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Reverse inequalities involving two relative operator entropies and two relative entropies

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Abstract. We shall discuss relation among Tsallis relative operator entropy $T_p(A,B)$, the relative operator entropy $\hat{S}(A|B)$ by J.I.Fujii-Kamei, the Tsallis relative entropy $D_p(A||B)$ by Furuichi-Yanagi-Kuriyama and the Umegaki relative entropy $S(A,B)$. We show the following result: *Let A and B be strictly positive definite matrices such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $h = \frac{M_1M_2}{m_1m_2} > 1$ and $p \in (0,1]$. Then the following inequalities hold:*

$$\left(\frac{1-K(p)}{p}\right)(\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^{p} + D_p(A||B) \geq -\mathrm{Tr}[T_p(A|B)] \geq D_p(A||B)$$

where $K(p)$ is the generalized Kantorovich constant defined by

$$K(p) = \frac{(h^p-h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p$$

and the first inequality is the reverse one of the second known inequalty, in particular

$$\log S(1)\mathrm{Tr}[A] + S(A,B) \geq -\mathrm{Tr}[\hat{S}(A|B)] \geq S(A,B)$$

where $S(1)$ is the Specht ratio defined by

$$S(1) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$$

and the first inequality is the reverse one of the second known inequalty.

It is known that $K(p) \in (0,1]$ for $p \in (0,1]$ and $S(1) > 1$.

§1. Introduction

A capital letter means a $n \times n$ complex matrix and $\mathrm{Tr}[X]$ means the trace on the matrix X. A matrix X is said to be *strictly positive definite* if X is positive definite and invertible (denoted by $X > 0$). Let A and B be strictly positive definite matrices. Umegaki relative entropy $S(A,B)$ in [11] is defined by

$$S(A,B) = \mathrm{Tr}[A(\log A - \log B)] \tag{1.1}$$

and the relative operator entropy $\hat{S}(A|B)$ in [2] is defined by

$$\hat{S}(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \tag{1.2}$$

as an extension of [10]. Very recently, Tsallis relative operator entropy $T_p(A|B)$ in Yanagi-Kuriyama-Furuichi [12] is defined by

$$T_p(A|B) = \frac{A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^p A^{\frac{1}{2}} - A}{p} \tag{1.3}$$

for $p \in (0,1]$ and also Tsallis relative entropy $D_p(A||B)$ in Furuichi-Yanagi-Kuriyama [4] is defined by

$$D_p(A||B) = \frac{Tr[A] - Tr[A^{1-p}B^p]}{p} \tag{1.4}$$

for $p \in (0,1]$. Next we shall state the following results on $-Tr[T_p(A|B)]$, $D_p(A||B)$, $-\mathrm{Tr}[\hat{S}(A|B)]$ and $S(A,B)$.

Theorem A. (Genaralized Peierls-Bogoliubov inequality [4]) *Let $A, B > 0$ and also let $p \in (0,1]$. Then the following inequality holds:*

$$D_p(A||B) \geq \frac{\mathrm{Tr}[A] - (\mathrm{Tr}[A])^{1-\mathrm{p}}(\mathrm{Tr}[B])^{\mathrm{p}}}{p}. \tag{1.5}$$

Theorem B. *Let $A, B > 0$. The following inequality holds:*

$$-\mathrm{Tr}[\mathrm{T_p}(A|B)] \geq D_p(A||B) \quad \textit{for } p \in (0,1] \tag{1.6}$$

and

$$-\mathrm{Tr}[\hat{\mathrm{S}}(A|B)] \geq S(A,B). \tag{1.7}$$

Theorem C. [4] *The following properties hold:*

$$\lim_{p\to 0} T_p(A|B) = \hat{S}(A|B) \tag{1.8}$$

and

$$\lim_{p\to 0} D_p(A||B) = S(A,B). \tag{1.9}$$

Let $h > 1$. The genaralized Kantrovich constant $K(p)$ is defined by

$$K(p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p \tag{1.10}$$

for any real number p and it is known that $K(p) \in (0,1]$ for $p \in [0,1]$. Also $S(p)$ is defined by

$$S(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}} \tag{1.11}$$

for any real number p. In particular $S(1) = \dfrac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is said to be *the Specht ratio* and $S(1) > 1$ is well known. We state the known results on the generalized Kantorovich constant $K(p)$ and Specht ratio $S(1)$ (for example, [6]). *Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities* (1.12), (1.13) *and* (1.14) *hold for every unit vector x and* (1.12) *is equivalent to* (1.13)*:*

$$K(p)(Ax, x)^p \geq (A^p x, x) \geq (Ax, x)^p \quad \text{for any } p > 1 \text{ or any } p < 0. \tag{1.12}$$

$$(Ax, x)^p \geq (A^p x, x) \geq K(p)(Ax, x)^p \quad \text{for any } 1 \geq p > 0. \tag{1.13}$$

$$S(1)\Delta_x(A) \geq (Ax, x) \geq \Delta_x(A). \tag{1.14}$$

where the determinant $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by $\Delta_x(A) = \exp\langle((\log A)x, x)\rangle$ and (1.14) is shown in [3].

Theorem D. (i) *$K(p)$ is symmetric with respect to $p = \frac{1}{2}$ and $K(p)$ is an increasing function of p for $p \geq \frac{1}{2}$, and, $K(p)$ is a decreasing function of p for $p \leq \frac{1}{2}$, and* $K(0) = K(1) = 1$.

(ii) *$K(p) \geq 1$ for $p \geq 1$ or $p \leq 0$, and $1 \geq K(p) \geq \frac{2h^{\frac{1}{4}}}{h^{\frac{1}{2}}+1}$ for $p \in [0, 1]$.*

(iii) *$S(p)$ is symmetric with respect to $p = 0$ and $S(p)$ is an increasing function of p for $p \geq 0$, and, $S(p)$ is a decreasing function of p for $p \leq 0$ and $S(0) = 1$.*

(iv) $S(1) = e^{K'(1)} = e^{-K'(0)}$.

(iv) of Theorem D is shown in [Proposition 1, 5] and (i), (ii) and (iii) are shown in [6].

For two strictly positive definite matrices A, B and $p \in [0, 1]$, p-power mean $A\sharp_p B$ is defined by

$$A\sharp_p B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^p A^{\frac{1}{2}}$$

and we remark that $A\sharp_p B = A^{1-p}B^p$ if A commutes with B.

§2. Reverse inequalities involving $-Tr[T_p(A|B)]$, $D_p(A||B)$, $-\mathrm{Tr}[\hat{S}(A|B)]$ and $S(A,B)$

We shall show the following reverse inequalities involving $-Tr[T_p(A|B)]$, $D_p(A||B)$, $-\mathrm{Tr}[\hat{S}(A|B)]$ and $S(A, B)$.

Theorem 1. *Let A and B be strictly positive definite matrices such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$. Put $h = \frac{M_1 M_2}{m_1 m_2} > 1$ and $p \in (0, 1]$. Then the following inequalities hold:*

$$\left(\frac{1-K(p)}{p}\right)(\mathrm{Tr}[A])^{1-p}(\mathrm{Tr}[B])^{p}+D_p(A||B) \tag{2.1}$$
$$\geq -\mathrm{Tr}[T_p(A|B)]$$
$$\geq D_p(A||B)$$

where $K(p)$ is the generalized Kantorovich constant defined in (1.10) *and the first inequality is the reverse one of the second inequalty.*

Corollary 2. *Let A and B be strictly positive definite matrices such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $h = \frac{M_1M_2}{m_1m_2} > 1$. Then the following inequalities hold:*

$$\log S(1)\mathrm{Tr}[A] + S(A,B) \tag{2.2}$$
$$\geq -\mathrm{Tr}[\hat{S}(A|B)]$$
$$\geq S(A,B)$$

where $S(1)$ is the Specht ratio defined in (1.11) *and the first inequality is the reverse one of the second inequalty.*

We remark that the second inequality in (2.1) is known in (1.6) of Theorem B and also the second inequality in (2.2) is well known in (1.7) of Theorem B. Also we remark that $\dfrac{1-K(p)}{p} \geq 0$ in Theorem 1 by (ii) of Theorem D.

Remark. By using (iv) of Theorem D and Theorem C, (2.1) of Theorem 1 implies (2.2) of Corollary 2. By the same way, It is interesting to point out that by also using (iv) of Theorem D. (1.13) implies (1.14) as follows:

$$(Ax,x) \geq (A^px,x)^{\frac{1}{p}} \geq K(p)^{\frac{1}{p}}(Ax,x) \quad \textit{for any } 1 \geq p > 0.$$

and it is easily verified that $\lim_{p\to 0}(A^px,x)^{\frac{1}{p}} = \Delta_x(A)$ and $\lim_{p\to 0} K(p)^{\frac{-1}{p}} = S(1)$ by (iv) of Theorem D, so that (1.13) implies (1.14).

At the end of this remark, we would like to emphasize that $\lim_{p\to 0}\dfrac{1-K(p)}{p} = \log S(1)$ of (3.4) in the step (2.1) $\Longrightarrow$ (2.2) and $\lim_{p\to 0} K(p)^{\frac{-1}{p}} = S(1)$ in the step (1.13) $\Longrightarrow$ (1.14) are both derived from $S(1) = e^{K'(1)} = e^{-K'(0)}$ by (iv) of Theorem D.

The results in this paper will appear elsewhere.

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