数理解析研究所講究録 1455 巻 2005 年 65-69

Functions related to some geometrical properties of Banach Spaces

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Abstract. We introduce some functions $\varphi_X(\tau)$ as a generalization (or refinement) of the *Schäffer constant S(X)* of Banach spaces X, and investigate geometrical properties of Banach spaces such as uniform non-squareness and uniform convexity in terms of those functions. The normal structure coefficient $N(X)$ is also estimated by the function $\varphi_X(\tau)$.

バナッハ空間 X の幾何学的性質の度合いを記述しようとすれば，幾何学的定数（あるいは関数）の考察が必要となる．James 定数 $J(X)$ と Schäffer 定数 $S(X)$ は uniform non-squareness の度合いを表し，modulus of convexity $\delta_X(\epsilon)$ は一様凸性の度合いを表す．Clarkson [4] が導入した von Neumann-Jordan 定数 $C_{NJ}(X)$ は，ヒルベルト空間の特徴づけに関連した概念であるが，最近では $J(X)$ との関係も考察されている (cf.[12])．また，$C_{NJ}(X)$ を用いて uniform non-squareness の特徴づけもできる (cf.[16])．しかしながら，これらの定数 $J(X)$, $S(X)$, $C_{NJ}(X)$ を用いて一様凸性などの幾何学的性質を記述することはできない．他方，不動点定理に関連した重要な概念である一様正規構造は正規構造係数で記述される．バナッハ空間 X が一様正規構造をもつための十分条件が，James 定数 $J(X)$ や von Neumann-Jordan 定数 $C_{NJ}(X)$ との関連で知られている (cf.[7,12,14])．本講演では，Schäffer 定数 $S(X)$ の概念を一般化（精密化）した関数 $\varphi_X(\tau)$ を導入し，uniform convexity, uniform non-squareness などの特徴づけ，更には，正規構造係数 $N(X)$ の $\varphi_X(\tau)$ による評価等を考察する．以下 X をバナッハ空間とする．

1. Definitions (i) X is called *uniformly non-square in the sense of James* when there exists $\delta > 0$ such that

$$\min(\|x+y\|, \|x-y\|) \le 2(1-\delta) \quad \text{if} \quad \|x\| = \|y\| = 1.$$

(ii) The *James constant* is defined by

$$J(X) := \sup \{ \min(\|x+y\|, \|x-y\|) : \ \|x\| = \|y\| = 1\}.$$

(iii) X is called *uniformly non-square in the sense of Schäffer* when there exists $\lambda > 1$ such that

$$\max(\|x+y\|, \|x-y\|) \ge \lambda \quad \text{if} \quad \|x\| = \|y\| = 1.$$

(iv) The *Schäffer constant* is defined by

$$S(X) := \inf\{\max(\|x+y\|, \|x-y\|) : \|x\| = \|y\| = 1\}.$$

(v) The *von Neumann-Jordan* (NJ-) *constant* of a Banach space X is the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C \quad for\ \forall\ (x,y) \neq (0,0)$$

holds; we denote it by $C_{NJ}(X)$.

(vi) The *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\right\} \quad (0 \leq \epsilon \leq 2).$$

X is called *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \leq 2$, and *q-unifromly convex* $(2 \leq q < \infty)$ if there is $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^q$ for all $0 < \epsilon \leq 2$.

It is obvious that X is uniformly non-square in the sense of James, resp., Schäffer if and only if $J(X) < 2$, resp., $S(X) > 1$. Since $J(X)S(X) = 2$ for any Banach space X (cf.[3,12]), these two notions are equivalent. It is known that X is uniformly non-square if and only if $C_{NJ}(X) < 2$ (cf.[16]). Let us recall that X is super-reflexive if any Banach space finitely representable in X is reflexive. It is well-known that if X is uniformly convex, or more generally, uniformly non-square, then X is reflexive. It is easy to see that if X is uniformly non-square, then any Banach space finitely representable in X is uniformly non-square. Thus, any uniformly non-square Banach space is super-reflexive (cf.[9]). Enflo [5] showed that X is super-reflexive if and only if X admits an equivalent uniformly convex norm. Pisier [13] also showed that if X is super-reflexive, then X admits an equivalent q-uniformly convex norm for some $2 \leq q < \infty$.

2. Definitions (i) A Banach space X is said to have *normal structure* if $r(K) < \mathrm{diam}(K)$ for every non-singleton closed bounded convex subset K of X, where $\mathrm{diam}(K) := \sup\{\|x-y\| : x, y \in K\}$ and $r(K) := \inf\{\sup\{\|x-y\| : y \in K\} : x \in K\}$.

(ii) The *normal structure coefficient* of X (Bynum [2]) is the number:

$$N(X) = \inf\{\mathrm{diam}(K)/r(K) : K \subset X \text{ bounded and convex}, \mathrm{diam}(K) > 0\}.$$

Obviously, $1 \leq N(X) \leq 2$. The space X is said to have *uniform normal structure* if $N(X) > 1$. It is well-known that if X has uniform normal structure, then X has fixed point property (cf.[8]). Gao and Lau [7] showed that if $J(X) < 3/2$, then X has uniform normal structure. Prus [14] even estimated the normal structure coefficient $N(X)$ by $J(X)$. Kato, Maligranda and Takahashi [12] also estimated the normal structure coefficient $N(X)$ by $C_{NJ}(X)$, and showed that if $C_{NJ}(X) < 5/4$, then X as well as its dual X' have the uniform normal structure.

3. Definitions (*Schäffer type constants*): We define for $\tau \geq 0$

$$S_{X,p}(\tau) := \begin{cases} \inf\left\{\left(\dfrac{\|x+\tau y\|^p + \|x-\tau y\|^p}{2}\right)^{1/p} : \|x\| = \|y\| = 1\right\} & \text{if } 1 < p < \infty, \\ \inf\left\{\max(\|x+\tau y\|, \|x-\tau y\|) : \|x\| = \|y\| = 1\right\} & \text{if } p = \infty. \end{cases}$$

Let X be a Banach space (of dimension at least 2). Let φ be a strictly convex and strictly increasing function defined on $[0,\infty)$ with values in $[0,\infty)$ (such a function is continuous on $[0,\infty)$). For simplicity, we assume that $\varphi(0) = 0$, $\varphi(1) = 1$.

4. Definition (*Generalized Schäffer type constant*): For $\tau \geq 0$ let

$$\varphi_X(\tau) = \inf\left\{\frac{\varphi(\|x+\tau y\|) + \varphi(\|x-\tau y\|)}{2} : \|x\| = \|y\| = 1\right\}$$

5. Remark If $\varphi(t) = t^p, 1 < p < \infty$, then $\varphi^{-1}(\varphi_X(\tau)) = S_{X,p}(\tau)$, where $S_{X,p}(\tau)$ is the *Schäffer type constant.*

6. Proposition $\varphi_X(\tau)$ is continuous and non-decreasing for $0 \leq \tau < \infty$.

7. Theorem X is uniformly non-square if and only if $\varphi_X(\tau) > 1$ for some $0 < \tau < 1$.

8. Corollary Let $1 < p \leq \infty$. The following are equivalent.
(1) X is uniformly non-square.
(2) $S_{X,p}(1) > 1$.
(3) $S_{X,p}(\tau) > 1 \quad (0 < \exists\tau < 1)$.
(4) $S_{X,p}(\tau) > \tau \quad (1 < \exists\tau < \infty)$.

9. Theorem X is uniformly convex if and only if $\varphi_X(\tau) > 1$ for any $0 < \tau < 1$.

10. Corollary Let $1 < p \leq \infty$. The following are equivalent.
(1) X is uniformly convex.
(2) $S_{X,p}(\tau) > 1 \quad (0 < \forall\tau < 1)$.
(3) $S_{X,p}(\tau) > \tau \quad (1 < \forall\tau < \infty)$.

11. Theorem $N(X) \geq \varphi^{-1}(\varphi_X(1/2))$. In particular, if $\varphi_X(1/2) > 1$, then X has uniform normal structure.

12. Corollary Let $1 < p \leq \infty$. Then

$$N(X) \geq S_{X,p}(1/2).$$

It is easy to see that if $C_{NJ}(X) < 5/4$, then $S_{X,2}(1/2) > 1$. Since $C_{NJ}(X) = C_{NJ}(X')$, we have

13. Corollary If $C_{NJ}(X) < 5/4$, then X as well as X' have the uniform normal structure.

14. Theorem Let $1 < p \leq \infty$ and $2 \leq q < \infty$. The following are equivalent.

(1) X is q-uniformly convex.

(2) There is $C > 0$ such that

$$S_{X,p}(\tau) \geq (1 + C\tau^q)^{1/q} \quad \text{for all } \tau \geq 0.$$

15. Theorem Let $2 \leq p < \infty$. Then the following are equivalent.

(1) X is isometric to a Hilbert space.

(2) $S_{X,p}(\tau) = (1 + \tau^2)^{1/2}$ for all $\tau \geq 0$.

16. Remark If X is a Hilbert space, then for all $\tau \geq 0$

$$S_{X,p}(\tau) = \left(\frac{|1+\tau|^p + |1-\tau|^p}{2}\right)^{1/p} \quad \text{if } 1 < p < 2.$$

Hence, the above theorem is false if $1 < p < 2$. Finally we calculate $S_{X,p}(\tau)$ in L_r-spaces.

17. Theorem Let X be an L_r-space with $\dim X \geq 2$.

(1) Let $1 < r \leq 2$ and $1/r + 1/r' = 1$. Then for all $\tau \geq 0$

$$S_{X,p}(\tau) = \left(\frac{|1+\tau|^r + |1-\tau|^r}{2}\right)^{1/r} \quad \text{if } r \leq p \leq \infty.$$

(2) Let $2 \leq r < \infty$ and $1/r + 1/r' = 1$. Then for all $\tau \geq 0$

$$S_{X,p}(\tau) = (1 + \tau^r)^{1/r} \quad \text{if } r' \leq p \leq \infty.$$

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