SEVERAL REVERSE INEQUALITIES OF OPERATORS

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ABSTRACT. In this report, we show reverse inequalities to Araki’s inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if $A$ and $B$ are positive operators on a Hilbert space $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\|$$

for all $0 < p < 1$,

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let $A$ and $B$ be positive operators on a Hilbert space $H$. The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [8], Furuta showed that the Cordes inequality

$$\|A^pB^p\| \leq \|AB\|^p \quad \text{for } 0 < p < 1$$

is equivalent to the Löwner-Heinz inequality (cf.[14])

$$A \geq B \geq 0 \quad \text{implies } A^p \geq B^p \quad \text{for } 0 < p < 1$$

(cf. [5]). In [1], Araki showed a trace inequality which entails the following inequality:

$$\|B^pA^pB^p\| \leq \|BAB\|^p \quad \text{for } 0 < p < 1.$$ 

Moreover, it was shown in [6, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [9] showed the following Kantorovich type inequalities: If $A$ and $B$ are positive operators with $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies } K(m, M, p)A^p \geq B^p \quad \text{for } p > 1,$$

where a generalized Kantorovich constant $K(m, M, p)$ [3, 7, 11] is defined as

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p$$

for all real numbers $p$.

We here cite Furuta's textbook [10] as a pertinent reference to Kantorovich inequalities.

Also, Yamazaki [16] showed the following difference type reverse inequalities of the Löwner-Heinz inequality: If $A$ and $B$ are positive operators with $0 < mI \leq B \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies } C(m, M, p) + A^p \geq B^p \quad \text{for } p > 1,$$
where the constant \( C(m, M, p) \) \([12, 16]\) is defined as

\[
C(m, M, p) = (p - 1) \left( \frac{M^p - m^p}{p(M - m)} \right)^{\frac{1}{p-1}} + \frac{Mm^p - mM^p}{M - m}
\]

for all real numbers \( p \).

In this report, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If \( A \) and \( B \) are positive operators with \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then the following inequalities hold

\[
K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\| \quad \text{for } 0 < p < 1,
\]

\[
K(m^2, M^2, p)^{1/2} \|AB\| \leq \|A^pB^p\| \quad \text{for } 0 < p < 1,
\]

respectively. We moreover show that reverse inequalities (4), (8) and (9) are mutually equivalent.

2. Main Results

First of all, we present our main theorem which is a reverse inequality to Araki's inequality (3).

**Theorem 1.** If \( A \) and \( B \) are positive operators on \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then for each \( \alpha > 0 \)

\[
\|BAB\|^p \leq \alpha \|B^pA^pB^p\| + \beta(m, M, p, \alpha) \|B\|^{2p} \quad \text{for all } 0 < p < 1,
\]

or equivalently

\[
\|B^pA^pB^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha) \|B\|^2 \quad \text{for all } p > 1,
\]

where

\[
\beta(m, M, p, \alpha) = \begin{cases} 
\frac{p-1}{p} \left( \frac{M^p - m^p}{p(M - m)} \right)^{\frac{1}{p-1}} & \text{if } \frac{M^p - m^p}{pM^{p-1}(M - m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M - m)}, \\
(1 - \alpha)M & \text{if } 0 < \alpha \leq \frac{M^p - m^p}{pM^{p-1}(M - m)}, \\
(1 - \alpha)m & \text{if } \alpha \geq \frac{M^p - m^p}{pm^{p-1}(M - m)}.
\end{cases}
\]

If we choose \( \alpha \) satisfying \( \beta(m, M, p, \alpha) = 0 \) in Theorem 1, then we have the following ratio type reverse inequalities.

**Corollary 2.** If \( A \) and \( B \) are positive operators on \( H \) such that \( 0 < mI \leq A \leq MI \) for some scalars \( m < M \), then

\[
K(m, M, p) \|BAB\|^p \leq \|B^pA^pB^p\| \quad \text{for } 0 < p < 1,
\]

or equivalently

\[
\|BAB\|^p \leq K(m, M, p) \|B^pA^pB^p\| \quad \text{for } p > 1,
\]

where \( K(m, M, p) \) is defined as (5) in §1.
If we put $\alpha = 1$ in Theorem 1, then we have the following difference type reverse inequalities.

**Corollary 3.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\[ \|BAB\|_p - \|B^pA^pB^p\| \leq -C(m, M, p)\|B\|^{2p} \quad \text{for } 0 < p < 1, \]

or equivalently

\[ \|B^pA^pB^p\|^{\frac{1}{p}} - \|BAB\| \leq -C(m^p, M^p, \frac{1}{p})\|B\|^2 \quad \text{for } p > 1, \]

where $C(m, M, p)$ is defined as (7) in §1.

As special cases of Corollary 2 and Corollary 3, we have the following corollary.

**Corollary 4.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\[ \|B^2A^2B^2\| \leq \frac{(M + m)^2}{4Mm}\|BAB\|^2. \]

\[ \|B^2A^2B^2\|^\frac{1}{2} - \|BAB\| \leq \frac{(M - m)^2}{4(M + m)}\|B\|^2. \]

\[ \frac{2\sqrt{Mm}}{\sqrt{M + m}}\|BAB\|^\frac{1}{2} \leq \|B^\frac{1}{2}A^\frac{1}{2}B^\frac{1}{2}\|. \]

\[ \|BAB\|^\frac{1}{2} - \|B^\frac{1}{2}A^\frac{1}{2}B^\frac{1}{2}\| \leq \frac{(\sqrt{M} - \sqrt{m})^2}{4(\sqrt{M} + \sqrt{m})}\|B\|. \]

Since $\|X^*X\| = \|X\|^2$ for an operator $X$, we obtain the following reverse inequality to the Cordes inequality by Corollary 2.

**Theorem 5.** If $A$ and $B$ are positive operators on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

\[ K(m^2, M^2, p)^\frac{1}{2}\|AB\|^p \leq \|A^pB^p\| \quad \text{for all } 0 < p < 1, \]

or equivalently

\[ \|A^pB^p\| \leq K(m^2, M^2, p)^\frac{1}{2}\|AB\|^p \quad \text{for all } p > 1. \]

In particular,

\[ \sqrt{\frac{2\sqrt{Mm}}{M + m}}\|AB\|^\frac{1}{2} \leq \|A^\frac{1}{2}B^\frac{1}{2}\|. \]

and

\[ \|A^2B^2\| \leq \frac{M^2 + m^2}{2Mm}\|AB\|^2. \]

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.
Theorem 6. For a given $p > 1$, the following are mutually equivalent: For all $A, B \geq 0$ and $0 < mI \leq A \leq MI$

(A) $A \geq B \geq 0$ implies $K(m, M, p)A^p \geq B^p$.
(B) $\|A^pB^p\| \leq K(m^2, M^2, p)^{1/2}\|AB\|^p$.
(C) $\|B^pA^pB^p\| \leq K(m, M_1, p)\|BAB\|^p$.

(B') $K(m^2, M^2, 1/p)^{1/2}\|AB\|^p \leq \|A^pB^p\|$.
(C') $K(m, M, 1/p)\|BAB\|^p \leq \|B^pA^pB^p\|$.

3. Lemmas

We start with the following three lemmas before we give proofs of the results in §2.

Let $A$ be a positive operator on a Hilbert space $H$ and $x$ a unit vector in $H$. Then it follows from Hölder-McCarthy inequality that

$$ (Ax, x) \leq (A^p x, x)_1^\frac{1}{p} \quad \text{for all } p > 1. $$

By using the Mond-Pečarić method [12, 13], we have the following reverse inequality of (25) [15, 4]:

Lemma 7. If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $\alpha > 0$

$$ (A^p, x)_1^\frac{1}{p} \leq \alpha(Ax, x) + \beta(m, M, p, \alpha) \quad \text{for all } p > 1 $$

holds for every unit vector $x \in H$, where $\beta(m, M, p, \alpha)$ is defined as (12) in Theorem 1.

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m, M, p, \alpha)$ and $f(t) = (at+b)^{\frac{1}{p}} - \alpha t$ for $a = \frac{M^p-m^p}{M-m}$ and $b = \frac{Mm^p-mM^p}{M-m}$, then we have $f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$. It follows that the equation $f'(t) = 0$ has exactly one solution $t_0 = \frac{\alpha p}{a} - \frac{b}{a}$. If $m \leq t_0 \leq M$, then we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0)$ since $f''(t) = \frac{a^2(1-p)}{p^2(t+b)^{\frac{1}{p}-2}} < 0$ and the condition $m \leq t_0 \leq M$ is equivalent to the condition

$$ \frac{M^p - m^p}{pm^{p-1}(M-m)} \leq \alpha \leq \frac{M^p - m^p}{pm^{p-1}(M-m)}. $$

If $M \leq t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)M$ for $t_0 = M$. Similarly, we have $\beta = \max_{m \leq t \leq M} f(t) = f(t_0) = (1-\alpha)m$ for $t_0 = m$ if $t_0 \leq m$. Hence it follows that

$$ (at + b)^{\frac{1}{p}} - \alpha t \leq \beta \quad \text{for all } t \in [m, M]. $$

Since $t^p$ is convex for $p > 1$, it follows that $t^p \leq at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \leq aA + b$ and hence $(A^p, x)^{\frac{1}{p}} \leq a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

$$ (A^p, x)_1^\frac{1}{p} - \alpha(Ax, x) \leq \max_{m \leq t \leq M} f(t) = \beta(m, M, p, \alpha). $$

$\square$
By Lemma 7, we have the following estimates of both the difference and the ratio in the inequality (25).

**Lemma 8.** If $A$ is a positive operator on $H$ such that $0 < mI \leq A \leq MI$ for some scalars $0 < m < M$, then for each $p > 1$

\begin{align}
(27) & \quad (A^p x, x) \leq K(m, M, p)^{\frac{1}{p}} (Ax, x) \\
(28) & \quad (A^p x, x)^{\frac{1}{p}} - (Ax, x) \leq -C(m^p, M^p, \frac{1}{p})
\end{align}

hold for every unit vector $x \in H$, where $K(m, M, p)$ is defined as (5) in §1 and $C(m, M, p)$ is defined as (7) in §1.

**Proof.** If we choose $\alpha$ satisfying $\beta(m, M, p, \alpha) = 0$ in Lemma 7, then we have $\alpha = K(m, M, p)^{\frac{1}{p}}$. If we put $\alpha = 1$ in Lemma 7, then we have $\beta(m, M, p, 1) = -C(m^p, M^p, \frac{1}{p})$. \hfill \Box

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ if $p = 2$.

We summarize some important properties of a generalized Kantorovich constant [3, 11].

**Lemma 9.** Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.

(i) $K(m, M, p) = K(M, m, p)$ for all $p \in \mathbb{R}$.

(ii) $K(m, M, p) = K(m, M, 1-p)$ for all $p \in \mathbb{R}$.

(iii) $K(m, M, 0) = K(m, M, 1) = 1$ for all $p \in \mathbb{R}$.

(iv) $K(m, M, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$.

(v) $K(m^p, M^p, \frac{2}{p})^{\frac{1}{p}} = K(m^p, M^p, \frac{1}{p})^{-\frac{1}{p}}$ for $pr \neq 0$.

4. PROOF OF RESULTS

Based on Lemmas in the preceding section, we give proofs of the results mentioned in the second section.

**Proof of Theorem 1.**

For every unit vector $x \in H$, it follows that

\[
((BAB)^p x, x) \leq (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1
\]

\[
\leq \left( (A^p)^{\frac{1}{p}} \frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|} \right)^{p} \|Bx\|^{2p}
\]

\[
\leq \left( \alpha(A^p Bx, Bx)\frac{Bx}{\|Bx\|}, \frac{Bx}{\|Bx\|} \right) + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p}
\]

\[
= \alpha(A^p Bx, Bx)\|Bx\|^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p}
\]

\[
= \alpha \left( B^p A^p B^p \frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|} \right) \|Bx\|^{2p-2} \|B^{1-p}x\|^{2} + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|Bx\|^{2p}
\]
and
\[ \|Bx\|^2 \|B^{1-p}x\|^2 = (B^2x, x)^{p-1}(B^{2-2p}x, x) \leq (B^2x, x)^{p-1}(B^{2}x, x)^{1-p} = 1 \quad \text{by} \quad 0 < 1 - p < 1. \]

By combining two inequalities above, we have
\[
\|BAB\|^p = \|(BAB)^p\| \leq \alpha \|B^p A^p B^p\| + \beta(m^p, M^p, \frac{1}{p}, \alpha) \|B\|^{2p}
\]
and hence we have the desired inequality (10).

Next, we show (10)\(\Rightarrow\)(11). For \(p > 1\), since \(0 < \frac{1}{p} < 1\), it follows from (10) that
\[
\|BAB\|^{\frac{1}{p}} \leq \alpha \|B^\frac{1}{p} A^\frac{1}{p} B^\frac{1}{p}\| + \beta(m^\frac{1}{p}, M^\frac{1}{p}, p, \alpha) \|B\|^\frac{2}{p},
\]
By replacing \(A\) by \(A^p\) and \(B\) by \(B^p\) in the above inequality respectively, we have
\[
\|B^p A^p B^p\|^{\frac{1}{p}} \leq \alpha \|BAB\| + \beta(m, M, p, \alpha) \|B^p\|^\frac{2}{p},
\]
and so we have the desired inequality (11). Similarly we can show (11)\(\Rightarrow\)(10). Therefore (10) is equivalent to (11).
\(\square\)

**Proof of Corollary 2.**
For \(p > 1\), if we put \(\beta(m, M, p, \alpha) = 0\) in Theorem 1, then it follows that
\[
\frac{p-1}{p} \left( \frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{p-1}} + \alpha^{\frac{p}{p-1}} \frac{(Mm^p - mM^p)}{M^p - m^p} = 0,
\]
and hence
\[
\alpha^{\frac{p}{p-1}} = -\frac{p-1}{p} \left( \frac{M^p - m^p}{p(M-m)} \right)^{\frac{1}{p-1}} \frac{M^p - m^p}{Mm^p - mM^p}.
\]
Therefore, we have
\[
\alpha^p = \frac{M^p - m^p}{p(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{Mm^p - mM^p} \right)^{p-1}
= K(m, M, p)
\]
and we obtain the desired inequality (14).

For \(0 < p < 1\), since \(1/p > 1\), it follows from (14) that
\[
\|BAB\|^{\frac{1}{p}} \leq K(m, M, \frac{1}{p})\|B^\frac{1}{p} A^\frac{1}{p} B^\frac{1}{p}\|.
\]
By replacing \(A\) and \(B\) by \(A^p\) and \(B^p\) respectively, then we have
\[
\|B^p A^p B^p\|^{\frac{1}{p}} \leq K(m^p, M^p, \frac{1}{p})\|BAB\|.
\]
Hence it follows from Lemma 9 that
\[
\|B^p A^p B^p\| \leq K(m^p, M^p, \frac{1}{p})\|BAB\|^p
\leq K(m, M, p)^{-1}\|BAB\|^p,
\]
and we have the desired inequality (13). Similarly we have the implication (13)\(\Rightarrow\)(14).  \(\square\)
Proof of Corollary 3.
If we put $\alpha = 1$ in Theorem 1, then it follows that
\[
\beta(m^p, M^p, \frac{1}{p}, 1) = \frac{\frac{1}{p} - 1}{\frac{1}{p}} \left( \frac{M - m}{\frac{1}{p}(M^p - m^p)} \right)^{\frac{p}{1-p}} + \frac{M^p m - m^p M}{M - m} = (1-p) \left( \frac{p(M - m)}{M^p - m^p} \right)^{\frac{p}{1-p}} + \frac{M^p m - m^p M}{M - m} = -C(m, M, p).
\]
Similarly it follows that $\beta(m, M, \frac{1}{p}) = -C(m^p, M^p, \frac{1}{p})$. Hence we have the equivalence (15) $\iff$ (16).

Proof of Corollary 4.
In Corollary 2 and 3, we have only to put $p = 2$ and $p = 1/2$.

Proof of Theorem 5
By Corollary 2, it follows that
\[
K(m, M, p) \|A^{\frac{1}{2}}B\|^{2p} \leq \|A^pB^p\|^2.
\]
By replacing $A$ by $A^2$, we have
\[
K(m^2, M^2, p) \|AB\|^{2p} \leq \|A^pB^p\|^2.
\]
Therefore we have (21). Similarly, we have the equivalence (21) $\iff$ (22).

Proof of Theorem 6
The proof is divided into three parts, namely the equivalence $(A) \implies (B) \implies (C) \implies (A)$, $(B) \iff (B')$ and $(C) \iff (C')$.

$(A) \implies (B)$. It follows that
\[
(A) \iff \|A^{-\frac{1}{2}}B\| \leq 1 \implies \|A^{-\frac{5}{2}}B^\frac{5}{2}\|^2 \leq K(m, M, p)
\]
\[
\iff \|A^\frac{1}{2}B\| \leq 1 \implies \|A^\frac{p}{2}B^p\|^2 \leq K(m^{-1}, m^{-1}, p) = K(m, M, p)
\]
\[
\iff \|AB\| \leq 1 \implies \|A^pB^p\| \leq K(m^2, M^2, p).
\]
If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that
\[
\|A^pB_1^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \iff \|A^pB^p\| \leq K(m^2, M^2, p)^{\frac{1}{2}} \|AB\|^p.
\]

$(B) \implies (C)$. If we replace $A$ by $A^{\frac{1}{2}}$ in (A), then it follows that
\[
\|A^\frac{p}{2}B^p\| \leq K(m, M, p)^{\frac{1}{2}} \|A^{\frac{1}{2}}B\|^p.
\]
Square both sides, we have
\[
\|B^pA^pB^p\| \leq K(m, M, p)\|BAB\|^p.
\]

$(C) \implies (A)$. If we replace $B$ by $B^{\frac{1}{2}}$ and $A$ by $A^{-1}$ in (C), then it follows that
\[
\|B^\frac{p}{2}A^{-p}B^\frac{p}{2}\| \leq K(m^{-1}, m^{-1}, p)\|B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\|^p.
\]
By rearranging it, we have
\[
\|A^{-\frac{5}{2}}B^pA^{-\frac{5}{2}}\| \leq K(m, M, p)\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|^p.
\]
Since $A \geq B \geq 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq 1$ that
\[
\|A^{-\frac{p}{2}}B^{p}A^{-\frac{1}{2}}\| \leq K(m, M, p)
\]
and hence
\[
B^{p} \leq K(m, M, p)A^{p}.
\]

$(B) \iff (B')$: If we replace $A$ and $B$ by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in $(B)$ respectively, then it follows that

\[
(B) \iff \|AB\| \leq K\left(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p\right)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\||AB|^{\frac{1}{p}} \
\leq K\left(m^{\frac{2}{p}}, M^{\frac{2}{p}}, p\right)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \tag{B'}
\]

Similarly we have $(C) \iff (C')$ and so the proof is complete. \hfill \Box

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