NON-DEGENERATE BILINEAR FORMS AND FIBER
FUNCTORS

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ABSTRACT. We shall review a categorical approach to the basic
quantum group $SL_q(2, \mathbb{C})$ or $SU_q(2)$ based on the preprint S. Ya-
magami, Fiber functors on Temperley-Lieb categories,

1. CLASSICAL TANNAKA DUALITY

The notion of tensor category is traced back to the celebrated work
of T. Tannaka on a duality theory of compact groups.

Given a compact group $G$, let

$\mathcal{R}ep(G) =$ the category of finite-dimensional unitary representations of $G$,

which is referred to as the Tannaka dual of $G$ and provides a typical example of tensor categories:

- Given two objects (representations) $V, W$, another object $V \otimes W$
is associated functorially so that $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
- There is a special object $I$, the trivial representation of $G$, which
  satisfies $I \otimes V = V \otimes I$ for any $V$.
- We have the operation of taking costragradient representation
  $V \rightarrow V^*$, which is categorically characterized by the existence
  of morphisms $\epsilon : V \otimes V^* \rightarrow I$ and $\delta : I \rightarrow V^* \otimes V$
such that the compositions

$$
\begin{align*}
V & \xrightarrow{1_V \otimes \delta} V \otimes V^* \otimes V \xrightarrow{\epsilon \otimes 1_V} V, \\
V^* & \xrightarrow{\delta \otimes 1_{V^*}} V^* \otimes V \otimes V^* \xrightarrow{1_{V^*} \otimes \epsilon} V^*
\end{align*}
$$

are identities.

The Tannaka dual has the special feature that it is realized as a sub-
category of $\mathcal{V}ec$, the category of finite-dimensional vector spaces, or
$\mathcal{H}ilb$, the category of finite-dimensional Hilbert spaces.

The celebrated Tannaka duality states that the group $G$ itself is
recovered by looking at the categorical information on representations.
Here are some important generalizations to quantum groups.
Unitary version (Woronowicz, 1988):

Compact quantum groups
\(\Longleftrightarrow\) rigid C*-tensor categories \(\subset \mathcal{H}ilb\).

Algebraic version (Ulbrich, 1990):

Algebraic quantum groups
\(\Longleftrightarrow\) rigid tensor categories \(\subset \text{Vec}\).

Typical examples are \(SU_q(2) (q \in \mathbb{R}^x)\) and \(SL_q(2, \mathbb{C}) (q \in \mathbb{C}^x)\).

2. FIBER FUNCTORS

We shall here split the relevant information in Tannaka duality into two parts. Given an abstract tensor category \(T\), a fiber functor on \(T\) is, by definition, a faithful tensor functor \(F : T \to \text{Vec}\). Each object \(X\), which is considered to an abstract label, produces a finite-dimensional vector space \(F(X)\) in such a way that \(F(X \otimes Y) = F(X) \otimes F(Y)\). In other words, a fiber functor is a kind of representation of an abstract tensor category \(T\) in terms of the concrete tensor category \(\text{Vec}\) or \(\mathcal{H}ilb\).

Viewing quantum groups this way, we are naturally lead to the problem of their representation theoretical classifications: Again we have two stages.

- Classify abstract tensor categories.
- Classify fiber functors on a tensor category up to equivalences.

Here 'equivalences' are with respect to a natural equivalence, say \(\{\varphi_X\}\), preserving tensor products.

\[
\begin{align*}
F(X) & \xrightarrow{\varphi_X} F'(X) \\
F(f) & \downarrow F'(f) \\
F(Y) & \xrightarrow{\varphi_Y} F'(Y)
\end{align*}
\]

\(\varphi_{X \otimes Y} = \varphi_X \otimes \varphi_Y\).

3. TEMPERLEY-LIEB CATEGORIES

Generally classification is a difficult problem for tensor categories because it consists of determining moduli for non-linear equations. We shall here restrict ourselves to the simple but fundamental case of Temperley-Lieb category \(\mathcal{T}L_d\) (\(d \in C^x\) being a complex parameter), which is the linearization of the monoidal category of Kauffman's
monoids and turns out to be the representation category of the quantum group $SL_q(2,\mathbb{C})$.

$$\mathcal{T}_d \equiv \mathcal{R}ep(SL_q(2,\mathbb{C})), \quad \text{with} \quad d = -q - q^{-1}.$$  

By definition, Kauffman's monoids are isotopy classes of planar strings stretched out between upper and lower boundaries of a strip. Let $K_{m,n}$ be the set of Kauffman's monoids with $m$ and $n$ vertices placed on upper and lower boundaries respectively. Here is a figure for the case $K_{2,4}$.

Objects in $\mathcal{T}_d$ are labeled by the natural numbers $\{0, 1, 2, \ldots\}$ with hom-sets given by $\mathcal{T}_d(n, m) = \mathbb{C}[K_{m,n}]$, the free vector space generated by the set $K_{m,n}$. We regard each $n$ as representing $n$-th tensor product of the object $V$ labeled by 1:

$$n \iff V \otimes \cdots \otimes V.$$

The structure of tensor category is then defined as follows:

- The operation of composition is given by the concatenation of monoids with each closed circle replaced by the complex number $d$.

$$\circ \quad = \quad d \quad \circ$$

- Tensor product is defined by the horizontal juxtaposition of base monoids:

$$(\mathrm{I} \cap) \otimes (\mathrm{I} \cap) = \mathrm{I} \cap.$$ 

Note here that

$$\mathcal{T}_d \not\cong \mathcal{T}_{d'} \quad \text{if} \quad d \neq d'.$$

With this geometrical presentation of Temperley-Lieb categories, we have
Theorem

\[ \bigcup_{d \in \mathbb{C}^{x}} \left\{ \text{fiber functors on } \mathcal{T}_{d} \right\} / \sim \cong \{ \text{non-degenerate bilinear forms} \} / \sim \]

\[ B \sim B' \iff B' = t^{T}Bt \text{ for some invertible linear map } T. \]

Basic arcs in Kauffman’s monoids are replaced with non-degenerate bilinear forms by fiber functors.

With a choice of linear basis, non-degenerate bilinear forms are represented by invertible matrices and the above classification problem can be dealt with by the following result:

**Williamson-Wall Theorem:** Let

\[ \Theta : GL(n, \mathbb{C}) \ni B \mapsto ^{t}B^{-1}B \in GL(n, \mathbb{C}). \]

Then

\[ GL(n, \mathbb{C})/ \sim \cong \Theta(GL(n, \mathbb{C}))/\text{similarity}. \]

An invertible matrix \( M \) belongs to \( \Theta(GL(n, \mathbb{C})) \) if and only if

1. \( \mu_{M}(z) = \mu_{M}(z^{-1}) \) for \( z \in \mathbb{C}^{x} \),
2. \( \mu_{M}^{(k)}(1) \) is even for even \( k \),
3. \( \mu_{M}^{(k)}(-1) \) is even for odd \( k \).

Here

\[ \mu_{M}(z) = (\mu_{M}^{(1)}(z), \mu_{M}^{(2)}(z), \ldots), \]

denotes the multiplicity function of the matrix \( M \): \( \mu_{M}^{(k)}(z) \) is the multiplicity of \( z \)-Jordan block of size \( k \) in \( M \). Note here that the parameter is related to \( \Theta(B) \) by the formula

\[ d = \text{trace}(\Theta(B)) = \text{trace}(^{t}B^{-1}B). \]

**Example:** We have the following identifications.

\[ GL(2, \mathbb{C})/ \sim = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \bigg/ (q \leftrightarrow q^{-1}) \bigg/ \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \]

with

\[ \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \leftrightarrow \text{SL}_{q^{-1}}(2, \mathbb{C}), \]
\[ \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \text{Woronowicz' Hopf algebra}. \]
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More generally, algebraic quantum groups of Dubois-Violette and Launer (1990), together with their representation categories, are classified in this way.

4. UNITARY FIBER FUNCTORS

A C*-tensor category is, by definition, a tensor category with a compatible C*-structure. It is well-known that the Temperley-Lieb category $\mathcal{T}_d$ is a C*-tensor category if and only if $d \in \mathbb{R}$ and $|d| \geq 2$. Moreover, the C*-structure is unique up to natural equivalences.

We can work out a similar characterization of unitary fiber functors on the Temperley-Lieb C*-category $\mathcal{T}_d$:

\[
\{\text{unitary fiber functors on } \mathcal{T}_d\} / \sim \cong \left\{ \Phi : V \to \overline{V}; \Phi^{-1} = \frac{d}{|d|} \overline{\Phi} \right\} / \sim,
\]

where the equivalence relation $\sim$ is defined by $\Phi \sim {}^tU\Phi U$ with $U$ a unitary operator on $V$.

In terms of an eigenvalue list of the positive part $|\Phi|$ of $\Phi$, we have the following description:

**Theorem:**
An equivalence class of unitary fiber functors $\Leftrightarrow$ an unordered sequence $\{h_j > 0\}$ such that

\[
\begin{align*}
\{h_j^{-1}\} &= \{h_j\}, \\
\text{tr}(|\Phi|^2) &= |d|, \\
(d/|d|)^m &= 1, \\
|d| &= \dim \ker(|\Phi| - 1).
\end{align*}
\]

(i) $n = 2$: For each $|d| \geq 2$, $\exists$ a unique

\[
\{h, h^{-1}\}, h \geq 1, h^2 + h^{-2} = |d|.
\]

(ii) $n = 3$: For $d \geq 3$, a new choice

\[
\{h, h^{-1}, 1\}, h \geq 1, h^2 + h^{-2} + 1 = d.
\]

(iii) $n = 2k$: For $|d| \geq n$, $\exists$ a unique

\[
\{h_j, h_j^{-1}\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = |d|.
\]
(iv) $n = 2k + 1$: For $d \geq n$,
$$\{h_j, h_j^{-1}, 1\}, h_j \geq 1, \sum_j (h_j^2 + h_j^{-2}) = d - 1.$$ 

Set $t_j = h_j^2 + h_j^{-2} - 2$,
$$\sum_j t_j = |d| - 2k \text{ or } \sum_j t_j = d - 2k - 1.$$ 

Then the parameter space (moduli) of unitary fiber functors is the $k - 1$-dimensional simplex
$$\{(t_1, \ldots, t_k); t_j \geq 0, \sum t_j = r\}/S_k.$$ 

In this way, we have multiparameter families of compact quantum groups, which turns out to be Wang-Banica’s universal quantum group of orthogonal type.