UNIFORM NON-ℓ_1^n-NESS OF ψ-DIRECT SUMS
OF BANACH SPACES $X \oplus_\psi Y$

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Abstract. We shall characterize the uniform non-ℓ_1^n-ness of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y, where ψ is a convex function on the unit interval satisfying certain conditions. In particular the previous result for uniform non-squareness will be derived as a corollary. To do this we shall present a result on monotonicity property of absolute norms on $\mathbb{C}^2$.

1. INTRODUCTION AND PRELIMINARIES

It is said to be absolute normalized norm on $\mathbb{C}^2$ if

$$\|(z,w)\| = \|(|z|,|w|)\| \quad \text{and} \quad \|(1,0)\| = \|(0,1)\| = 1. \tag{1}$$

Let ψ be a convex function on $[0, 1]$ satisfying

$$\psi(0) = \psi(1) = 1 \quad \text{and} \quad \max\{1-t,t\} \le \psi(t) \le 1 \quad (0 \le t \le 1). \tag{2}$$

We define a norm on $\mathbb{C}^2$ by

$$\|(z,w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \ne (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{3}$$

Then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ ([3]).

Using this absolute norm $\|\cdot\|_\psi$, Takahashi, Kato and Saito [17] introduced the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y as their direct sum $X \oplus Y$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi \tag{4}$$

and they proved the strict convexity of ψ-direct sum $X \oplus_\psi Y$ of two Banach spaces X and Y are characterized. Also Saito and Kato[14] characterized the uniformly convexity of ψ-direct sum $X \oplus_\psi Y$. On the other hand, Saito-Kato-Takahashi [15] showed that all absolute, normalized norms on $\mathbb{C}^2$ are uniformly non-square except the ℓ_1- and ℓ_∞-norms. And recently the present authors [11] characterized the uniform non-squareness of $X \oplus_\psi Y$.

In this paper, under the assumption that X and Y are not uniformly non-ℓ_1^{n-1}, we shall show that $X \oplus_\psi Y$ is uniformly non-ℓ_1^n if and only if X and Y are uniformly non-ℓ_1^n and the norm corresponding to ψ is neither ℓ_1- nor ℓ_∞-norms. As the case $n = 2$ the previous result in [11] concerning the uniform non-squareness of these spaces is obtained.

A Banach space X is said to be *uniformly non-ℓ_1^n* (cf. [1, 12]) provided there exists ϵ $(0 < \epsilon < 1)$ such that for any $x_1, \cdots, x_n \in S_X$, the unit sphere of X, there exists $\theta = (\theta_j)$ of n signs ± 1 for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1 - \epsilon).$$

When $n = 2$ X is called *uniformly non-square* ([8]; cf. [1, 12]). Formally, we consider that every Banach space is not uniformly non-ℓ_1^1.

Uniformly non-ℓ_1^n-ness is introduced by Beck[2] to prove the strong law of large numbers in Banach spaces. Since then, this property has been playing important role in probablity in abanach spaces and related fields.

The following fundamental fact was proved in Brown [5].

Proposition A ([5], 1.6). *Let X and Y be Banach spaces. If X is uniformly non-ℓ_1^n, then X is uniformly non-ℓ_1^{n+1} for every $n \in \mathbf{N}$.*

2. MONOTONICITY PROPERTY OF ABSOLUTE NORMS

In this section we discuss the monotonicity property of absolute norms on $\mathbb{C}^2$ for later use. Recall first the following facts.

Lemma 1 ([2, p.36, Lemma 2]). *Let* $\| \cdot \| \in N_a$.
(i) *If* $|p| \leq |r|$ *and* $|q| \leq |s|$, *then* $\|(p, q)\| \leq \|(r, s)\|$.
(ii) *If* $|p| < |r|$ *and* $|q| < |s|$, *then* $\|(p, q)\| < \|(r, s)\|$.

The following assertion is not true in general for a norm $\| \cdot \| \in N_a$:

Let $|p| \leq |r|$ *and* $|q| \leq |s|$. *If* $|p| < |r|$ *or* $|q| < |s|$, *then* $\|(p, q)\| < \|(r, s)\|$.

$$(5)$$

Proposition 2 (Takahashi, Kato and Saito [17]). *Let* $\psi \in \Psi$. *Then the following assertions are equivalent:*
(i) *If* $|z| \leq |u|$ *and* $|w| < |v|$, *or* $|z| < |u|$ *and* $|w| \leq |v|$, *then* $\|(z, w)\|_\psi < \|(u, v)\|_\psi$.
(ii) $\psi(t) > \psi_\infty(t)$ *for all* $t \in (0, 1)$.

A more precise (component-wise) result is given in [17]. Next we present a condition on specified (z, w) and (u, v) for which the above assertion (i) is valid component-wise for a general $\psi \in \Psi$.

Proposition 3. *Let* $\psi \in \Psi$ *and let* (z, w), $(u, v) \in \mathbb{C}^2$.
(i) *Let* $|z| < |u|$ *and* $|w| = |v|$. *Then* $\|(z, w)\|_\psi = \|(u, v)\|_\psi$ *if and only if* $\|(z, w)\|_\psi = |w|$.
(ii) *Let* $|z| = |u|$ *and* $|w| < |v|$. *Then* $\|(z, w)\|_\psi = \|(z, v)\|_\psi$ *if and only if* $\|(z, w)\|_\psi = |z|$.

3. UNIFORM NON-ℓ_1^n-NESS OF $X \oplus_\psi Y$

We need a sequence of lemmas. Our first lemma is of independent interest as it provides a couple of inequalities which are more sharp than the triangle inequality.

Lemma 2. *Let* $x_1, x_2, \ldots, x_n$ *be arbitrary nonzero elements in a Banach space* X. *Let* $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ *and* $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$. *Then*

$$\left[\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \|x_{j_0}\|\right] \leq \sum_{j=1}^{n} \|x_j\| \leq \left[\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \|x_{j_1}\|\right].$$

$$(6)$$

Lemma 3. *Let* $\{x_1^{(k)}\}_k, \ldots, \{x_n^{(k)}\}_k$ *be arbitrary nonzero sequences in a Banach space* X *for which their norms are convergent to non-zero limits. Then the following are equivalent.*

(i) $\displaystyle\lim_{k \to \infty} \left\|\sum_{j=1}^{n} x_j^{(k)}\right\| = \lim_{k \to \infty} \sum_{j=1}^{n} \|x_j^{(k)}\|$.

(ii) $\displaystyle\lim_{k \to \infty} \left\|\sum_{j=1}^{n} \frac{x_j^{(k)}}{\|x_j^{(k)}\|}\right\| = n$.

Lemma 4. *Let* $\{x_1^{(k)}\}_k, \ldots, \{x_n^{(k)}\}_k$ *be arbitrary nonzero sequences in a Banach space* X *for which their norms are convergent. Then the following are equivalent.*

(i) $\displaystyle\lim_{k \to \infty} \left\|\sum_{j=1}^{n} x_j^{(k)}\right\| = \lim_{k \to \infty} \sum_{j=1}^{n} \|x_j^{(k)}\|$.

(ii) $\displaystyle\lim_{k \to \infty} \left\|\alpha x_1^{(k)} + \sum_{j=2}^{n} x_j^{(k)}\right\| = \lim_{k \to \infty} \left[\alpha \|x_1\| + \sum_{j=2}^{n} \|x_j^{(k)}\|\right]$ *for all* $\alpha > 0$.

(iii) $\displaystyle\lim_{k \to \infty} \left\|\alpha x_1^{(k)} + \sum_{j=2}^{n} x_j^{(k)}\right\| = \lim_{k \to \infty} \left[\alpha \|x_1\| + \sum_{j=2}^{n} \|x_j^{(k)}\|\right]$ *for some* $\alpha > 0$.

Now we are in a position to present the main result.

Theorem 1. *Let* X *and* Y *be Banach spaces and* $\psi \in \Psi$. *If* X *and* Y *are unformly non-ℓ_1^n and* $\psi \neq \psi_1, \psi_\infty$, *then* $X \oplus_\psi Y$ *is uniformly non-ℓ_1^n. If* $X \oplus_\psi Y$ *is unformly non-ℓ_1^n and neither* X *nor* Y *are uniformly non-ℓ_1^{n-1}, then the converse is hold.*

The same is true for the uniformly non-squareness of these spaces.

Theorem 2 *Let X and Y be Banach spaces. The following are equivalent.*

(i) $X \oplus_{\psi_1} Y$ *is uniformly non-ℓ_1^n.*

(ii) *There exist $n_1, n_2 \in \mathbf{N}$ such that $n_1 + n_2 = n - 1$, X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

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