{\frac{1}{\epsilon(\mathfrak{p}_i)}\}}$$

for all $\sigma \in I'$. Then we have

$$|c_i((s_\sigma)_{\sigma \in I'}) - c_i((s'_\sigma)_{\sigma \in I'})| \leq p^{-(m + \lambda') \min_{1 \leq i \leq s}\{\frac{1}{\epsilon(\mathfrak{p}_i)}\}}$$

for all $i \geq 1$, where we put $\lambda' := \min_{1 \leq i \leq s}\{1 - \frac{e(\mathfrak{p}_i)}{p-1}\}$.

**Proof.** Since $S(G; \Gamma_1(N))$ can be regarded as a direct summand of $S^h$ via the isomorphism (10) in Definition 1.3 (1), it is enough to show the statement for the coefficients $c'_i$ of the characteristic power series $P'((\xi_\sigma)_{\sigma \in I'}, X)$ of $T(\pi)$ on $S^h$. Note that both $S_{(s,t)}$ and $S_{(s',t')}$ can be generated by the same orthonormal basis over $K(s,t)$ and $K(s',t')$, respectively. For $M = \prod_{\tau \in I \backslash I'} X^a_{\tau} Y^b_{\tau} \prod_{\sigma \in I'} x_\sigma^m$ and

$$\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}, \gamma_2 \in \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N),$$

with $\gamma_1, \gamma_2 \in \Gamma_1(N)$, we see that

(17) $[\gamma]_{(s,t)} \cdot M = \prod_{\tau \in I \backslash I'} \det(\gamma_{\tau})^{n_\tau} \left( \prod_{\sigma \in I'} j_\sigma(\gamma_\sigma)^{s_\sigma} \det(\gamma_1 \gamma_2)^{t_\sigma} \right)$

$$\times \left( \prod_{\tau \in I \backslash I'} X^a_{\tau} Y^b_{\tau} \right)^{t_\tau} \prod_{\sigma \in I'} (\gamma_\sigma \cdot x_\sigma)^{m_\sigma}$$

and

(18) $[\gamma]_{(s',t')} \cdot M = \prod_{\tau \in I \backslash I'} \det(\gamma_{\tau})^{n_\tau} \left( \prod_{\sigma \in I'} j_\sigma(\gamma_\sigma)^{s'_\sigma} \det(\gamma_1 \gamma_2)^{t_\sigma} \right)$

$$\times \left( \prod_{\tau \in I \backslash I'} X^a_{\tau} Y^b_{\tau} \right)^{t_\tau} \prod_{\sigma \in I'} (\gamma_\sigma \cdot x_\sigma)^{m_\sigma}.$$


By the assumption that $|s_{\sigma} - s'_{\sigma}| \leq p^{-\frac{m}{\epsilon(\mathfrak{p}_{i})}}$ for each $\sigma \in I'$ with $i$ such that $\sigma \in I_{i}$, we can write in $\mathbb{C}_{p}$
\[
s'_{\sigma} = s_{\sigma} + (\pi_{i}^{'\sigma})^{m}u_{\sigma} \quad \text{and} \quad t'_{\sigma} = t_{\sigma} - \frac{u_{\sigma}}{2}(\pi_{i}^{'\sigma})^{m}
\]
with some $u_{\sigma} \in \mathcal{O}_{\mathbb{C}_{p}}$ by Definition 1.2. Then we have
\[
j_{\sigma}(\gamma_{\sigma})^{s_{\acute{\sigma}}} = j_{\sigma}(\gamma_{\sigma})^{s_{\sigma}}(j_{\sigma}(\gamma_{\sigma})^{(\pi_{\mathrm{i}}^{'\sigma})^{m}})^{u_{\sigma}}
\]
and
\[
\det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_{\acute{\sigma}}} = \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_{\sigma}}(\det(\gamma_{1\sigma}\gamma_{2\sigma})^{(\pi_{i}^{'\sigma})^{m}}1^{u_{\vec{2}}})
\]
Noting that $j_{\sigma}(\gamma_{\sigma})$ and $\det(\gamma_{1\sigma}\gamma_{2\sigma})$ are of the form $1 + \pi_{i}^{\sigma}a$ with some $a$ with norm $|a| \leq 1$, by (17), (18) and (19) and the formula (7) in Section 1.2, we can calculate that for each $\sigma \in I'$ with $i$ such that $\sigma \in I_{i}$,
\[
|j_{\sigma}((s_{\sigma})_{\sigma\in I'}) - j_{\sigma}((s_{\sigma}')_{\sigma\in I'})| \leq p^{-(m+\lambda')\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_{i})}\}}
\]
for all $i \geq 1$.
\[\square\]

Let $(s_{\sigma})_{\sigma\in I'}, (s'_{\sigma})_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\emptyset)$. By Proposition 2.4, we see that
\[N_{(s,t)}(x), N_{(s',t')} (x) \geq \mu(x).\]
We put
\[\nu(x) := \frac{2\lambda g'}{(g'+1)(g'+2)h_{n}}(\frac{g'!}{h_{n}})\frac{1}{g'}(x^{\frac{1}{g'}} - (h_{n}(g'+1))^{\frac{1}{g'}})
\]
for $x \in \mathbb{R}_{\geq 0}$. Then $\nu$ is a strictly monotone increasing function, and we have
\[\nu(0) < 0 \quad \text{and} \quad \lim_{x\to\infty} \nu(x) = \infty.
\]
Moreover, the inverse function
\[\nu^{-1}(x) = h_{n}\left(\frac{(g'+1)(g'+2)^{2}}{2\lambda g'(g'!)}\frac{1}{g'}x + (g'+1)^{\frac{1}{g'}}\right)^{\frac{1}{g'}}
\]
of \( \nu \) is also a monotone increasing function on \( \mathbb{R}_{\geq 0} \) and \( \nu^{-1}(x) \geq 0 \) for \( x \geq 0 \). For \( \alpha \in \mathbb{Q}_{\geq 0} \), we put
\[
m(\alpha) := \frac{\max_{1 \leq i \leq s} \{e(p_i)\}}{\min_{1 \leq i \leq s} \{e(p_i)\}} [\alpha \nu^{-1}(\alpha)].
\]
By Proposition 2.5, we then see that if \( |s_\sigma - s'_\sigma| \leq p^{-m(\alpha)} \) for all \( \sigma \in I' \), then
\[
|c_i((s_\sigma)_{\sigma \in I'}) - c_i((s'_\sigma)_{\sigma \in I'})| \leq p^{-\min_{1 \leq i \leq s} \{e(p_i)\} \{\max_{1 \leq i \leq s} \{e(p_i)\}\} [\alpha \nu^{-1}(\alpha)] + \lambda'}
\]
for all \( i \geq 1 \). Since we can replace \( \mathbb{Z}_p \) (resp. \( m_\nu(\alpha) + 1 \)) by \( \mathcal{O}_{\mathbb{C}_p} \) (resp. \( \min_{1 \leq i \leq s} \{\frac{1}{e(p_i)}\} \{\max_{1 \leq i \leq s} \{e(p_i)\}\} [\alpha \nu^{-1}(\alpha)] + \lambda'\} \)) in the statement of [14, Lemma 4.1], we have the following

**Proposition 2.6.** Assume the condition (ram). For any \( \alpha \in \mathbb{Q}_{\geq 0} \), we put
\[
m(\alpha) := \frac{\max_{1 \leq i \leq s} \{e(p_i)\}}{\min_{1 \leq i \leq s} \{e(p_i)\}} [\alpha h_{n'}(\frac{(g' + 1)(g' + 2)^2}{2\lambda g'(g')!})^2 + (g' + 1)^{\frac{1}{g'}}].
\]
If \( (s_\sigma)_{\sigma \in I'}, (s'_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n',v)}(\mathbb{C}_p) \) satisfy \( |s_\sigma - s'_\sigma| \leq p^{-m(\alpha)} \) for all \( \sigma \in I' \), then the slope-\( \alpha \)-part of the Newton polygons of \( P((s_\sigma)_{\sigma \in I'}, X) \) and \( P((s'_\sigma)_{\sigma \in I'}, X) \) are equal.

By combining this proposition with [12, Corollary of Section IV.4], we obtain the following

**Theorem 2.7.** Assume the condition (ram). Let \( \alpha \in \mathbb{Q}_{\geq 0} \) and \( (s_\sigma)_{\sigma \in I'}, (s'_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n',v)}(\mathbb{C}_p) \). If \( |s_\sigma - s'_\sigma| \leq p^{-m(\alpha)} \) for all \( \sigma \in I' \), then we have
\[
\dim_{\mathbb{C}_p} S_{(s,t)}(G; \Gamma_1(N))_{\mathbb{C}_p}^\alpha = \dim_{\mathbb{C}_p} S_{(s',t')}((s'_\sigma)_{\sigma \in I'}, X)_{\mathbb{C}_p}^\alpha.
\]
Further, by Theorem 2.3, we then have immediately the following

**Corollary 2.8.** Assume the condition (ram). If \( (n'_\sigma)_{\sigma \in I'}, (n''_\sigma)_{\sigma \in I'} \in \mathcal{W}_{(n',v)}(\mathbb{C}_p) \) satisfy the conditions that \( |n'_\sigma - n''_\sigma| \leq p^{-m(\alpha)} \) for all \( \sigma \in I' \) and \( \nu, \nu'' > \alpha \), then we have
\[
\dim_{\mathbb{C}_p} S_{(n',v')}^{\alpha}(G; \Gamma_1(N); \mathbb{C}_p) = \dim_{\mathbb{C}_p} S_{(n'',v'')}^{\alpha}(G; \Gamma_1(N); \mathbb{C}_p).
\]

**Remark 2.2.** In Corollary 2.8, we need to assume the condition (ram) to apply the modified Wan's lemma with the positive rational number \( \lambda' \). This corollary is a generalization of Coleman's result [5, Theorem B3.4] which gives a solution to a conjecture of Gouvêa and Mazur [7, Conjecture 1 in Section 5].

**Remark 2.3.** Kassaei [11] has constructed overconvergent \( \mathcal{P} \)-adic modular forms on quaternion algebras defined over any totally real field \( F \) which are unramified at \( \mathcal{P} \) and exactly one infinite place, where \( \mathcal{P} \) is a
prime ideal of $F$ above $p$ whose residue field has cardinality $>3$. Then he has also showed the local constancy of dimensions of the spaces of overconvergent forms ([11, Theorem 1.1]).

References