Schauder's Fixed Point Theorems in Complete Metric Spaces and Fuzzy Boundary Value Problems on an Infinite Interval

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Abstract

Aims of our study are follows: One is to prove that a complete metric space of fuzzy numbers becomes a Banach space under a condition that the metric has a homogeneous property. Another is to give sufficient conditions that a subset in the complete metric space and an into continuous mapping on the subset have at least one fixed point by applying Schauder's fixed point theorem. Finally we discuss a sufficient conditions for the existence of solutions of fuzzy differential equations on an infinite interval with boundary conditions.

1 Complete Metric Space of Fuzzy Numbers

Denote \( I = [0,1] \). The following definition means that a fuzzy number can be identified with a membership function.

**Definition 1** Denote a set of fuzzy numbers with bounded supports and strict fuzzy convexity by

\[ F_{b}^{st} = \{ \mu : \mathbb{R} \to I \text{ satisfying (i)-(iv) below} \}. \]

(i) \( \mu \) has a unique number \( m \in \mathbb{R} \) such that \( \mu(m) = 1 \) (normality);

(ii) \( \text{supp}(\mu) = \text{cl}(\{\xi \in \mathbb{R} : \mu(\xi) > 0\}) \) is bounded in \( \mathbb{R} \) (bounded support);

(iii) \( \mu \) is strictly fuzzy convex on \( \text{supp}(\mu) \) as follows:

(a) if \( \text{supp}(\mu) \neq \{m\} \), then

\[ \mu(\lambda \xi_1 + (1-\lambda)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)] \]

for \( \xi_1, \xi_2 \in \text{supp}(\mu) \) with \( \xi_1 \neq \xi_2 \) and \( 0 < \lambda < 1 \); 

(b) if \( \text{supp}(\mu) = \{m\} \), then \( \mu(m) = 1 \) and \( \mu(\xi) = 0 \) for \( \xi \neq m \);

(iv) \( \mu \) is upper semi-continuous on \( \mathbb{R} \) (upper semi-continuity).

It follows that \( \mathbb{R} \subset F_{b}^{st} \). Because \( m \) has a membership function as follows:

\[ \mu(m) = 1; \quad \mu(\xi) = 0 (\xi \neq m) \]  

Then \( \mu \) satisfies the above (i)-(iv).
In usual case a fuzzy number \( x \) satisfies fuzzy convex on \( \mathbb{R} \), i.e.,
\[
\mu(\lambda \xi_{1} + (1 - \lambda) \xi_{2}) \geq \min[\mu(\xi_{1}), \mu(\xi_{2})]
\]
for \( 0 \leq \lambda \leq 1 \) and \( \xi_{1}, \xi_{2} \in \mathbb{R} \). Denote \( \alpha \)-cut sets by
\[
L_{\alpha}(\mu) = \{ \xi \in \mathbb{R} : \mu(\xi) \geq \alpha \}
\]
for \( \alpha \in I \). When the membership function is fuzzy convex, then we have the following remarks.

**Remark 1** The following statements (1) - (4) are equivalent each other, provided with (i) of Definition 1.

1. (1.2) holds;
2. \( L_{\alpha}(\mu) \) is convex with respect to \( \alpha \in I \);
3. \( \mu \) is non-decreasing in \( \xi \in (-\infty, m) \), non-increasing in \( \xi \in [m, +\infty) \), respectively;
4. \( L_{\alpha}(\mu) \subseteq L_{\beta}(\mu) \) for \( \alpha > \beta \).

**Remark 2** The above condition (iiia) is stronger than (1.2). From (iiia) it follows that \( \mu(\xi) \) is strictly monotonously increasing in \( \xi \in [\min \text{supp}(\mu), m] \). Suppose that \( \mu(\xi_{1}) \geq \mu(\xi_{2}) \) for \( \xi_{1} < \xi_{2} \leq m \). From Remark 1(iii), it follows that \( \mu(\xi_{1}) = \mu(\xi_{2}) \) for some \( \xi_{1} < \xi_{2} \), so we get \( \mu(\xi) = \mu(\xi_{1}) = \mu(\xi_{2}) \) for \( \xi \in [\xi_{1}, \xi_{2}] \). This contradicts with Definition 1 (iiia). Thus \( \mu \) is strictly monotonously increasing. In the similar way \( \mu \) is strictly monotonously decreasing in \( \xi \in [m, \max \text{supp}(\mu)] \). This condition plays an important role in Theorem 1.

We introduce the following parametric representation of \( \mu \in \mathcal{F}_{\text{b}}^{st} \) as
\[
\begin{align*}
x_{1}(\alpha) &= \min L_{\alpha}(\mu), \\
x_{2}(\alpha) &= \max L_{\alpha}(\mu)
\end{align*}
\]
for \( 0 < \alpha \leq 1 \) and
\[
\begin{align*}
x_{1}(0) &= \min \text{supp}(\mu), \\
x_{2}(0) &= \max \text{supp}(\mu).
\end{align*}
\]

In the following example we illustrate typical types of fuzzy numbers.

**Example 1** Consider the following \( L - R \) fuzzy number \( x \in \mathcal{F}_{\text{b}}^{st} \) with a membership function as follows:
\[
\mu(\xi) = \begin{cases} 
L(\frac{m-\xi}{\ell})_{+} & (\xi \leq m) \\
R(\frac{\xi-m}{\epsilon})_{+} & (\xi > m)
\end{cases}
\]
Here it is said that \( m \in \mathbb{R} \) is a center and \( \ell > 0, \epsilon > 0 \) are spreads. \( L, R \) are \( I \)-valued functions. Let \( L(\xi)_{+} = \max(L(\xi), 0) \) etc. We identify \( \mu \) with \( x = (x_{1}, x_{2}) \). As long as there exist \( L^{-1} \) and \( R^{-1} \), we have \( x_{1}(\alpha) = m - L^{-1}(\alpha) \ell \) and \( x_{2}(\alpha) = m + R^{-1}(\alpha) \epsilon \).

Let \( L(\xi) = -c_{1}\xi + 1 \), where \( c_{1} > 0 \) and \( |x_{1} - m| \leq \ell \). We illustrate the following cases (i)-(iv).

(i) Let \( R(\xi) = -c_{2}\xi + 1 \), where \( c_{2} > 0 \). Then \( c_{2}\ell(x_{2} - m) = c_{1}r(m - x_{1}) \).

(ii) Let \( R(\xi) = -c_{2}\sqrt{\xi} + 1 \), where \( c_{2} > 0 \). Then \( c_{2}\ell(x_{2} - m)^{2} = c_{1}r^{2}(m - x_{1}) \).

(iii) Let \( R(\xi) = -c_{2}\xi^{2} + 1 \), where \( c_{2} > 0 \). Then \( c_{2}\ell^{2}(x_{2} - m) = c_{1}^{2}r(x_{1} - m)^{2} \).
Let $c$ be a real number such that $0 < c < 1$. Denote

$$L(\xi) = \begin{cases} 
1 & (\xi = 0) \\
-c\xi + c & (0 < \xi \leq 1)
\end{cases}$$

and let $R(\xi) = L(\xi)$. Then we have $\ell(x_2 - m) = r(m - x_1)$ for $|x_1 - m| \leq \ell$. The representation of $x = (x_1, x_2)$ is as follows:

$$
\begin{align*}
 x_1(\alpha) &= m - (1 - \frac{\alpha}{c})\ell \\
 x_2(\alpha) &= m + (1 - \frac{\alpha}{c})r \\
 x_1(\alpha) &= x_2(\alpha) = m & (c \leq \alpha \leq 1)
\end{align*}
$$

The membership function is given by as follows:

$$
\mu(\xi) = \begin{cases} 
0 & (\xi < x_1(0), \xi > x_2(0)) \\
x_1^{-1}(\xi) & (x_1(0) \leq \xi < m) \\
x_2^{-1}(\xi) & (m < \xi \leq x_2(0))
\end{cases}
$$

Denote by $C(I)$ the set of all the continuous functions on $I$ to $\mathbb{R}$. The following theorem shows a membership function is characterized by $x_1, x_2$.

**Theorem 1** Denote the left-, right-end points of the $\alpha$--cut set of $\mu \in F_b^{et}$ by $x_1(\alpha), x_2(\alpha)$, respectively. Here $x_1, x_2 : I \rightarrow \mathbb{R}$. The following properties (i)-(iii) hold.

(i) $x_1, x_2 \in C(I)$;

(ii) $\max_{\alpha \in I} x_1(\alpha) = x_1(1) = m = \min_{\alpha \in I} x_2(\alpha) = x_2(1)$;

(iii) $x_1, x_2$ are non-decreasing, non-increasing on $I$, respectively, as follows:

(a) there exists a positive number $c \leq 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $\alpha \in [0, c]$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $\alpha \in [c, 1]$;

(b) $x_1(\alpha) = x_2(\alpha) = m$ for $\alpha \in I$;

Conversely, under the above conditions (i) - (iii), if we denote

$$
\mu(\xi) = \sup_{\alpha \in I} \{\xi : x_1(\alpha) \leq \xi \leq x_2(\alpha)\} \quad (1.3)
$$

for $\xi \in \mathbb{R}$, then $\mu \in F_b^{st}$.

**Remark 3** From the above Condition (i) a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve over $\mathbb{R}^2$ and $x_1(\alpha) \leq x_2(\alpha)$ for $\alpha \in I$.

In what follows we denote $\mu = (x_1, x_2)$ for $\mu \in F_b^{et}$. The parametric representation of $\mu$ is very useful in calculating binary operations of fuzzy numbers and analyzing qualitative behaviors of fuzzy differential equations.

Let $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be an $\mathbb{R}$--valued function. The corresponding binary operation of two fuzzy numbers $x, y \in F_b^{et}$ to $g(x, y) : F_b^{et} \times F_b^{et} \rightarrow F_b^{et}$ is calculated by the extension principle of Zadeh. The membership function $\mu_{g(x,y)}$ of $g$ is as follows:

$$
\mu_{g(x,y)}(\xi) = \sup_{\xi = g(\xi_1, \xi_2)} \min(\mu_x(\xi_1), \mu_y(\xi_2))
$$
Here \( \xi, \xi_1, \xi_2 \in \mathbb{R} \) and \( \mu_x, \mu_y \) are membership functions of \( x, y \), respectively. From the extension principle, it follows that, in case where \( g(x, y) = x + y \),

\[
\begin{align*}
\mu_{x+y}(\xi) &= \max_{\xi = \xi_1 + \xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\
&= \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i), i = 1, 2\} \\
&= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}.
\end{align*}
\]

Thus we get \( x + y = (x_1 + y_1, x_2 + y_2) \). In the similar way \( x - y = (x_1 - y_2, x_2 - y_1) \).

Denote a metric by

\[
d_{\infty}(x, y) = \sup_{\alpha \in I} \max(|x_1(\alpha) - y_1(\alpha)|, |x_2(\alpha) - y_2(\alpha)|)
\]

for \( x = (x_1, x_2), y = (y_1, y_2) \in F_b^{st} \).

Theorem 2 \( F_b^{st} \) is a complete metric space in \( C(I)^2 \).

2 Induced Linear Spaces of Fuzzy Numbers

According to the extension principle of Zadeh, for respective membership functions \( \mu_x, \mu_y \) of \( x, y \in F_b^{st} \) and \( \lambda \in \mathbb{R} \), the following addition and a scalar product are given as follows:

\[
\begin{align*}
\mu_{x+y}(\xi) &= \sup\{\alpha \in [0,1] : \xi = \xi_1 + \xi_2, \xi_1 \in L_\alpha(\mu_x), \xi_2 \in L_\alpha(\mu_y)\}; \\
\mu_{\lambda x}(\xi) &= \left\{ \begin{array}{ll}
\mu_x(\xi/\lambda) & (\lambda \neq 0) \\
0 & (\lambda = 0, \xi \neq 0) \\
\sup_{\eta \in \mathbb{R}} \mu_x(\eta) & (\lambda = 0, \xi = 0)
\end{array} \right.
\end{align*}
\]

In [5] they introduced the following equivalence relation \( (x,y) \sim (u, v) \) for \( (x,y),(u,v) \in F_b^{st} \times F_b^{st} \), i.e.,

\[
(x,y) \sim (u,v) \iff x + v = u + y.
\] (2.4)

Putting \( x = (x_1, x_2), y = (y_1, y_2), u = (u_1, u_2), v = (v_1, v_2) \) by the parametric representation, the relation (2.4) means that the following equations hold.

\[
x_i + v_i = u_i + y_i \quad (i = 1, 2)
\]

Denote an equivalence class by \( [x,y] = \{(u,v) \in F_b^{st} \times F_b^{st} : (u,v) \sim (x,y)\} \) for \( x, y \in F_b^{st} \) and the set of equivalence classes by

\[
F_b^{st}/\sim = \{[x,y] : x, y \in F_b^{st}\}
\]

such that one of the following cases (i) and (ii) hold:

(i) if \( (x,y) \sim (u,v) \), then \( [x,y] = [u,v] \);

(ii) if \( (x,y) \not\sim (u,v) \), then \( [x,y] \cap [u,v] = \emptyset \).

Then \( F_b^{st}/\sim \) is a linear space with the following addition and scalar product

\[
[x,y] + [u,v] = [x+u, y+v]
\] (2.5)

\[
\lambda[x,y] = \left\{ \begin{array}{ll}
[\lambda x, \lambda y] & (\lambda \geq 0) \\
[(-\lambda)y, (-\lambda)x] & (\lambda < 0)
\end{array} \right.
\] (2.6)
for $\lambda \in \mathbb{R}$ and $[x, y], [u, v] \in \mathcal{F}^0 \sim$. They denote a norm in $\mathcal{F}^0 \sim$ by

$$||[x, y]|| = \sup_{\alpha \in I} d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)).$$

Here $d_H$ is the Hausdorff metric is as follows:

$$d_H(L_\alpha(\mu_x), L_\alpha(\mu_y)) = \max\left\{ \sup_{\xi \in L_\alpha(\mu_x)} \inf_{\eta \in L_\alpha(\mu_y)} |\xi - \eta|, \sup_{\eta \in L_\alpha(\mu_y)} \inf_{\xi \in L_\alpha(\mu_x)} |\xi - \eta| \right\}$$

It can be easily seen that $||[x, y]|| = d_\infty(x, y)$.

Note that $||[x, y]|| = 0$ in $\mathcal{F}^0 \sim$ if and only if $x = y$ in $\mathcal{F}^0$.

3 Schauder’s Fixed Point Theorem in Complete Metric Spaces

In the following theorem we show that the complete metric space $\mathcal{F}^0$ has an induced Banach space.

**Theorem 3** Let $S$ be a bounded closed subset in $\mathcal{F}^0_\mathrm{b}$. Assume that $S$ contains any segments of $x, y \in S$, i.e., $\lambda x + (1 - \lambda)y \in S$ for $\lambda \in I$. Let $V$ be an onto continuous mapping on $S$. Assume that the closure $\overline{V}(S)$ is compact in $\mathcal{F}^0_\mathrm{b}$. Then $V$ has at least one fixed point $x$ in $\overline{S}$, i.e., $V(x) = x$.

In the following theorem complete metric spaces have at least one fixed point of the induced Banach space.

**Theorem 4** Let $\mathcal{F}$ be a complete metric space with a metric $d$. Assume that $\mathcal{F}$ is closed under addition and scalar product, and that $d(\lambda x, 0) = |\lambda|d(x, 0)$ for the scalar product $\lambda x$ and $\lambda \in \mathbb{R}, x \in \mathcal{F}$. Denote $X = \{[x, 0]: x, 0 \in \mathcal{F}\}$. Here $[x, y]$ for $x, y \in \mathcal{F}$ are equivalence classes of (2.4) and 0 is the origin. Then $X$ is a Banach space concerning addition (2.5), scalar product (2.6) and norm $||[x, 0]|| = d(x, 0)$ for $[x, 0] \in X$.

Moreover let $S$ be a bounded closed subset in $\mathcal{F}$. Assume that $S$ contains any segments of $x, y \in S$ in the same meaning of Theorem 3. Let $V$ be an into continuous mapping on $S$. Assume that the closure $\overline{V}(S)$ is compact in $\mathcal{F}$. Then $V$ has at least one fixed point in $S$.

4 FBVP on Infinite Intervals

In this section we deal with the following FBVP on an infinite interval:

$$\frac{dx}{dt} = p(t)x + f(t, x), \quad x(\infty) = c \quad (4.7)$$

Here $p : \mathbb{R}_+ \rightarrow \mathcal{F}^0_\mathrm{b}, f : \mathbb{R}_+ \times \mathcal{F}^0 \rightarrow \mathcal{F}^0$ are continuous functions. Let denote $\mathbb{R}_+ = [0, \infty)$ and $c \in \mathcal{F}^0$. The following assumptions play important roles in considering the existence of solutions of (4.7).

**Assumption.**

(A1) Assume that

$$\int_0^\infty d(p(s), 0)ds = K < \infty.$$

(A2) There exist positive real numbers $a, r, R$ and integrable function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$d(f(t, x), 0) \leq m(t) \text{ for } (t, x) \in \mathbb{R}_+ \times \mathcal{S}_1;$$

$$\int_0^\infty m(s)ds \leq rR;$$

$$[R + N_p(a + || L || R)]K < 1.$$
Here

\[ S_1 = \{ x \in F^\epsilon_t : d(x, O) \leq \min(\alpha r, r) \} \]

and \( N^p \) is independent on the function \( p \).

\( L : C^\lim_r \to F^\epsilon_t \) is a linear operator as \( L(x) = x(\infty) \) and

\[ C^\lim_r = \{ x \in C(R_+ : F^\epsilon_t) : \exists x(\infty), d(x, 0) \leq r \} \].

(A3) There exists no solution of

\[ \frac{dx}{dt} = p(t)x, L(x) = 0 \]

except for the zero solution.

We expect the following existence theorem for solutions of FBVP on the infinite interval.

Under assumptions (A1) - (A3) we expect that there exists at least one solution of (4.7) in \( C^\lim_r \) for any \( c \in S_1 \) by applying the Schauder's fixed point theorem in \( C^\lim_r \).

References


