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費用負担戦略を基にしたネットワーク形成ゲームの強ナッシュ均衡について

On strong Nash equilibria of a network formation game with an endogenous cost allocation rule

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Abstract

This paper studies an endogenous network formation game based on well-known Connections Model first introduced by Jackson and Wolinsky (1996). We assume that the cost to construct a link can be shared between two players on the both ends through a bilateral negotiation process. Although this game always has a Nash equilibrium in which the underlying network is efficient, there also exist equilibria with inefficient networks. To solve this multiplicity of equilibria, we introduce a stronger equilibrium with a special type of symmetry, that we call indiscriminate Strong Nash Equilibrium (ISNE). It is shown that the ISNE necessarily achieves efficient networks and attains the core of the associated cooperative game.

Keywords: Network formation game; Connections Model; Endogenous cost allocation; Efficient network; Strong Nash Equilibrium

1 Introduction

Network Formation Game (NFG) is a fast growing domain of game theory. The literature includes Aumann and Myerson (1988), Jackson and Wolinsky (1996), and Slikker and van den Nouweland (2000). Among the existing studies, there are two major problems drawing researchers' attention. The first problem is determining the type or topology of network constructed in equilibrium. The second is identifying payoff structures, or rules of negotiation among players, that guarantee the equilibrium network to be efficient, i.e., socially optimal. In fact, it is not easy to find such a rule.

Our model is based upon the model introduced by Jackson and Wolinsky (1996), which is often referred to as Connections Model. In their model, the value of network connection between two nodes declines geometrically as the distance between the two increases. Here the distance is defined as the number of links constituting the shortest path between the two. The distance is sometimes "hops" in the context of telecommunication network design. In the original Connections Model, the cost to establish a link is split evenly to the adjacent nodes.

One of major weakness in the original Connections Model is that it does not guarantee the network in equilibrium to be efficient. In order to overcome this problem, we make a small modification to the original Connections Model. We allow some degree of freedom to negotiate on allocation of the link cost between two players on the both ends. It will be shown that this small modification enables to control the NFG so that it can always achieve an efficient network.

In our model, each player declares the player on the other edge of a link the upper limit of cost he / she can incur to construct the link. The link is decided to be constructed if the sum of declared costs exceeds the total construction cost. The negotiation tool of these two players is how much to declare as the upper limit. This bilateral negotiation is done simultaneously among all the player pairs and this game is therefore formulated as a game in strategic form.

We first show that Nash equilibrium which achieves an efficient network always exists in this modified Connections Model. However there also exists equilibrium which may lead to an inefficient network. Hence we consider a stronger equilibrium, Strong Nash Equilibrium (SNE). Unfortunately SNE alone is not sufficient to obtain an efficient network. However, by focusing onto a class of SNE where each player declares the same cost preference to the other players sharing links, it can be shown that such SNE guarantees the efficiency of the underlying network. We call the class of equilibrium to be indiscriminate SNE. The indiscriminate SNE in our game is explicitly identified in a closed form, and is always included in the core of the associated cooperative game although indiscriminate SNE is originally a solution of non-cooperative game.

To the end of this section, we would like to note that the endogenous assumption we added is practical one in many telecommunication applications, such as network formation of the Internet. The bilateral interconnection between Internet Service Providers (ISPs) is often referred as peering, and the cost allocation between the two is determined through negotiation. The quality of service of communication declines rapidly as number of hops between the source and the destination increases, which matches the assumption of Connections Model. Hence, our model will provide many insights to scholars studying Internet economics. For further discussion on interconnection issues among Internet service providers, see Bailey (1997).

This paper is organized as follows. In the next section, we review the existing literature related to NFG, identifying the position of our study. In Section 3, we formally introduce our modified Connections Model and provide some important property of the equilibrium. In Section 4, we provide conditions under which the efficient network can be achieved in equilibrium. In

Section 5, we discuss Strong Nash Equilibrium of our game and its efficiency. Finally, in Section 6, we consider a cooperative game derived from the modified Connections Model, and explore relationships between cooperative solutions and Strong Nash equilibria. Throughout this paper, we omit all proofs for theorems, which are available upon request.

2 Literature Review

Since the pioneering work of Myerson on the stability of networks was published in 1977, many models related to Network Formation Games (NFGs) have been introduced. They are categorized into two groups, exogenous models and endogenous models. They differ in timing in which players' payoffs are determined.

In the exogenous models, players' payoffs are predetermined corresponding to the network topology, and no negotiation on cost sharing is allowed. The only decision variable each player has is whom he / she wants to have direct links to. Jackson and Wolinsky (1996) introduced Connections Model, which is the basis of our model, and consider whether both the stability and the efficiency of the network can be compatible, or not. Their model assumes an exogenous cost sharing rule where the two players adjacent to the established link bear the cost evenly. Their conclusion is that the efficiency and the stability are not necessarily compatible. Jackson and Nouweland (2003) introduced a stronger stability concept in which network in equilibrium is not deviated from by any coalition among players. They concluded that relationship between strong stable network and efficient network in general depends on allocation rule that is given exogenously. However if a component-wise egalitarian allocation is employed, strong stable networks and efficient networks coincide.

In endogenous models, players negotiate on how much link cost to be shared. Our study in fact belongs to this group. Our model is based on Connections Model which is originally exogenous, but incorporates endogenous cost negotiation mechanism among players. Our study is closely related to Bala and Goyal (2000), and Bloch and Jackson (2004). The model of Bala and Goyal is an endogenous Connections Model but their model allows him / her to establish the link without agreement of the other player on the other edge when a player accepts to bear the whole cost of the link.

Bloch and Jackson (2004) consider an endogenous game with a bilateral negotiation process and characterize structure of Nash equilibrium. They also propose a sufficient condition under which efficient networks are supportable in equilibrium. A major difference of their study from ours is that like Slikker and Nouweland (2001) payoff itself is subject to negotiation in their model while our model allows players to negotiate only over the cost sharing. We focus our study on Connections Model and investigate Strong Nash Equilibrium as well as Nash equilibrium. In fact, we find a class of equilibrium which necessarily achieve efficient networks and in which the corresponding payoff necessarily attains a core allocation of the related cooperative game.

3 Model Description and Characteristics of Nash Equilibrium

In this section, we formally introduce our model. Our model is based on Connections Model first introduced by Jackson and Wolinsky (1996). We incorporate a negotiation process among players on cost sharing into the original Connections Model. In other words, our model is an endogenous version of Connections Model.

Let $N = \{1, 2, \ldots, n\}$ be the set of nodes of the network. Each node represents a player of the game. A link can be established for any node pair i and j. The construction cost for a link (i, j) connecting players i and j is c (> 0). In our model, the cost c for link (i, j) can be shared between player i and j. This endogenous nature of the game is the biggest difference from the original Connections Model.

The negotiation between players i and j is done in the following manner. First, player i declares player j the maximum amount of the cost that he / she incurs to construct link (i, j). We denote the declared cost by $x_{ij}(0 \leq x_{ij} \leq c)$. The strategy of player i is summarized as a vector $\boldsymbol{x}_i = \{x_{ij}\}$. For convenience, we denote the set of all $\boldsymbol{x}_i, i \in N$ by $\boldsymbol{x}$. Player j follows the same procedure and the two players take their actions simultaneously.

Link (i, j) is constructed if and only if the sum of the declared costs exceeds the construction cost for the connection. When the link is constructed, the actual cost c is split proportionally to declared cost x_{ij}. That is, the cost player i has to pay is $cx_{ij}/(x_{ij} + x_{ji})$. The resulting network constructed through the game is given by $G(\boldsymbol{x}) = (N, L(\boldsymbol{x}))$ where $L(\boldsymbol{x}) = \{(i, j) | x_{ij} + x_{ji} \geq c\}$.

In summary, given network $G(\boldsymbol{x}) = (N, L(\boldsymbol{x}))$, the payoff $r_i(\boldsymbol{x})$ of player i is

$$r_i(\boldsymbol{x}) \equiv \sum_{j \in N} \delta^{t_{ij}(G(\boldsymbol{x}))} - \sum_{j:(i,j) \in L(\boldsymbol{x})} c \frac{x_{ij}}{x_{ij} + x_{ji}} \tag{1}$$

where δ is a constant value satisfying $0 < \delta < 1$, and $t_{ij}(G(\boldsymbol{x}))$ is the number of links on the shortest path from i to j in $G(\boldsymbol{x})$. Here we define $t_{ij}(G(\boldsymbol{x})) = \infty$ if there is no path between i and j.

The definition of an equilibrium in the non-cooperative setting, i.e., Nash equilibrium, is given as a strategy profile $\boldsymbol{x}^*$ satisfying

$$r_i(\boldsymbol{x}^*) \geq r_i(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \text{ such that } 0 \leq x_{ik} \leq c \text{ and } \boldsymbol{x}_k = \boldsymbol{x}_k^* \text{ for } k \neq i.$$

4 Non-cooperative Implementation of Efficient Network

One of concerns among NFG researchers is whether or not Nash equilibrium achieves the socially optimal network. That question is negatively proved in the original Connections Model. Our game, on the other hand, guarantees the existence of a cost allocation vector $\boldsymbol{x}^*$ for which the efficient network G^* becomes an equilibrium. Before we prove this claim, we give the definition of efficient networks and an important lemma regarding topologies of efficient networks in possible conditions.

Definition 4. 1 *A network G^* is called to be efficient if* $G^* = \arg\max_G(\sum_{i \in N}\sum_{j \in N} \delta^{t_{ij}(G)} - \sum_{(i,j) \in L} c)$.

Lemma 4. 1 *(Bala and Goyal, 2000)*

Case 1: *If* $2(\delta - \delta^2) > c$, *then the efficient network is given by the* complete graph *of degree n, where every node is directly connected to every other node. We denote it by K_n.*

Case 2: *If* $2(\delta - \delta^2) < c < 2\delta + (n-2)\delta^2$, *then the efficient network is given by a* star graph *of degree n, where there is a* center *node and all links are between the center node and each other node. We denote it by $P_{n,\bar{i}}$ where $\bar{i}$ represents the center node of the network.*

Case 3: *If* $2\delta + (n-2)\delta^2 < c$, *then the efficient network is given by the* empty graph *where there is no link in the network. We denote it by $\emptyset_n$.*

The following theorem provide conditions under which the efficient network becomes Nash equilibrium in **Cases 1** through **3** described in Lemma 4. 1 above.

Theorem 4. 1 *For any $\delta > 0$, there exists a Nash equilibrium $\boldsymbol{x}^*$ for which $G(\boldsymbol{x}^*)$ is efficient. In fact, the following allocation rules satisfy the claim in* **Cases 1** *through* **3** *described in Lemma 4. 1 , respectively.*

Case 1: *For any x^*_{ij} satisfying* $\max(0, c - (\delta - \delta^2)) \leq x^*_{ij} \leq \min(\delta - \delta^2, c)$, *set* $x^*_{ji} = c - x^*_{ij}$.

Case 2: *For a player $\bar{i}$, who eventually becomes the center node, set $x^*_{\bar{i}i}$ so that* $\max(0, c-(\delta+(n-2)\delta^2)) \leq x^*_{\bar{i}i} \leq \min(c, \delta)(i \neq \bar{i})$. *For the other players, set* $x^*_{i\bar{i}} = c - x^*_{\bar{i}i}$ *and* $x^*_{ij} = 0$ $(j \neq \bar{i})$.

Case 3: *Set $x^*_{ij} = 0$ for any pair i, j.*

5 Indiscriminate Strong Nash Equilibrium and Its Efficiency

In Theorem 4. 1 one sees that there exist multiple equilibria in **Case 1** and **Case 2** which achieve efficient networks. However, there also exist equilibria with inefficient networks. This fact suggests that in order to predict an outcome of our game it does not suffice to analyze Nash equilibrium only. Hence, in this section we consider a refinement of Nash equilibrium. We discuss a stronger equilibrium concept, Strong Nash Equilibrium (SNE), which allow coalitions of players to deviate from equilibrium.

In this section we denote strategy of a coalition $S \subseteq N$ by $\boldsymbol{x}_S$ which consists of $|S| \times n$ component. We write the set of feasible $\boldsymbol{x}_S$ as X_S. We first define the concept of the deviation by a coalition from a strategy.

Definition 5. 1 *For a given strategy profile $\tilde{\boldsymbol{x}} \in X_N$ and a coalition $S \subseteq N$, we write $\tilde{\boldsymbol{x}}_{N-S} \in X_{N-S}$ as a vector constituting with components $\tilde{\boldsymbol{x}}_k$ of $\tilde{\boldsymbol{x}}$ such that $k \in N - S$.*

We call that the strategy $\tilde{\boldsymbol{x}}$ can be deviated from by the coalition S if and only if there exists a strategy $\hat{\boldsymbol{x}}_S \in X_S$ such that $r_i(\hat{\boldsymbol{x}}_S, \tilde{\boldsymbol{x}}_{N-S}) \geq r_i(\tilde{\boldsymbol{x}})$ for all $i \in S$ and $r_j(\hat{\boldsymbol{x}}_S, \tilde{\boldsymbol{x}}_{N-S}) > r_j(\tilde{\boldsymbol{x}})$ for at least one player $j \in S$.

Definition 5. 2 *A strategy $\boldsymbol{x}^* \in X_N$ is called to be in Strong Nash Equilibrium (SNE) if there is no coalition $S \subseteq N$ that can deviate from $\boldsymbol{x}^*$.*

In what follows, we examine efficiency of resulting networks in SNE. First, we gives an example demonstrating that SNE does not necessarily guarantee the resulting network to be efficient. Secondly, we introduce a class of *indiscriminate strategies* for which each player declares the same cost for every adjacent link to the node. Focusing SNGs onto ones with indiscriminate strategies, it is possible to explicitly identify strategy profiles in equilibrium. Furthermore it is shown that the resulting network is necessarily efficient.

Before proceeding formal definition of indiscriminate strategies, we give an example in which SNE generates an inefficient network. Namely SNE alone is not sufficient to obtain efficient network in equilibrium of NFG of this sort.

Example 5. 1 Consider a game with four players $N = \{1, 2, 3, 4\}$. Suppose it holds that $2\delta - 2\delta^3 < c < 2\delta + 2\delta^2 (= 2\delta + (n-2)\delta^2)$. Then, let us consider strategy such that $x_{12}^* = c - (\delta - \delta^3)$, $x_{21}^* = \delta - \delta^3$, $x_{23}^* = x_{32}^* = \frac{c}{2}$, $x_{34}^* = \delta - \delta^3$, $x_{43}^* = c - (\delta - \delta^3)$, and $x_{ij}^* = 0$ for others. The underlying network corresponding to $\boldsymbol{x}^*$ is a straight tandem network $1-2-3-4$. This is in fact inefficient but SNE.

The SNE in Example 5. 1 generates an inefficient network. However, this SNE seems unrealistic because players 2 and 3 declare different amounts of cost for their adjacent links. Our game assumes simultaneous move of players and then it seems difficult for the players to predict the network topology in equilibrium in advance. Therefore it might be difficult for each player to adjust cost declaration link-by-link. Hence we consider a natural assumption that each player declares the same amount of cost for his/her adjacent links.

Definition 5. 3 *A strategy $\boldsymbol{x}$ is said to be indiscriminate to adjacent links when for each player i, it holds that $x_{ij} = x_{ik}$ for $j, k (j \neq k)$ such that $(i, j) \in L(\boldsymbol{x})$ and $(i, k) \in L(\boldsymbol{x})$.*

In what follows, we show that indiscriminate SNE guarantees networks in equilibrium to be efficient. We for a moment focus on analysis for Case 2 because Cases 1 and 3 are relatively simple and straight forward. The following lemma characterizes payoffs of players for an indiscriminate SNE in Case 2.

Lemma 5. 1 *In* **Case 2**,*we suppose that x^* is an indiscriminate SNE. Then, for any player i with no less than two links in $G(x^*)$, it holds that*

$$x^*_{ij} \geq \delta - \delta^2$$

for any link $(i, j) \in L(x^*)$.

By using Lemma 5. 1 , we show the following lemma, which guarantees the efficiency of network derived in indiscriminate SNE.

Lemma 5. 2 *In* **Case 2**, *let x^* be an indistriminate SNE. Then, $G(x^*)$ is efficient.*

Now, we are ready to state the main theorem of this paper, which fully describes the set of all indiscriminate SNE's.

Theorem 5. 1 *Let C be the set of strategies x^* defined as follows.*
Case 1: For any i, j pair, set $x^*_{ij} = \frac{c}{2}$.
Case 2: For a given player $\bar{i}$, assign an arbitary $x^*_{\bar{i}}$ such that $\max(c - (\delta + (n-2)\delta^2), \delta - \delta^2) \leq x^*_{\bar{i}} \leq \min(c, \delta)$, and set $x^*_{\bar{i}i} = x^*_{\bar{i}}$ for any $i \neq \bar{i}$. For other players, set $x^*_{i\bar{i}} = c - x^*_{\bar{i}}$ and $x^*_{ij} = 0$ for $j \neq \bar{i}$.
Case 3: For any i, j pair, set $x^*_{ij} = 0$.

Then C coincides with the set of all indiscriminate SNE's. Furthermore, any strategy in C guarantees the efficiency of the underlying network.

6 Implementation of the Solutions of Cooperative NFG

In this section, we examine a further capability of the indiscriminate SNE. Consider a cooperative situation where all players jointly construct an efficient network. We can analyze the stable cost allocation among players for the efficient network by a cooperative game theoretic approach. Our purpose is to investigate the strong Nash implementation of the solutions of the cooperative NFG.

Now we define a cooperative game with characteristics function $V = (N, R)$ specified below.

$$R(S) = \max_{G_S}\left(\sum_{i \in S} \sum_{j \in S} \delta^{d_{ij}(G_S)} - \sum_{(i,j) \in L_S} c\right)$$

where $G_S = (S, L_S)$ is the set of possible networks formed by members of a coalition S. It is obvious that the network achieving the value $R(S)$ is the one specified in Lemma 4. 1 with degree $|S|$.

Of interest is whether a strong Nash solution can always realize a core allocation of game V, or not. Intuitively SNE and core of the cooperative game resemble each other. However, as

Jackson and Nouweland (2003) mentioned, it is not obvious how they are related in setting of network formation game.

This is partially due to the structure of payoff transfer among players. When some players decide to leave a coalition, the cooperative games require the members of the coalition to form a new network among them, whereas SNE allows them to maintain links with players outside the coalition. This observation raises a conjecture that the core includes the set of SNE. One may have an opposite impression. The core allows extreme flexibility for transferring payoffs among players although our network formation game only allows sharing of link cost among each player pair. Hence the core may not coincide with SNE.

We however prove that any payoff vectors associated with the set of indiscriminate SNE C are included in the core of the cooperative game V. In other words, indiscriminate SNE generates payoffs in the core of the game V.

Theorem 6. 1 *If $\boldsymbol{x}^* \in C$, then $r_i(\boldsymbol{x}^*)$ is a core of cooperative game V.*

Theorem 6. 1 shows that indiscriminate SNE always achieves a core allocation of associated cooperative game. Unfortunately, an indiscriminate strategy which achieves a core allocation is not necessarily included in SNE. The following is a counter example.

Example 6. 1 Consider a game with three players $N = \{\bar{i}, i, j\}$ and assume $2\delta > c > 2\delta - 2\delta^2$. Now the topology of any efficient network is a star. If $x^*_{\bar{i}i} = x^*_{\bar{i}j} = \frac{c}{2} - \delta^2$, $x^*_{i\bar{i}} = x^*_{j\bar{i}} = \frac{c}{2} + \delta^2$, and $x^*_{ij} = x^*_{ji} = 0$, then $\boldsymbol{x}^*$ is indiscriminate and furthermore becomes a core allocation.

However, since $2\delta > c$, $\max((c - (\delta + \delta^2), \delta - \delta^2) = \delta - \delta^2$ and $x^*_{\bar{i}i} = \frac{c}{2} - \delta^2 < \delta - \delta^2$ hold. By Theorem 5. 1 , $\boldsymbol{x}^*$ is not an SNE.

7 Concluding Remarks

We have studied an endogenous NFG based on Connections Model originally introduced Jackson and Wolinsky (1996). Although the original model does not guarantee the efficiency of the equilibrium network, it is proven that the efficiency and the stability are compatible in our model. The major problem here is that there also exist equilibria with inefficient networks.

To challenge the problem, we focus on equilibrium with indiscriminate strategies. Then it is shown that indiscriminate SNE guarantees the efficiency of underlying network. Furthermore, the set of indiscriminate SNE is fully described in a closed form. Although the indiscriminate SNE is originally non-cooperative solution, it is in fact included in the core of the associated cooperative game. This result is a demonstration about relationship between non-cooperative and cooperative games in the framework of endogenous Connections Model.

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