数理解析研究所講究録 1459 巻 2005 年 74-81

Types of von Neumann algebras arising from boundary actions of hyperbolic groups

大阪教育大学　岡安　類 (OKAYASU Rui)
Osaka Kyoiku University

1 Introduction

In this report, we discuss the joint work with Masaki Izumi and Sergey Neshveyev. A non-degenerate finitely supported probability measure on a non-elementary hyperbolic group defines a quasi-invariant harmonic measure on the boundary of the group. The corresponding von Neumann algebra is well-known to be an injective factor of type III. We discuss what can be said about the S-invariant of the factor, or in other words, about the ratio set of the orbit equivalence relation on the boundary.

2 Preliminary

2.1 Martin and Poisson boundaries

We first introduce the notations of Martin and Poisson boundaries. We refer the reader to e.g. [W] for details.

Let Γ be a countable discrete group and μ a finitely supported probability measure on Γ. We assume that μ is non-degenerate in the sense that the semigroup generated by the support of μ coincides with Γ. The measure μ defines a random walk on Γ with transition probabilities $p(x,y)=\mu(x^{-1}y)$. The n-step transition probabilities are given by

$$p^{(n)}(x,y)=\mu^{(n)}(x^{-1}y),$$

where $\mu^{(n)}$ is the n-fold convolution of μ with itself. We assume that the random walk is transient, that is, the Green function

$$G(x,y)=\sum_{n=0}^{\infty}p^{(n)}(x,y)$$

is finite for every $x,y\in\Gamma$.

The Martin kernel is defined by

$$K(x,y) = \frac{G(x,y)}{G(e,y)},$$

where $e \in \Gamma$ is the unit element. The Martin compactification $\overline{\Gamma}$ of Γ is the smallest compactification such that $\Gamma \subset \overline{\Gamma}$ is discrete and the functions $K(x,\cdot)$, $x \in \Gamma$, extend to continuous functions on $\overline{\Gamma}$. The Martin boundary is $\partial_M\Gamma = \overline{\Gamma}\backslash\Gamma$. The left action of Γ on itself extends to a continuous action on $\overline{\Gamma}$.

Let $\Omega = \prod_{n=0}^{\infty}\Gamma$ be the path space of our random walk. For any point $g \in \Gamma$ we have the Markov measure $\mathbb{P}_g$ defined on paths starting at g. For $\mathbb{P}_g$-a.e. path $\underline{x} = \{x_n\}_{n\geq 0} \in \Omega$ the sequence $\{x_n\}_{n\geq 0}$ converges to a point on the boundary, so we get a measurable map $\Omega \to \partial_M\Gamma$. We denote by ν_g the image of $\mathbb{P}_g$ under this map. The measure space $(\partial_M\Gamma, \nu_e)$ is called the Poisson boundary of Γ. We will write ν instead of ν_e. The measures $\{\nu_g\}_{g\in\Gamma}$ are mutually equivalent, and we have

$$K(g,\omega) = \frac{d\nu_g}{d\nu}(\omega) = \frac{dg\nu}{d\nu}(\omega),$$

where $g\nu$ is the measure defined by $g\nu(X) = \nu(g^{-1}X)$.

2.2 Hyperbolic groups

We next recall basic facts about hyperbolic groups. We refer the reader to e.g. [GH] for details. Let Γ be a finitely generated group with a symmetric finite set S of generators. We denote by $|g|$ the word length and by $d(x,y) = |x^{-1}y|$ the word metric with respect to S. We denote the ball with center x and radius r by

$$B(x,r) = \{g \in \Gamma \mid d(x,g) \leq r\}.$$

For a subset $\Delta \subset \Gamma$, we write

$$N(\Delta,r) = \{g \in \Gamma \mid d(g,\Delta) \leq r\}.$$

The Gromov product is defined by the formula

$$(x|y)_z = \frac{1}{2}(d(x,z) + d(y,z) - d(x,y))$$

for $x, y, z \in \Gamma$. When z is the unit element e, we simply write $(x|y) = (x|y)_e$.

Let $\delta \geq 0$. The group Γ is said to be δ-hyperbolic if

$$(x|y) \geq \min\{(x|z), (y|z)\} - \delta$$

for every $x, y, z \in \Gamma$. One says that Γ is hyperbolic if there is $\delta \geq 0$ such that Γ is δ-hyperbolic.

If Γ is δ-hyperbolic, then every geodesic triangle $\triangle = \{\alpha, \beta, \gamma\}$ in Γ is 4δ-slim, that is,

$$\alpha \subset N(\beta \cup \gamma, 4\delta), \quad \beta \subset N(\gamma \cup \alpha, 4\delta), \quad \gamma \subset N(\alpha \cup \beta, 4\delta).$$

A sequence $\{x_i\}_{i \geq 1}$ in Γ is said to converge to infinity if

$$\lim_{i,j \to \infty} (x_i | x_j) = \infty.$$

Two sequences $\{x_i\}_{i \geq 1}$ and $\{y_i\}_{i \geq 1}$ converging to infinity are said to be equivalent if

$$\lim_{i,j \to \infty} (x_i | y_j) = \infty.$$

The Gromov boundary $\partial\Gamma$ is defined as the set of equivalence classes of sequences converging to infinity. If p is a point in $\partial\Gamma$, we say that a sequence $\{x_i\}_{i \geq 1}$ in Γ converges to p if this sequence belongs to p. The Gromov product $(p|q)$ for $p, q \in \Gamma \cup \partial\Gamma$ is defined by

$$(p|q) = \sup \liminf_{i,j \to \infty} (x_i | y_i)$$

where the sup above runs over all sequences $\{x_i\}_{i \geq 1}$ converging to p and $\{y_i\}_{i \geq 1}$ converging to q.

Recall that $\Gamma \cup \partial\Gamma$ is compact equipped with the base $\{B(x, r)\} \cup \{V_r(p)\}$, where

$$V_r(p) = \{q \in \Gamma \cup \partial\Gamma \mid (q|p) > r\}.$$

For $p, q \in \Gamma \cup \partial\Gamma$, we denote by $[p, q]$ the set of all geodesic segments (or rays, lines) between p and q.

3 Main result

We always assume that Γ is a non-elementary hyperbolic group (that is, it does not have a cyclic subgroup of finite index) and consider a random walk on it defined by a finitely supported non-degenerate probability measure μ. Then its Martin boundary coincides with the Gromov boundary by a result of Ancona [A].

We denote by $\mathcal{R}(\Gamma, \mu) = \mathcal{R}(\Gamma, \partial\Gamma, \nu)$ the orbit equivalence relation defined by the action of Γ on $(\partial\Gamma, \nu)$.

Recall that the ratio set $r(\Gamma, \partial\Gamma, \nu)$ consists of all $\lambda \geq 0$ such that for any $\varepsilon > 0$ and any subset $A \subset \partial\Gamma$ of positive measure there are $g \in \Gamma$ and $B \subset A$ with positive measure such that $gB \subset A$ and

$$\lambda e^{-\epsilon} < \frac{dg^{-1}\nu}{d\nu}(\omega) < \lambda e^{\varepsilon} \qquad \text{for } \omega \in B.$$

Note that $\mathcal{R}(\Gamma, \mu)$ is ergodic, amenable and of type III by [K]. Hence $\{0, 1\} \subset r(\Gamma, \partial\Gamma, \nu)$, and $r(\Gamma, \partial\Gamma, \nu) \setminus \{0\}$ is a closed multiplicative subgroup of $(0, +\infty)$. One says that $\mathcal{R}(\Gamma, \mu)$ is of type III_0, III_λ $(0 < \lambda < 1)$ or III_1 depending on whether this group is $\{1\}$, $\{\lambda^n\}_{n\in\mathbb{Z}}$ or $(0, +\infty)$. Recall also that for $0 < \lambda \leq 1$ there is only one amenable ergodic equivalence relation of type III_λ.

We can now formulate our main result.

Theorem 1 *Let Γ be a non-elementary hyperbolic group and μ be a finitely supported non-degenerate probability measure on Γ. Then $\mathcal{R}(\Gamma, \mu)$ is never of type III_0.*

We will briefly explain how we prove the above theorem in next two sections.

4 An observation

It is known that every infinite order element $g \in \Gamma$ acts on $\partial\Gamma$ as a hyperbolic homeomorphism, i.e., there are exactly two fixed points g^+ and g^- in $\partial\Gamma$ such that g^+ is stable and g^- is unstable. For any open subsets $U^\pm \subset \partial\Gamma$ with $g^+ \in U^+$ and $g^- \in U^-$, it holds that $g^n(\partial\Gamma \setminus U^-) \subset U^+$

for sufficiently large $n \geq 0$. If $h \in \Gamma$ then $g^n h \to g^+$ and $g^{-n}h \to g^-$ as $n \to +\infty$.

For any infinite order element $g \in \Gamma$ we define

$$r(g) = K(g^{-1}, g^+).$$

Note that being a non-zero positive harmonic function, $K(\cdot, \xi)$ is nowhere vanishing, so that $r(g) > 0$. We can also write

$$\begin{aligned} r(g) &= \lim_{n\to\infty} \frac{G(e, g^{n+1})}{G(e, g^n)} = \lim_{n\to\infty} G(e, g^n)^{1/n} \\ &= \lim_{n\to\infty} \frac{F(e, g^{n+1})}{F(e, g^n)} = \lim_{n\to\infty} F(e, g^n)^{1/n} \\ &= \sup_n F(e, g^n)^{1/n}. \end{aligned}$$

We put $r(g) = 1$ for any finite order element $g \in \Gamma$.

Lemma 2 *The function r on Γ is a class function satisfying $r(g^k) = r(g)^k$ for $k \in \mathbb{N}$. If μ is symmetric, then $r(g) = r(g^{-1})$.*

Example 3 *Consider the simple random walk defined by the canonical symmetric generating set S of the free group $\mathbb{F}_N$. Then*

$$F(e, s) = \frac{1}{2N - 1}$$

for $s \in S$, see e.g. [L, Section 2a]. It follows that

$$F(e, g) = (2N - 1)^{-|g|}$$

for any $g \in \mathbb{F}_N$. We can then conclude that

$$r(g) = (2N - 1)^{-\ell_g},$$

where ℓ_g is the minimal length of elements in the conjugacy class of g.

Lemma 4 *If $g \in \Gamma$ is an infinite order element, then $0 < r(g) < 1$.*

Hence to prove Theorem 1, it suffices to show that

$$r(g) \in r(\Gamma, \partial\Gamma, \nu).$$

We now fix an infinite order element $g \in \Gamma$. Let $\varepsilon > 0$. For any non-empty open subset $U \subset \partial\Gamma$, since the action of Γ on $\partial\Gamma$ is minimal, there is $h \in \Gamma$ such that $hg^+ \in U$. Note that $(hgh^{-1})^+ = hg^+$. Then thanks to the cocyle property, we have

$$\frac{dhg^{-1}h^{-1}\nu}{d\nu}(hg^+) = K(hg^{-1}h^{-1}, hg^+) = K(g^{-1}, g^+) = r(g).$$

Since $K(hg^{-1}h^{-1}, \cdot\,)$ is continuous on $\partial\Gamma$, there exists a open neighbourhood V of hg^+ such that $V \cup hg^{-1}h^{-1}V \subset U$ and

$$r(g)e^{-\varepsilon} < \frac{dhg^{-1}h^{-1}\nu}{d\nu}(\omega) < r(g)e^{-\varepsilon}$$

for any $\omega \in V$.

Therefore by the above observation, to prove Theorem 1, it suffices to replace the above topological arguments by measurable ones. For our purpose, next results play important roles in the proof. The proof is inspired by Bowen's computation of the ratio set of a Gibbs measure in [B, Lemma 8]. (See also Theorem 7.)

5 Key results

5.1 Multiplicativity of the Green function along geodesic segments

For any points x, y and z we have $F(x,z)G(z,y) \leq G(x,y)$, where

$$F(x,z) = \frac{G(x,z)}{G(z,z)}$$

is the probability that a path starting at x hits z. The main technical result of Ancona [A] needed to identify the Martin boundary of a hyperbolic group with its Gromov boundary is that up to a factor the converse inequality is also true if z lies on a geodesic segment $\alpha \in [x,y]$. We need a slightly stronger result saying that the same is true for the restriction of the random walk to any subset containing a sufficiently large neighbourhood of the segment. This is essentially contained in [A].

For a subset $\Delta \subset \Gamma$ consider the induced random walk on Δ (to be precise, to get a random walk we have to add a cemetery point to Δ).

We denote the corresponding quantities using the subscript Δ, so we write G_Δ and F_Δ.

Theorem 5 *There exist $R_0 > 0$ and $C \geq 1$ such that if $x, y \in \Gamma$ and $v \in \Gamma$ lies on a geodesic segment $\alpha \in [x, y]$, then*

$$G_\Delta(x,y) \leq CF_\Delta(x,v)G_\Delta(v,y)$$

for any $\Delta \subset \Gamma$ containing $N(\alpha, R_0)$.

5.2 The Hölder condition of the Martin kernel

In [L] Ledrappier proves that in the case of a free group the Martin kernel is Hölder continuous, which is a discrete analogue of a result of Anderson and Schoen [AS]. We need to extend this result to hyperbolic groups to attain our purpose.

Theorem 6 *There exist $0 \leq \tau < 1$ and for any $g \in \Gamma$ a constant $H_g \geq 0$ such that for $\xi, \eta \in \partial\Gamma$,*

$$|K(g,\xi) - K(g,\eta)| \leq H_g \tau^{(\xi|\eta)}.$$

5.3 A Gibbs-like property of a harmonic measure

For $\xi \in \partial\Gamma$ and $R > 0$, we define $U(\xi, R)$ to be the set of all $\eta \in \partial\Gamma$ such that for any pair of geodesic rays $\{x(n)\}_{n=0}^\infty \in [e, \xi]$ and $\{y(n)\}_{n=0}^\infty \in [e, \eta]$, we have

$$\lim_{n\to\infty} (x(n)|y(n)) > R.$$

Remark that the sequence $\{(x(n)|y(n))\}_{n=0}^\infty$ is non-decreasing and thus the above limit always exists. These sets are considered as hyperbolic versions of cylindric sets.

The following property of the harmonic measure ν on $\partial\Gamma$ reminds of a Gibbs measure.

Theorem 7 *There exists $D \geq 1$ such that for every $\xi \in \partial\Gamma$ and $\{x(n)\}_{n=0}^\infty \in [e, \xi]$, we have*

$$\frac{1}{D} \leq \frac{\nu(U(\xi,R))}{F(e,x(R))} \leq D \quad \textit{for } R \in \mathbb{N}.$$

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