$AW^*$-triples, partial *-isomorphisms and Morita equivalence

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Let $B$ and $C$ be two $AW^*$-subalgebras of an $AW^*$-algebra $A$. In this talk, we describe the relative position of $B$ and $C$ in $A$ (e.g., Morita equivalence, etc.) in terms of $AW^*$-subtriples of $A$, the normalizer of $B$ and $C$ in $A$, or partial *-isomorphisms between $B$ and $C$ (the definitions will be given below). Then, under a stronger assumption of $B$ and $C$ being monotone complete, we show that the set of these $AW^*$-subtriples is embedded naturally in an inverse semigroup associated with $B$ and $C$. The key idea is to regard $AW^*$-algebras and $AW^*$-triples as a generalization of projections and partial isometries.

1 Introduction

If $X$ is a linear subspace of a *-algebra $A$, the two conditions that $X^2 \subset X = X^*$ ($X$ being a *-subalgebra of $A$) and that $XX^*X \subset X$ (we call such an $X$ a subtriple of $A$) are regarded as a generalization of the notions of projection ($p^2 = p = p^*$) and partial isometry ($xx^*x = x$), respectively. Here, for $X, Y, Z \subset A$, we write $XY := \{xy : x \in X, y \in Y\}$, $X^2 := XX$, $X^* := \{x^* : x \in X\}$, and $XYZ := (XY)Z = X(YZ)$. For a subtriple $X$ of $A$ the sets $B := \text{lin } XX^*$ and $C := \text{lin } X^*X$ (lin denotes linear span) are *-subalgebras of $A$, and the relation among $B, C$ and $X$ is viewed as an analogue of the relation among Murray-von Neumann equivalent projections and the partial isometry implementing the equivalence.

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If we adjust the above situation slightly, we obtain the notions of strong Morita equivalence for $C^*$-algebras and Morita equivalence for von Neumann algebras in the sense of M. Rieffel [7]. That is, two $C^*$-algebra $B$ and $C$ are strong Morita equivalent if there exist a $C^*$-algebra $A$ containing $B$ and $C$ as $C^*$-subalgebras and a norm closed subtriple $X$ of $A$ such that $B = \overline{\text{lin}} X X^*$ and $C = \overline{\text{lin}} X^* X$. Two von Neumann algebras $B$ and $C$ are Morita equivalent if there exist a von Neumann algebra $A$ containing $B$ and $C$ as von Neumann subalgebras and a $\sigma$-weakly closed subtriple $X$ of $A$ such that $B = \overline{\text{lin}}^\sigma X X^*$ and $C = \overline{\text{lin}}^\sigma X^* X$. Here $\{\}$, $\overline{\{\}}$ denote respectively norm closure and $\sigma$-weak closure. The linking algebra technique ([3]) shows that the definitions of (strong) Morita equivalences above are equivalent to the usual ones, which are defined in terms of imprimitivity bimodules.

Subtriples in the above sense arise naturally in the theory of operator algebras, The following fact, which will be worked out in another paper, was a motivation for considering them and introducing an inverse semigroup structure of them in [4]. Let $A$ be a von Neumann algebra and let $\{X_g\}_{g \in G}$ be a family of $\sigma$-weakly closed linear subspaces of $A$ indexed by a discrete group $G$ such that

$$X^*_g = X_{g^{-1}}, \quad X_{g_1} X_{g_2} \subseteq X_{g_1 g_2}, \quad \forall g, g_1, g_2 \in G$$

(such a family corresponds to each coaction of $G$ on $A$). Then each $X_g$ is a subtriple of $A$, the algebraic direct sum $A := \bigoplus_{g \in G} X_g$, with the product and involution inherited from $A$, is a $G$-graded *-algebra, and under a certain technical assumption, $A$ (and $A$ also if $\{X_g\}$ is associated with a coaction of $G$) is viewed as the twisted crossed product $B \rtimes_{\theta, u} G$ with respect to a twisted action $(\theta, u)$ of $G$ on the von Neumann subalgebra $B := X_e$, and is described in terms of only $B$ and $G$. Indeed, it follows from Theorem 1 below that each $X_g = Bs_g B$ for some $s_g \in \text{PI} A$ (partial isometries of $A$). If $B$ is $\sigma$-finite and properly infinite, then we may take $s_g$ so that $X_g = Bs_g = s_g B$ and $s^*_g = s_{g^{-1}}$, and $\theta : G \to \text{PAut} B$ (the set of all partial *-automorphims of $B$, i.e., *-isomorphisms between reduced subalgebras of $B$) and $u : G \times G \to \text{PI} B$ are defined by $\theta_g := \text{Ad} s_g : s_g^* s_g B \to s_g s^*_g B$, $x \mapsto s_g x s^*_g$, and $u(g_1, g_2) := s_{g_1} s_{g_2} (s_{g_1 g_2})^*$ so that $\theta_{g_1} \circ \theta_{g_2} = \text{Ad} u(g_1, g_2) \circ \theta_{g_1 g_2}$, $u$ satisfies the 2-cocycle condition, and the product and involution in $A$ are given in terms of $(\theta, u)$.

The work in this talk was intended to generalize, and simplify the proofs of, part of the results in [4].

2 Invertible bimodules and normalizers

In this section $A$ denotes a fixed $AW^*$-algebra ([5], [1]), and $S(A)$ denotes the set of all $AW^*$-subalgebras of $A$. 
Definition (Invertible bimodule, MR-equivalence in an $AW^*$-algebra).

(i) For $B, C \in S(A)$ we call $X \subset A$ an invertible $B$-$C$-bimodule in $A$ if

$$L(X) := \begin{bmatrix} B & X \\ X^* & C \end{bmatrix} \subset M_2(A)$$

is an $AW^*$-subalgebra of $M_2(A)$.

Here $M_2(A)$ (the algebra of $2 \times 2$ matrices over $A$) is an $AW^*$-algebra ([2]). Then $BX + XC \subset X, XX^* \subset B, X^*X \subset C$; hence

(1) $X$ is both a sub-$B$-$C$-bimodule and a subtriple of $A$,

$\exists$ left inner product $\langle \cdot, \cdot \rangle_l : X \times X \to B, (x, y) \mapsto xy^*$,

$\exists$ right inner product $\langle \cdot, \cdot \rangle_r : X \times X \to C, (x, y) \mapsto x^*y$,

$\exists$ triple product $[\cdot, \cdot, \cdot] : X \times X \times X \to X, [x, y, z] := xy^*z$;

(2) $\exists h \in \text{Proj} Z(B)$ (resp. $\exists k \in \text{Proj} Z(C)$): $M_l(X) := M(K_l(X)) = hB, M_r(X) := M(K_r(X)) = kC$, where $\text{Proj}(\cdot)$ denotes the set of projections, $Z(\cdot)$ denotes the center, $K_l(X) := \text{lin} XX^*$ (resp. $K_r(X) := \text{lin} XX^*$) is a norm closed two-sided ideal of $B$ (resp. $C$), and $M(\cdot)$ denotes the multiplier algebra of a $C^*$-algebra.

We write $\text{INV}_A(B, C)$ for the set of all invertible $B$-$C$-bimodules in $A$.

(ii) We call $B, C \in S(A)$ MR (Morita-Rieffel) equivalent in $A$ and write $B \sim_A C$ if $\exists X \in \text{INV}_A(B, C): B = M_l(X), C = M_r(X)$.

Definition (Normalizer). For $B, C \in S(A)$ we call the following sets the normalizer and the regular normalizer of $B, C$ in $A$, respectively ($\text{PI} A$ denotes the set of all partial isometries in $A$):

$$N_A(B, C) := \{x \in A : xC \subset B, x^*B \subset C, xx^* \in B, x^*x \in C\},$$

$$RN_A(B, C) := \{s \in \text{PI} A \cap N_A(B, C) : \exists h \in \text{Proj} Z(B), \exists k \in \text{Proj} Z(C) : h \leq ss^*, k \leq s^*s, s = hs + sk\}.$$ 

Theorem 1. Let $A$ be an $AW^*$-algebra and $B, C \in S(A)$.

(i) For $X \subset A, X \in \text{INV}_A(B, C) \iff \exists s \in RN_A(B, C): X = BsC$.

In this case, $M_l(X) = C_B(ss^*)B, M_r(X) = CC(s^*s)C$ ($C_B(\cdot)$ and $C_C(\cdot)$ denote the the central cover of a projection in $B$ and in $C$, respectively);

$$\text{PI} X := X \cap \text{PI} A = \{usv : u \in \text{PI} B, v \in \text{PI} C, u^*u \leq ss^*, vv^* \leq s^*s, u^*u = sv^*sv\}.$$ 

(ii) For $s, t \in RN_A(B, C), BsC = BtC \iff \exists u \in \text{PI} B, v \in \text{PI} C : t = usv, u^*u = ss^*, s^*s = vv^*.$

(iii) $B \sim_A C \iff \exists s \in RN_A(B, C): C_B(s^*s) = 1_B, C_C(s^*s) = 1_C.$

(iv) $N_A(B, C) = B \cdot RN_A(B, C) \cdot C = \cup\{X : X \in \text{INV}_A(B, C)\}.$
3 Invertible bimodules and partial *-isomorphisms

Definition ((Abstract) invertible bimodule). (i) Let $B$ and $C$ be $AW^{*}$-algebras. We call a linear space $X$ an invertible $B$-$C$-bimodule if it is a $B$-$C$-bimodule and there exist maps $\langle \cdot, \cdot \rangle_{l}: X \times X \to B$, $\langle \cdot, \cdot \rangle_{r}: X \times X \to C$ such that $L(X) := \begin{bmatrix} B & X \\ X^{*} & C \end{bmatrix}$ is an $AW^{*}$-algebra with the following product and involution:

$$
\begin{bmatrix}
    b_{1} & x_{1} \\
    y^{*}_{1} & c_{1}
\end{bmatrix}
\begin{bmatrix}
    b_{2} & x_{2} \\
    y^{*}_{2} & c_{2}
\end{bmatrix} =
\begin{bmatrix}
    b_{1}b_{2} + \langle x_{1}, y_{2} \rangle_{l} & b_{1}x_{2} + x_{1}c_{2} \\
    y^{*}_{1}b_{2} + c_{1}y^{*}_{2} & \langle y_{1}, x_{2} \rangle_{r} + c_{1}c_{2}
\end{bmatrix},
\begin{bmatrix}
    b_{1} & x_{1} \\
    y^{*}_{1} & c_{1}
\end{bmatrix}^{*} =
\begin{bmatrix}
    b_{1}^{*} & y_{1} \\
    x_{1}^{*} & c_{1}^{*}
\end{bmatrix}.
$$

Here $X^{*}$ denotes the set of all $x^{*}$, $x \in X$, which is made into a $C$-$B$-bimodule by the following operations:

$$
\lambda x^{*} = (\lambda x)^{*}, \quad cx^{*}b = (b^{*}xc^{*})^{*} \quad (\lambda \in \mathbb{C}, b \in B, c \in C, x \in X).
$$

Then it follows that $\langle \cdot, \cdot \rangle_{l}$ and $\langle \cdot, \cdot \rangle_{r}$ satisfy the usual properties of inner products.

(ii) We call two $AW^{*}$-algebras $B$ and $C$ MR (Morita-Rieffel) equivalent and write $B \sim C$ if $\exists$ invertible $B$-$C$-bimodule $X$: $M_{l}(X) = B$, $M_{r}(X) = C$.

We write $\text{INV}(B, C)$ for the set of all invertible $B$-$C$-bimodules. If, in particular, $B = C$, we abbreviate this to $\text{INV}(B) := \text{INV}(B, B)$, and call its element an invertible $B$-bimodule.

(iii) We call a map $\tau : X \to Y$ between $X, Y \in \text{INV}(B, C)$ a module monomorphism if it is a $B$-$C$-bimodule map and preserves the inner products (i.e., $\tau(bxc) = b\tau(x)c$,

$$
\langle \tau(x), \tau(y) \rangle_{l} = \langle x, y \rangle_{l}, \langle \tau(x), \tau(y) \rangle_{r} = \langle x, y \rangle_{r}, \forall x, y \in X, b \in B, c \in C).
$$

A surjective module monomorphism is called a module isomorphism.

We call $X, Y \in \text{INV}(B, C)$ isomorphic and write $X \cong Y$ if $\exists$ a module isomorphism $X \to Y$.

$$
X \in \text{INV}(B, C) \Rightarrow X^{*} \in \text{INV}(C, B), (X^{*})^{*} = X. \text{ Here we define the inner products in } X^{*} \text{ by } \langle x^{*}, y^{*} \rangle_{l} := \langle x, y \rangle_{r}, (X^{*})^{*} := \langle x^{*}, y^{*} \rangle_{r} := \langle x, y \rangle_{l} \in B (x, y \in X).
$$

Definition (Partial *-isomorphism). By a partial *-isomorphism between $AW^{*}$-algebras $C$ and $B$ we mean a *-isomorphism of the form $\theta : r(\theta)Cr(\theta) \to l(\theta)Bl(\theta)$, where $r(\theta) \in \text{Proj} C$ and $l(\theta) \in \text{Proj} B$. We call the partial *-isomorphism $\theta$ positive (resp. negative) if $r(\theta) \in \text{Proj} Z(C)$ (resp. $l(\theta) \in \text{Proj} Z(B)$); central if it is both positive and negative; and regular if $\exists$ positive $\theta_{1}$, $\exists$ negative $\theta_{2}$: $\theta = \theta_{1} \oplus \theta_{2}$. Here, when two partial *-isomorphisms $\theta_{i}$, $i = 1, 2$, satisfy the condition $C_{C}(r(\theta_{1}))C_{C}(r(\theta_{2})) = 0 = C_{B}(l(\theta_{1}))C_{B}(l(\theta_{2}))$ (hence $r(\theta_{1}) + r(\theta_{2}))C(r(\theta_{1}) + r(\theta_{2})) = r(\theta_{1})Cr(\theta_{2}) + r(\theta_{2})Cr(\theta_{2})$, it follows that $\theta^{*} = -\theta$. The map $\theta$ is called a partial *-isomorphism if $\theta^{*} = \theta$.
and similarly for \( l(\cdot) \), a partial \(*\)-isomorphism \( \theta_1 \oplus \theta_2 \) is defined by

\[
r(\theta_1 \oplus \theta_2) := r(\theta_1) + r(\theta_2), \quad l(\theta_1 \oplus \theta_2) := l(\theta_1) + l(\theta_2),
\]

\[
(\theta_1 \oplus \theta_2)(x_1 + x_2) := \theta_1(x_1) + \theta_2(x_2), \quad x_i \in r(\theta_i)C r(\theta_i).
\]

We write \( \text{PISom}(B, C) \) for the set of all partial \(*\)-isomorphisms between \( C \) and \( B \), and \( \text{PISom}(B, C)^+, \text{PISom}(B, C)^-, \text{PISom}(B, C)^0 \), and \( \text{RPISom}(B, C) \) for the sets of all positive, negative, central, and regular ones, respectively.

**Definition (Invertible bimodule associated with a regular partial \(*\)-isomorphism).** For \( \theta = \theta_1 \oplus \theta_2 \in \text{RPISom}(B, C) \) with \( \theta_1 \) positive and \( \theta_2 \) negative we define \( \langle \theta \rangle \in \text{INV}(B, C) \) as the set \( Bl(\theta_1) \oplus r(\theta_2)C \) with the following module operation, inner products, and triple product:

\[
\forall b \in B, \forall c \in C, \forall x_1, y_1, z_1 \in Bl(\theta_1), \forall x_2, y_2, z_2 \in r(\theta_2)C:
\]

\[
\delta \cdot (x_1 \oplus x_2) \cdot c := bx_1 \theta_1(r(\theta_1)c) + \theta_2^{-1}(l(\theta_2)b)x_2c,
\]

\[
\langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle_l := x_1y_1^* + \theta_2(x_2y_2^*) \in B,
\]

\[
\langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle_r := \theta_1^{-1}(x_1^*y_1) + x_2y_2 \in C,
\]

\[
[x_1 \oplus x_2, y_1 \oplus y_2, z_1 \oplus z_2] := x_1y_1^*z_1 \oplus x_2y_2^*z_2 = \langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle_l \cdot (z_1 \oplus z_2)
\]

\[
= (x_1 \oplus x_2) \cdot \langle y_1 \oplus y_2, z_1 \oplus z_2 \rangle_r.
\]

**Definition (Equivalence for partial \(*\)-isomorphisms).** Define \( \theta, \psi \in \text{PISom}(B, C) \) to be equivalent, \( \theta \sim \psi \), if \( \exists u \in \text{PIL}B, \exists v \in \text{PIL}C: v^*v \leq r(\psi) \leq C_C(v^*v), vv^* \leq r(\theta) \leq C_C(vv^*), \theta(vv^*) = u^*u, \psi \circ vCv^*v = (\text{Ad}u) \circ \theta \circ (\text{Ad}v) \).

We denote by \( [\theta] \) the equivalence class in \( \text{PISom}(B, C) \) containing \( \theta \), and by \( [S] \) the set of all equivalence classes containing elements of \( S \subset \text{PISom}(B, C) \).

**Proposition 2.** If \( B \) and \( C \) are \( AW^* \)-algebras, then \( \forall X \in \text{INV}(B, C), \exists \theta \in \text{RPISom}(B, C): X \cong \langle \theta \rangle \), and hence \( \exists \) bijection \( \text{INV}(B, C) \leftrightarrow [\text{RPISom}(B, C)], [BSC] \leftrightarrow [\text{Ad}s] \).

**Theorem 3.** Let \( B \) and \( C \) be monotone complete \( C^* \)-algebras (and hence \( AW^* \)-algebras; here a \( C^* \)-algebra is called **monotone complete** if every bounded increasing net in its self-adjoint part has a supremum).

(i) The following conditions are equivalent:

1. \( B \sim C \) (MR-equivalent);
2. \( \exists \theta \in \text{RPISom}(B, C): C_B(l(\theta)) = 1_B, C_C(r(\theta)) = 1_C; \)
3. \( \exists \theta \in \text{PISom}(B, C): \) as in (2).

(ii) \( \exists \) monotone complete \( C^* \)-algebra \( D \) containing \( B \) and \( C \) as monotone closed \( C^* \)-subalgebras: each element of \( \text{INV}(B, C) \) or \( \text{PISom}(B, C) \) is realized via a partial isometry...
of $D$, i.e., $\forall X \in \text{INV}(B, C), \exists s \in \text{RN}_D(B, C)$ (the normalizer of $B$, $C$ in $D$): $X \cong BsC$, $\forall \theta \in \text{PIsom}(B, C), \exists s \in \text{RN}_D(B, C): \theta \sim \text{Ad}s$.

(iii) $\exists$ inverse semigroup $S$ (cf., e.g., [6]): $[\text{INV}(B, C)] \cong [\text{PIsom}(B, C)]$ is a subtriple of $S$. Here $T \subset S$ is called a subtriple of $S$ if $TT^{-1}T = T$ (and so the triple product $[x, y, z] := x^{y^{-1}}z$ is defined in $T$). Moreover the triple products in $[\text{INV}(B, C)]$ and in $[\text{PIsom}(B, C)]$ are described in terms of the tensor product of bimodules and the composition of maps, respectively.

4 $AW^*$-triples

Definition ($AW^*$-triple). (i) Let $A$ be an $AW^*$-algebra. We call $X \subset A$ an $AW^*$-subtriple of $A$ if $\exists B, C \in \text{S}(A)$: $X \in \text{INV}_A(B, C)$.

(ii) By an $AW^*$-triple we mean an $AW^*$-subtriple of some $AW^*$-algebra. Here we identify two $AW^*$-triples $X$ and $Y$ if $\exists$ triple isomorphism $\tau : X \to Y$ (a linear bijection satisfying the condition

$$\tau([x, y, z]) = [\tau(x), \tau(y), \tau(z)], \forall x, y, z \in X,$$

i.e., we consider only the triple products forgetting the bimodule structures.

Proposition 4. (i) Every $AW^*$-triple $X$ is written in the following form:

$$X = X^{++} \oplus X^0 \oplus X^{--}, \quad X^{++} \cong A_1e, \quad X^0 \cong A_2, \quad X^{--} \cong fA_3,$$

where $A_i, i = 1, 2, 3$, are $AW^*$-algebras, $e \in \text{Proj} A_1$, $C_{A_1}(e) = 1_{A_1}$,

$$\exists h \in \text{Proj} Z(A_1), \exists u \in \text{PI} A_1 : \ he = uu^*, \ u^*u = h \Rightarrow h = 0;$$

$f \in \text{Proj} A_3, C_{A_3}(f) = 1_{A_3}, f$ satisfies the condition similar to the above; and the triple products in $A_1e, A_2, fA_3$ are given by $[x, y, z] := xy^*z$.

(ii) For $AW^*$-triples $X, Y$, write $X^{++} = A_1e$, $X^0 = A_2$, $X^{--} = fA_3$, $Y^{++} = B_1p$, $Y^0 = B_2$, $Y^{--} = qB_3$ as above. Then $\tau : X \to Y$ is a triple isomorphism $\iff$

$$\tau(X^{++}) = Y^{++}, \tau(X^0) = Y^0, \tau(X^{--}) = Y^{--}, \exists *\text{-isomorphisms } \alpha : A_1 \to B_1, \beta : A_2 \to B_2, \gamma : A_3 \to B_3, \exists u \in \text{PI} B_1, \alpha(e) = uu^*, \ u^*u = p, \exists v \in B_2: \text{unitary}, \exists w \in \text{PI} B_3, \gamma(f) = w^*w, \ ww^* = q$:

$$\tau|X^{++} = \alpha(\cdot)u, \quad \tau|X^0 = \beta(\cdot)v, \quad \tau|X^{--} = w\gamma(\cdot).$$
References


