数理解析研究所講究録 1458 巻 2005 年 38-45

Some Problems in Fourier Analysis and Matrix Theory

Rajendra Bhatia
Indian Statistical Institute
7, S. J. S. Sansanwal Marg
New Delhi - 110 016.
Email : rbh@isid.ac.in

We discuss some problems studied in diverse contexts but with a common theme: the use of Fourier analysis to evaluate norms of some special matrices.

Let $\mathbb{M}_n$ be the space of $n\times n$ matrices. For $A\in\mathbb{M}_n$ let

$$\|A\|=\sup\{\|Ax\|:x\in\mathbb{C}^n,\|x\|=1\},$$

be the usual operator norm of A. Let $A\circ X$ be the entrywise product of two matrices A and X and let

$$\|A\|_S=\sup\{\|A\circ X\|:\|X\|=1\}.$$

This is the norm of the linear map on $\mathbb{M}_n$ defined as $X\mapsto A\circ X$. Since $A\circ X$ is a principal submatrix of $A\otimes X$, we have $\|A\circ X\|\leq\|A\otimes X\|=\|A\|\,\|X\|$, and hence

$$\|A\|_S\leq\|A\|.$$

Let $\lambda_1, \ldots, \lambda_n$ be distinct real numbers and let

$$\delta = \min_{i \neq j} |\lambda_i - \lambda_j| \, .$$

Let H be the skew-symmetric matrix with entries h_{rs} defined as

$$h_{rs} = \begin{cases} 1/(\lambda_r - \lambda_s) & r \neq s \\ 0 & r = s. \end{cases} \tag{1}$$

Motivated by problems arising in number theory, Montgomery and Vaughan [5] proved the following.

Theorem 1. *The norm of the matrix H is bounded as*

$$\|H\| \ \leq \ c_1/\delta, \tag{2}$$

where

$$c_1 \ = \ \inf \left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \varphi \geq 0, \text{ and } \hat{\varphi}(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\} . \tag{3}$$

Here $\hat{\varphi}$ stands for the Fourier transform of φ. Further,

$$c_1 = \pi. \tag{4}$$

A very special case of this theorem is "Hilbert's inequality". Let $\lambda_j = j, \ \ j = 1, 2, \ldots.$ Then the (infinite) matrix H defined by (1) is called the Hilbert matrix. Hilbert showed that H defines a bounded operator on the space ℓ_2 and $\|H\| < 2\pi$. This was improved upon by Schur who showed that $\|H\| = \pi$. Different proofs of this fact were discovered by others, one using Fourier series by Toeplitz. (Matrices structured as H are now called Toeplitz matrices.)

In particular, this shows that the inequality (2) with $c_1 = \pi$ is sharp (in the sense that if it is to hold for all n, then no constant smaller than π would work).

Now suppose we have two real n-tuples $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ where for all i and j we have $\lambda_i \neq \mu_j$. Let

$$\delta = \min_{i,j} |\lambda_i - \mu_j| .$$

Let M be the matrix with entries m_{rs} defined as

$$m_{rs} = \frac{1}{\lambda_r - \mu_s}. \tag{5}$$

Motivated by problems arising in perturbation theory, Bhatia, Davis and McIntosh [1] proved the following.

Theorem 2. *The norm* $\|M\|_S$ *is bounded as*

$$\|M\|_S \leq c_2/\delta, \tag{6}$$

where

$$c_2 = \inf\left\{ \|\varphi\|_{L_1} : \varphi \in L_1(\mathbb{R}), \hat{\varphi}(\xi) = \frac{1}{\xi} \text{ for } |\xi| \geq 1 \right\}. \tag{7}$$

The constant c_2 had been evaluated earlier by Sz-Nagy [6] and we have

$$c_2 = \frac{\pi}{2}. \tag{8}$$

Note that the infimum in (7) is over a class of functions larger than the one in (3).

It has been shown by McEachin [4] that the inequality (6) is sharp with $c_2 = \pi/2$, and the extremal value is attained when the points $\{\lambda_i\}$ and $\{\mu_j\}$ are regularly spaced.

The resemblance between the two problems is striking and it is a natural curiosity to ask whether good expressions for the norms $\|M\|$ and $\|H\|_S$ may be found to supplement what is known.

In [1] the authors considered also the case where $\{\lambda_i\}$ and $\{\mu_j\}$ are n-tuples of complex numbers with the same restriction as before, viz.,

$$\delta = \min_{i,j} |\lambda_i - \mu_j| > 0.$$

They proved the following.

Theorem 3. *Let M be the matrix (with complex entries) defined as in* (5). *Then*

$$\|M\|_S \leq c_3/\delta, \tag{9}$$

where

$$c_3 = \inf\left\{ \|\phi\|_{L_1} : \varphi \in L_1(\mathbb{R}^2), \hat{\varphi}(\xi_1, \xi_2) = \frac{1}{\xi_1 + i\xi_2} \text{ for } \xi_1^2 + \xi_2^2 \geq 1 \right\}. \tag{10}$$

An attempt to calculate the constant c_3 was made by Bhatia, Davis and Koosis [2]. These authors first obtained another characterisation of c_3. Let C be the class of all functions g on $\mathbb{R}$ that satisfy the following conditions

(i) g is even,

(ii) $g(x) = 0$ for $|x| \geq 1$,

(iii) $\int_{-1}^{1} g(x)dx = 1$,

(iv) $\hat{g} \in L_1(\mathbb{R})$.

The following theorem was proved in [2]

Theorem 4.
$$c_3 = \inf\left\{ \int_0^\infty |\hat{g}| : g \in C \right\}. \tag{11}$$

Using this the following estimate was derived in [2]

$$c_3 \leq \frac{\pi}{2} \int_0^\pi \frac{\sin t}{t} dt < 2.90901. \tag{12}$$

The constant c_2 occurs in another context called *Bohr's inequality.* This says that if a function f and its derivative f' satisfy the following conditions

(i) $f \in L_1(\mathbb{R}), \quad f' \in L_\infty(\mathbb{R})$,

(ii) $\hat{f}(\xi) = 0$ for $|\xi| \leq \delta$.

Then

$$\|f\|_\infty \leq \frac{c_2}{\delta}\|f'\|_\infty, \tag{13}$$

and the inequality is sharp.

Attempts have been made to extend this result to functions of several variables. Hörmander and Bernhardsson [3] have shown that if f is a function on $\mathbb{R}^2$ satisfying conditions akin to (i) and (ii) above, then

$$\|f\|_\infty \leq \frac{c_3}{\delta}\|\nabla f\|_\infty. \tag{14}$$

With this motivation they tried to evaluate c_3. Like the authors of [2], they too first prove (11), and then use it more effectively to show that

$$2.903887282 < c_3 < 2.90388728275228. \tag{15}$$

It would surely be of interest to find the exact value of c_3, especially since the formulas (4) and (8) are so attractive.

Some other problems remain open. The estimate (6) has been shown to be sharp by McEachin [4]. The question about (9) does not seem to have been addressed. The matrix (5) when $\{\lambda_i\}$ and $\{\mu_i\}$ are points on the unit circle was considered in [1]. An extremal problem involving Fourier series instead of Fourier transforms as in (7) and (10) arises in this case. This too has not been solved.

References

[1] R. Bhatia, C. Davis and A. McIntosh, *Perturbation of spectral subspaces and solution of linear operator equations,* Linear Algebra Appl., 52/53 (1983) 45-67.

[2] R. Bhatia, C. Davis and P. Koosis, *An extremal problem in Fourier analysis with applications to operator theory,* J. Funct. Anal., 82 (1989) 138-150.

[3] L. Hörmander and B. Bernhardsson, *An extension of Bohr's Inequality,* in *Boundary Value Problems for Partial Differential Equations and Applications,* J. L. Lions and C. Baiocchi, eds., Masson, Paris, 1993.

[4] R. McEachin, *A sharp estimate in an operator inequality,* Proc. Amer. Math. Soc., 115 (1992) 161-165.

[5] H. L. Montgomery and R. C. Vaughan, *Hilbert's Inequality,* J. London Math. Soc., 8 (1974) 73-82.

[6] B. Sz.-Nagy, *Über die Ungleichung von H. Bohr,* Math. Nachr. 9 (1953) 225-259.