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An extension of Kantorovich inequality to n-operators

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ABSTRACT

In this report, we shall extend Kantorovich inequality. This is an estimate by using the geometric mean of n-operators which have been defined by Ando-Li-Mathias in [1]. As a related result, we obtain a reverse inequality of arithmetic-geometric means one of n-operators via Kantorovich constant. Moreover, we give a formula of geometric mean of n-touples of 2-by-2 matrices with a trace condition, and we shall obtain more precise results of extended Kantorovich inequality in case 2-by-2 matrices case.

This is based on the following preprint:
[Y] T. Yamazaki, *An extension of Kantorovich inequality to n-operators via the geometric mean by Ando-Li-Mathias*, preprint.

1. INTRODUCTION

In what follows a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive if $\langle Tx, x\rangle \geq 0$ holds for all $x \in \mathcal{H}$. For an operator T such that $0 < mI < T < MI$, the following inequality is called "Kantorovich inequality" [6, 7]:

$$(1.1) \qquad \langle Tx, x\rangle\langle T^{-1}x, x\rangle \leq \frac{(m+M)^2}{4mM} \quad \text{for } \|x\| = 1.$$

We call the constant $\frac{(m+M)^2}{4mM}$ Kantorovich constant. (1.1) is closely related to properties of convex functions, and many authors have given many results and comments [3, 5, 9, 10, 12]. It is well known that (1.1) is equivalent to the following form by replacing x with $\frac{T^{\frac{1}{2}}x}{\|T^{\frac{1}{2}}x\|}$ in (1.1):

$$(1.1') \qquad \langle T^2x, x\rangle \leq \frac{(m+M)^2}{4mM}\langle Tx, x\rangle^2 \quad \text{for } \|x\| = 1.$$

For positive invertible operators A and B, the geometric mean $A\sharp B$ of A and B is defined as follows [8]:

$$A\sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

$A\sharp B$ is an extension of the geometric mean $\sqrt{ab}$ of positive numbers a and b. It is well known that Kantorovich inequality is equivalent to the following inequality [2]: *Let*

A and B be positive invertible operators whose spectrums are contained in $[m, M]$ *with* $0 < m < M$. *Then*

$$(1.2) \qquad \langle Ax, x\rangle\langle Bx, x\rangle \le \frac{(m+M)^2}{4mM}\langle A\sharp Bx, x\rangle^2 \quad \text{for } x \in \mathcal{H}.$$

In this report, we call it "Kantorovich inequality of 2-operators."

Very recently, as an extension of $A\sharp B$, the geometric mean $G(A_1, A_2, \cdots, A_n)$ of n-touples of positive invertible operators A_i have been defined by T. Ando, C.-K. Li and R. Mathias [1] as follows:

Definition 1 (Geometric mean of n-operators [1]). *Let* A_i *be positive invertible operators for* $i = 1, 2, \cdots, n$. *Then the geometric mean* $G(A_1, A_2, \cdots, A_n)$ *is defined by induction as follows:*

(i) $G(A_1, A_2) = A_1 \sharp A_2$.

(ii) *Assume that the geometric mean of any* $n-1$*-touple of operators is defined. Let*

$$G((A_j)_{j\neq i}) = G(A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n),$$

and let sequences $\{A_i^{(r)}\}_{r=0}^{\infty}$ *be* $A_i^{(0)} = A_i$ *and* $A_i^{(r)} = G((A_j^{(r-1)})_{j\neq i})$. *If there exists* $\lim_{r\to\infty} A_i^{(r)}$, *and it does not depend on* i, *then we define the geometric mean of* n*-operators as*

$$\lim_{r\to\infty} A_i^{(r)} = G(A_1, A_2, \cdots, A_n).$$

In [1], it has been shown that for any positive invertible operators A_i for $i = 1, 2, \cdots, n$, there exists $\lim_{r\to\infty} A_i^{(r)}$ and

$$\lim_{r\to\infty} A_i^{(r)} = G(A_1, A_2, \cdots, A_n),$$

uniformly. In fact, they have shown it for n-matrices in [1]. But by their proof, we can understand that the result can be extended to Hilbert space operators.

The geometric mean defined above has the following properties in [1]:

(P1) Consistency with scalars. If A_i commute with each other, then

$$G(A_1, A_2, \cdots, A_n) = (A_1 A_2 \cdots A_n)^{\frac{1}{n}}.$$

(P2) Joint homogeneity. For positive numbers s_i,

$$G(s_1A_1, s_2A_2, \cdots, s_nA_n) = (s_1 s_2 \cdots s_n)^{\frac{1}{n}} G(A_1, A_2, \cdots, A_n).$$

(P3) Permutation invariance. For any permutation $\pi(A_1, A_2, \cdots, A_n)$ of $(A_1, A_2, \cdots, A_n)$,

$$G(\pi(A_1, A_2, \cdots, A_n)) = G(A_1, A_2, \cdots, A_n).$$

(P4) Monotonicity. If $A_i \ge B_i > 0$, then $G(A_1, A_2, \cdots, A_n) \ge G(B_1, B_2, \cdots, B_n)$.

(P5) Continuity above. For each i, if $\{A_{i,k}\}_{k=1}^{\infty}$ are monotonic decreasing sequences converging to A_i as $k \to \infty$, respectively, then

$$\lim_{k\to\infty} G(A_{1,k}, A_{2,k}, \cdots, A_{n,k}) = G(A_1, A_2, \cdots, A_n)\}.$$

(P6) Congruence invariance. For an invertible operator S,

$$G(S^*A_1S, S^*A_2S, \cdots, S^*A_nS) = S^*G(A_1, A_2, \cdots, A_n)S.$$

(P7) Joint concavity. The map $(A_1, A_2, \cdots, A_n) \mapsto G(A_1, A_2, \cdots, A_n)$ is jointly concave, i.e., for $0 < \lambda < 1$,

$$G(\lambda A_1 + (1-\lambda)B_1, \lambda A_2 + (1-\lambda)B_2, \cdots, \lambda A_n + (1-\lambda)B_n) \geq \lambda G(A_1, A_2, \cdots, A_n) + (1-\lambda)G(B_1, B_2, \cdots, B_n).$$

(P8) Self-duality. $G(A_1, A_2, \cdots, A_n) = G(A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1})^{-1}$.

(P9) Determinant identity. For positive invertible matrices A_i,

$$\det(A_1, A_2, \cdots, A_n) = (\det A_1 \cdot \det A_2 \cdots \det A_n)^{\frac{1}{n}}.$$

Moreover, $G(A_1, A_2, \cdots, A_n)$ satisfies the arithmetic-geometric means inequality:

$$G(A_1, A_2, \cdots, A_n) \leq \frac{A_1 + A_2 + \cdots + A_n}{n}.$$

For positive numbers a_i, as a reverse inequality of arithmetic-geometric means one, it is known the following inequality [11]: *For positive numbers a_i with $0 < m < a_i < M$,*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq S_h \sqrt[n]{a_1 a_2 \cdots a_n}. \tag{1.3}$$

holds, where $h = \frac{M}{m} > 1$ and $S_h = \dfrac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$. We call S_h the Specht's ratio, and there are a lot of properties of Kantorovich constant and Specht's ratio in [3, 4, 5]. We remark that Specht's ratio in (1.3) is the optimal constant.

In this report, we shall give an extension of Kantorovich inequality of 2-operators to one of n-operators via geometric mean by Ando-Li-Mathias. As a related result of it, we shall discuss on an extension of (1.3). These results are estimates via Kantorovich constant. Next, we shall show more precise estimations of them under some cases.

2. Main results

Theorem 2.1. *Let A_i be positive operators for $i = 1, 2, \cdots, n$ satisfying $0 < mI \leq A_i \leq MI$ with $m < M$. Then*

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left\{ \frac{(m+M)^2}{4mM} \right\}^{\frac{n-1}{2}} G(A_1, A_2, \cdots, A_n).$$

Theorem 2.2. *Let A_i be positive operators for $i = 1, 2, \cdots, n$ satisfying $0 < mI \leq A_i \leq MI$ with $0 < m < M$. Then*

$$\langle A_1 x, x \rangle \langle A_2 x, x \rangle \cdots \langle A_n x, x \rangle \leq \left\{ \frac{(m+M)^2}{4mM} \right\}^{\frac{n(n-1)}{2}} \langle G(A_1, A_2, \cdots, A_n)x, x \rangle^n$$

holds for all $x \in \mathcal{H}$.

Remark. In [1], the following inequality has been already shown: *For positive invertible operators A_i,*

$$\langle G(A_1, A_2, \cdots, A_n)x, x \rangle^n \leq \langle A_1 x, x \rangle \langle A_2 x, x \rangle \cdots \langle A_n x, x \rangle.$$

Hence Theorem 2.2 is a reverse inequality of the above one.

For positive invertible operators A and B, as a kind of distance between A and B, the following $R(A,B)$ is defined in [1]:

$$R(A,B) = \max\{r(A^{-1}B), r(B^{-1}A)\},$$

where $r(T)$ means the spectral radius of T. Especially, the following inequality holds:

$$R(A_i^{(1)}, A_k^{(1)}) = R(G((A_j)_{j\neq i}), G((A_j)_{j\neq k})) \leq R(A_i, A_k)^{\frac{1}{n-1}}. \tag{2.1}$$

To prove the above theorems, we shall show the following lemma:

Lemma 2.3. *Let A_i be positive invertible operators for $i = 1, 2, \cdots, n$, and $h = \max_{i,j} R(A_i, A_j)$. Then*

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \left(\frac{1+h}{2\sqrt{h}}\right)^{n-1} G(A_1, A_2, \cdots, A_n).$$

Proof. Here we shall introduce the proof of the cases $n = 2$ and 3. The complete proof is obtained in [Y].

In case $n = 2$. Let $X = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$, and

$$X = \int \lambda dE_\lambda$$

be the spectral decomposition of X. Since $h = R(A,B)$, then we have $\frac{1}{h} \leq \lambda \leq h$ and

$$\frac{1+X}{2} = \int \frac{1+\lambda}{2} dE_\lambda = \int \frac{1+\lambda}{2\sqrt{\lambda}} \sqrt{\lambda} dE_\lambda \leq \int \frac{1+h}{2\sqrt{h}} \sqrt{\lambda} dE_\lambda = \frac{1+h}{2\sqrt{h}} X^{\frac{1}{2}}.$$

Hence we have

$$\frac{1 + A^{\frac{-1}{2}} B A^{\frac{-1}{2}}}{2} \leq \frac{1+h}{2\sqrt{h}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}}.$$

Multiplying $A^{\frac{1}{2}}$ to both sides of this inequality we have

$$\frac{A+B}{2} \leq \frac{1+h}{2\sqrt{h}} A\sharp B = \frac{1+h}{2\sqrt{h}} G(A,B).$$

Next we shall prove the case $n = 3$. For a nonnegative integer r, we define A_r, B_r, C_r, h_r and K_r as follows:

$$\begin{aligned}
&A_0 = A \text{ and } A_r = G(B_{r-1}, C_{r-1}), \\
&B_0 = B \text{ and } B_r = G(C_{r-1}, A_{r-1}), \\
&C_0 = C \text{ and } C_r = G(A_{r-1}, B_{r-1}), \\
&h_0 = h \text{ and } h_r = \max\{R(A_r, B_r), R(B_r, C_r), R(C_r, A_r)\}, \\
&K_r = \frac{1+h_r}{2\sqrt{h_r}}.
\end{aligned} \tag{2.2}$$

Then by the case $n = 2$, we have

$$\begin{aligned}\frac{A+B+C}{3} &= \frac{1}{3}\left(\frac{A_0+B_0}{2}+\frac{B_0+C_0}{2}+\frac{C_0+A_0}{2}\right)\\ &\le \frac{1}{3}(K_0G(A_0,B_0)+K_0G(B_0,C_0)+K_0G(C_0,A_0))\\ &= K_0\frac{A_1+B_1+C_1}{3}\\ &\le K_0K_1\frac{A_2+B_2+C_2}{3}\\ &\vdots\\ &\le K_0K_1\cdots K_r\frac{A_{r+1}+B_{r+1}+C_{r+1}}{3}.\end{aligned}$$

Since

$$\lim_{r\to\infty} A_r = G(A,B,C),$$

we have

$$\lim_{r\to\infty}\frac{A_{r+1}+B_{r+1}+C_{r+1}}{3} = G(A,B,C).$$

So we have only to prove the following inequality:

$$\lim_{r\to\infty} K_0K_1\cdots K_r \le K_0^2.$$

By (2.1), we have

$$1 \le h_r \le h_{r-1}^{\frac{1}{2}} \le \cdots \le h_0^{(\frac{1}{2})^r}.$$

Since

$$\frac{1}{2}\left(\frac{1}{x}+x\right) \le \frac{1}{2}\left(\frac{1}{y^\alpha}+y^\alpha\right) \le \left\{\frac{1}{2}\left(\frac{1}{y}+y\right)\right\}^\alpha$$

holds for $1 \le x \le y^\alpha$ and $\alpha \in (0,1]$, we have

$$K_r = \frac{1+h_r}{2\sqrt{h_r}} = \frac{1}{2}\left(\frac{1}{\sqrt{h_r}}+\sqrt{h_r}\right) \le \left\{\frac{1}{2}\left(\frac{1}{\sqrt{h_0}}+\sqrt{h_0}\right)\right\}^{(\frac{1}{2})^r} = K_0^{(\frac{1}{2})^r}.$$

Therefore we obtain

$$K_0K_1\cdots K_r \le K_0^{1+\frac{1}{2}+\cdots+(\frac{1}{2})^r} \longrightarrow K_0^2 \quad \text{as } r\to\infty.$$

Hence we have

$$\frac{A+B+C}{3} \le \left(\frac{1+h}{2\sqrt{h}}\right)^2 G(A,B,C).$$

This completes the proof. □

Proof of Theorem 2.1. By putting $h = \frac{M}{m}$ in Lemma 2.3. we obtain Theorem 2.1. □

Proof of Theorem 2.2. By using Theorem 2.1 and arithmetic-geometric means inequality, we have

$$\begin{aligned}\prod_{i=1}^{n}\langle A_i x, x\rangle^{\frac{1}{n}} &\leq \frac{1}{n}\sum_{i=1}^{n}\langle A_i x, x\rangle \\ &= \left\langle \frac{1}{n}\sum_{i=1}^{n} A_i x, x\right\rangle \\ &\leq \left\{\frac{(m+M)^2}{4mM}\right\}^{\frac{n-1}{2}} \langle G(A_1, A_2, \cdots, A_n)x, x\rangle.\end{aligned}$$

This completes the proof. □

3. More precise estimations

In this section, we shall give more precise estimations than the results shown in section 2 under some cases.

Theorem 3.1. *Let A, B, C be positive operators whose spectrums are contained in $[m, M]$ with $0 < m < M$. Then*

$$\frac{A+B+C}{3} \leq \frac{h^2-1}{2h\log h} G(A, B, C),$$

where $h = \frac{M}{m} > 1$.

Proof. As in the proof of Lemma 2.3, we have

$$\frac{A+B+C}{3} \leq K_0 K_1 \cdots K_r \frac{A_{r+1}+B_{r+1}+C_{r+1}}{3},$$

where

$$K_r = \frac{h_r+1}{2\sqrt{h_r}} \text{ and } h_r = \max\{R(A_r, B_r), R(B_r, C_r), R(C_r, A_r)\}.$$

By (2.1), $1 \leq h_r \leq h_{r-1}^{\frac{1}{2}} \leq \cdots \leq h^{\frac{1}{2^r}}$, and we obtain

$$K_r = \frac{1}{2}\left(\frac{1}{h_r^{\frac{1}{2}}} + h_r^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\frac{1}{h^{\frac{1}{2^{r+1}}}} + h^{\frac{1}{2^{r+1}}}\right) = \frac{h^{\frac{1}{2^r}}+1}{2h^{\frac{1}{2^{r+1}}}}.$$

Hence we have

$$\begin{aligned} K_0 K_1 \cdots K_r &\leq \frac{h+1}{2h^{\frac{1}{2}}} \cdot \frac{h^{\frac{1}{2}}+1}{2h^{\frac{1}{4}}} \cdots \frac{h^{\frac{1}{2^r}}+1}{2h^{\frac{1}{2^{r+1}}}} \\ &= \frac{h+1}{2h^{\frac{1}{2}}} \cdot \frac{h^{\frac{1}{2}}+1}{2h^{\frac{1}{4}}} \cdots \frac{h^{\frac{1}{2^{r-1}}}-1}{2h^{\frac{1}{2^{r+1}}}(h^{\frac{1}{2^r}}-1)} \\ &= \frac{h^2-1}{2^{r+1}h^{1-\frac{1}{2^{r+2}}}(h^{\frac{1}{2^r}}-1)} \\ &\longrightarrow \frac{h^2-1}{2h\log h} \quad \text{as } n \to \infty, \end{aligned}$$

where the limit is given by $\lim_{n\to\infty} n(h^{\frac{1}{n}}-1) = \log h$.

This completes the proof. $\square$

Theorem 3.2. *Let A, B, C be positive invertible operators whose spectrums are contained in $[m, M]$ with $0 < m < M$. Then*

$$\langle Ax, x\rangle\langle Bx, x\rangle\langle Cx, x\rangle \leq \left(\frac{h^2-1}{2h\log h}\right)^3 \langle G(A,B,C)x, x\rangle^3,$$

where $h = \frac{M}{m} > 1$.

Theorem 3.2 is easily obtained by the same way to the proof of Theorem 2.2.

Remark. In Theorem 3.1, we obtain a more precise constant $\frac{h^2-1}{2h\log h}$ than Theorem 2.1. However, this is not less than the Specht's ratio in (1.3) as follows: First of all, we shall show

$$f(h) = (h-1)\log(h+1) - (h-1)\log 2 - h\log h + (h-1) \geq 0 \quad \text{for } h \geq 1. \tag{3.1}$$

By easy calculation, we have

$$f'(h) = \log(h+1) - \log h - \frac{2}{h+1} + 1 - \log 2$$

$$f''(h) = \frac{h-1}{h(h+1)^2} \geq 0 \quad \text{for } h \geq 1.$$

Since $f'(1) = 0$ and $f''(h) \geq 0$ holds for $h \geq 1$, $f'(h) \geq 0$ for $h \geq 1$. Then by $f(1) = 0$ and $f'(h) \geq 0$ for $h \geq 1$, we have (3.1).

Next, (3.1) is equivalent to

$$\frac{h}{h-1}\log h - 1 \leq \log\left(\frac{h+1}{2}\right),$$

i.e.,

$$\frac{h^{\frac{1}{h-1}}}{e} \leq \frac{h+1}{2h} \quad \text{for } h \geq 1.$$

Hence we obtain

$$S_h = \frac{h-1}{\log h}\cdot\frac{h^{\frac{1}{h-1}}}{e} \leq \frac{h-1}{\log h}\cdot\frac{h+1}{2h} = \frac{h^2-1}{2h\log h}.$$

The next theorem is a formula of geometric mean of n-touples of 2-by-2 matrices.

Theorem 3.3. *Let A_i be positive 2-by-2 matrices satisfying the following conditions: (i) $\det A_i = 1$ (ii) $tr(A_i^{-1}A_j) = c$ (constant) for $i, j = 1, 2, \cdots, n$. Then*

$$G(A_1, A_2, \cdots, A_n) = \frac{A_1 + A_2 + \cdots + A_n}{\sqrt{\det(A_1 + A_2 + \cdots + A_n)}}.$$

In [1], the formula of geometric mean of 2-touples of 2-by-2 matrices has been shown, and Theorem 3.3 is an extension of it. To prove the result, we prepare the following lemma:

Lemma 3.4. *Let A_i be positive 2-by-2 matrices with* $\det A_i = 1$ *for* $i = 1, 2, \cdots, n$. *Then*

$$\det(A_1 + A_2 + \cdots + A_n) = n + \sum_{1 \le i < j \le n} tr(A_i^{-1} A_j).$$

Especially, if $tr(A_i^{-1} A_j) = c$ *(constant) for* $i, j = 1, 2, \cdots, n$, *then*

$$\det(A_1 + A_2 + \cdots + A_n) = n + \frac{n(n-1)}{2} c. \tag{3.2}$$

Proof. Here, we shall introduce the proof in cases $n = 2$ and 3. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ b_2 & d_2 \end{pmatrix} \text{ and } C = \begin{pmatrix} a_3 & b_3 \\ b_3 & d_3 \end{pmatrix}.$$

In case $n = 2$. Since $\det A = \det B = 1$, we have

$$\begin{aligned} \det(A + B) &= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)^2 \\ &= (a_1 d_1 - b_1^2) + (a_2 d_2 - b_2^2) + (a_1 d_2 + a_2 d_1 - 2 b_1 b_2) \\ &= 2 + \mathrm{tr}(A^{-1} B). \end{aligned}$$

Next we shall show the case $n = 3$. By the case $n = 2$ and $\det C = 1$, we have

$$\begin{aligned} &\det(A + B + C) \\ &= (a_1 + a_2 + a_3)(d_1 + d_2 + d_3) - (b_1 + b_2 + b_3)^2 \\ &= (a_1 + a_2)(d_1 + d_2) - (b_1 + b_2)^2 \\ &\qquad + d_3(a_1 + a_2) + a_3(d_1 + d_2) - 2b_3(b_1 + B_2) + a_3 d_3 - b_3^2 \\ &= 2 + \mathrm{tr}(A^{-1} B) + (a_1 d_3 + a_3 d_1 - 2 b_1 b_3) + (a_2 d_3 + a_3 d_2 - 2 b_2 b_3) + 1 \\ &= 3 + \mathrm{tr}(A^{-1} B) + \mathrm{tr}(A^{-1} C) + \mathrm{tr}(B^{-1} C). \end{aligned}$$

It completes the proof. □

Proof of Theorem 3.3. Here we will prove it the case $n = 3$. The case $n = 2$ have been proven in [1].

Let A_r, B_r and C_r be the geometric means which have been introduced in (2.2). Firstly, we will prove that they can be written as the following form:

$$\begin{aligned} A_r &= \alpha_r A + \beta_r B + \beta_r C \\ B_r &= \beta_r A + \alpha_r B + \beta_r C \\ C_r &= \beta_r A + \beta_r B + \alpha_r C \end{aligned} \tag{3.3}$$

In case $r = 1$, by the case $n = 2$ and (3.2) in Lemma 3.4, we have

$$\begin{aligned} A_1 &= G(B, C) = \frac{B + C}{\sqrt{\det(B + C)}} = \frac{B + C}{\sqrt{2 + c}}, \\ B_1 &= G(C, A) = \frac{C + A}{\sqrt{\det(C + A)}} = \frac{C + A}{\sqrt{2 + c}}, \\ C_1 &= G(A, B) = \frac{A + B}{\sqrt{\det(A + B)}} = \frac{A + B}{\sqrt{2 + c}}. \end{aligned}$$

Hence we have only to set α_1 and β_1 as follows:

$$\alpha_1 = 0 \text{ and } \beta_1 = \frac{1}{\sqrt{2+c}}.$$

Assume that A_{r-1}, B_{r-1}, C_{r-1} can be written as the following form:

$$\begin{aligned} A_{r-1} &= \alpha_{r-1}A + \beta_{r-1}B + \beta_{r-1}C, \\ B_{r-1} &= \beta_{r-1}A + \alpha_{r-1}B + \beta_{r-1}C, \\ C_{r-1} &= \beta_{r-1}A + \beta_{r-1}B + \alpha_{r-1}C. \end{aligned} \tag{3.4}$$

By the case $n = 2$, we have

$$\begin{aligned} A_r &= \frac{B_{r-1} + C_{r-1}}{\sqrt{\det(B_{r-1} + C_{r-1})}}, \\ B_r &= \frac{C_{r-1} + A_{r-1}}{\sqrt{\det(C_{r-1} + A_{r-1})}}, \\ C_r &= \frac{A_{r-1} + B_{r-1}}{\sqrt{\det(A_{r-1} + B_{r-1})}}. \end{aligned} \tag{3.5}$$

We will show that $\det(A_{r-1} + B_{r-1}) = \det(B_{r-1} + C_{r-1}) = \det(C_{r-1} + A_{r-1})$. Note that by (P9), $\det A_{r-1} = \det B_{r-1} = \det C_{r-1} = 1$ and (3.4), we have

$$\begin{aligned} \{A_{r-1}\}^{-1} &= \alpha_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \beta_{r-1}C^{-1}, \\ \{B_{r-1}\}^{-1} &= \beta_{r-1}A^{-1} + \alpha_{r-1}B^{-1} + \beta_{r-1}C^{-1}, \\ \{C_{r-1}\}^{-1} &= \beta_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \alpha_{r-1}C^{-1}. \end{aligned} \tag{3.6}$$

Since $\mathrm{tr}(A^{-1}B) = \mathrm{tr}(B^{-1}C) = \mathrm{tr}(C^{-1}A) = c$, we have

$$\begin{aligned} \mathrm{tr}(\{A_{r-1}\}^{-1}B_{r-1}) &= \mathrm{tr}\left(\left\{\alpha_{r-1}A^{-1} + \beta_{r-1}B^{-1} + \beta_{r-1}C^{-1}\right\}\{\beta_{r-1}A + \alpha_{r-1}B + \beta_{r-1}C\}\right) \\ &= c\alpha_{r-1}^2 + (4+2c)\alpha_{r-1}\beta_{r-1} + (1+c)\beta_{r-1}^2, \end{aligned}$$

and also we can set

$$\mathrm{tr}(\{A_{r-1}\}^{-1}B_{r-1}) = \mathrm{tr}(\{A_{r-1}\}^{-1}C_{r-1}) = \mathrm{tr}(\{B_{r-1}\}^{-1}C_{r-1}) = c_{r-1}.$$

By (3.2) in Lemma 3.4, we have

$$\det(A_{r-1} + B_{r-1}) = \det(B_{r-1} + C_{r-1}) = \det(C_{r-1} + A_{r-1}) = 2 + c_{r-1}.$$

Hence by (3.5), we obtain

$$\begin{aligned} A_r &= G(B_{r-1}, C_{r-1}) \\ &= \frac{B_{r-1} + C_{r-1}}{\sqrt{\det(B_{r-1} + C_{r-1})}} \\ &= \frac{2\beta_{r-1}A + (\alpha_{r-1} + \beta_{r-1})B + (\alpha_{r-1} + \beta_{r-1})C}{\sqrt{2 + c_{r-1}}}. \end{aligned}$$

Here we set

$$\alpha_r = \frac{2\beta_{r-1}}{\sqrt{2 + c_{r-1}}} \text{ and } \beta_r = \frac{\alpha_{r-1} + \beta_{r-1}}{\sqrt{2 + c_{r-1}}}.$$

Then

$$A_r = \alpha_r A + \beta_r B + \beta_r C.$$

Similarly, we have (3.3).

Next, it has been shown that

$$\lim_{r\to\infty} A_r = \lim_{r\to\infty} B_r = \lim_{r\to\infty} C_r = G(A,B,C).$$

Hence by (3.3), we have

$$\lim_{r\to\infty} \alpha_r = \lim_{r\to\infty} \beta_r = \alpha > 0,$$

and

$$G(A,B,C) = \alpha(A+B+C).$$

Here by (P9), $\det(G(A,B,C)) = 1$, and we have

$$G(A,B,C) = \frac{A+B+C}{\sqrt{\det(A+B+C)}}.$$

□

By Theorem 3.3, we have an extension of Kantorovich inequality of n-touple of 2-by-2 matrices which is a more precise estimation than Theorem 2.1.

Theorem 3.5. *Let A_i be positive 2-by-2 matrices for $i = 1,2,\cdots,n$ satisfying $\det A_i = 1$, $tr(A_i^{-1}A_j) = c$ (constant) and $0 < mI \leq A_i \leq MI$ with $m < M$. Then*

$$\frac{A_1 + A_2 + \cdots + A_n}{n} \leq \frac{(m+M)^2}{4mM} G(A_1, A_2, \cdots, A_n).$$

Theorem 3.6. *Let A_i be positive 2-by-2 matrices for $i = 1,2,\cdots,n$ satisfying $\det A_i = 1$, $tr(A_i^{-1}A_j) = c$ (constant) and $0 < mI \leq A_i \leq MI$ with $m < M$. Then*

$$\langle A_1 x, x\rangle\langle A_2 x, x\rangle \cdots \langle A_n x, x\rangle \leq \left\{\frac{(m+M)^2}{4mM}\right\}^n \langle G(A_1, A_2, \cdots, A_n)x, x\rangle^n$$

holds for all $x \in C^2$.

To prove above results, we give the following inequality:

Lemma 3.7. *Let A_i be positive 2-by-2 matrices satisfying $\det A_i = 1$ and $0 < mI \leq A_i \leq MI$ with $m < M$. Then*

$$\det\left(\frac{A_1 + A_2 + \cdots + A_n}{n}\right) \leq \left\{\frac{(m+M)^2}{4mM}\right\}^2.$$

Proof. For each i, let $0 < m_i I \leq A_i \leq M_i I$ and $M = \max_i M_i$. Note that we have $m_i M_i = 1$ and $m = \frac{1}{M}$ by $\det A_i = 1$.

Let $A_i = \begin{pmatrix} a_i & b_i \\ b_i & d_i \end{pmatrix}$, and let

$$S = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \quad T = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \text{ and } x = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then we have $0 < mI \leq S \leq MI$ and $0 < mI \leq T \leq MI$ and

$$\begin{aligned}\det\left(\frac{A_1+A_2+\cdots+A_n}{n}\right) &\leq \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\left(\frac{d_1+d_2+\cdots+d_n}{n}\right)\\ &= \langle Sx,x\rangle\langle Tx,x\rangle\\ &\leq \frac{(m+M)^2}{4mM}\langle S\sharp Tx,x\rangle^2 \quad \text{by (1.2)}\\ &= \frac{(m+M)^2}{4mM}\left(\frac{\sqrt{a_1d_1}+\sqrt{a_2d_2}+\cdots+\sqrt{a_nd_n}}{n}\right)^2.\end{aligned}$$

Here by $M_i \leq M$ and $m = \dfrac{1}{M}$,

$$\sqrt{a_id_i} \leq \frac{a_i+d_i}{2} = \frac{\mathrm{tr}A_i}{2} = \frac{m_i+M_i}{2} = \frac{1}{2}\left(\frac{1}{M_i}+M_i\right) \leq \frac{1}{2}\left(\frac{1}{M}+M\right) = \frac{m+M}{2}.$$

Therefore we obtain

$$\begin{aligned}\det\left(\frac{A_1+A_2+\cdots+A_n}{n}\right) &\leq \frac{(m+M)^2}{4mM}\left(\frac{\sqrt{a_1d_1}+\sqrt{a_2d_2}+\cdots+\sqrt{a_nd_n}}{n}\right)^2\\ &\leq \frac{(m+M)^2}{4mM}\left(\frac{m+M}{2}\right)^2\\ &= \left\{\frac{(m+M)^2}{4mM}\right\}^2 \quad \text{by } mM = 1.\end{aligned}$$

It completes the proof. □

Proof of Theorem 3.5. By Theorem 3.3 and Lemma 3.7, we have

$$\begin{aligned}\frac{A_1+A_2+\cdots+A_n}{n} &= \frac{\sqrt{\det(A_1+A_2+\cdots+A_n)}}{n}G(A_1,A_2,\cdots,A_n)\\ &= \sqrt{\det\left(\frac{A_1+A_2+\cdots+A_n}{n}\right)}G(A_1,A_2,\cdots,A_n)\\ &\leq \frac{(m+M)^2}{4mM}G(A_1,A_2,\cdots,A_n).\end{aligned}$$

□

Proof of Theorem 3.6 is the same as one of Theorem 2.2.

Remark. It is not known whether the constant $\frac{(m+M)^2}{4mM}$ in Theorem 3.5 is optimal or not. But in [5, p. 224, Remark 8.1], it is known that for $0 < m < M$ and $h = \frac{M}{m} > 1$,

$$S_h \leq \frac{(m+M)^2}{4mM},$$

i.e., the constant $\dfrac{(m+M)^2}{4mM}$ in Theorem 3.5 is bigger than one of (1.3).

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